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Mathematical optimization and robust control synthesis

Xin Qi

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**Mathematical optimization
and
robust control synthesis**

by
Xin Qi

A dissertation submitted to the graduate faculty
in partial fulfillment of the requirements for the degree of
DOCTOR OF PHILOSOPHY

Major: Electrical Engineering (Control Systems)

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For the Major Program

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LIST OF SYMBOLS

c_0	The subspace of ℓ_∞ whose element x satisfies $\lim_{k \rightarrow \infty} x(k) = 0$.
$c_0^{m \times n}$	The space of matrix-valued right-sided real sequences such that each element $x \in c_0^{m \times n}$ is the matrix (x_{ij}) and each x_{ij} is in c_0 .
$\mathcal{H}_2$	The isometrically isomorphic image of ℓ_2 under the λ transform with the norm given by $\ \hat{x}(\lambda)\ _{\mathcal{H}_2} = \ x\ _2$.
$\mathcal{H}_\infty^{m \times n}$	The space of complex-valued matrix functions that are analytic in the open unit disc and bounded on the unit circle with the norm defined by $\ \hat{x}\ _{\mathcal{H}_\infty} := \text{ess sup}_\theta \sigma_{\max}[\hat{x}(e^{i\theta})]$.
ℓ_1	The Banach space of right sided absolutely summable real sequences with the norm given by $\ x\ _1 := \sum_{k=0}^{\infty} x(k) $.
$\ell_1^{m \times n}$	The Banach space of matrix-valued right-sided real sequences with the norm given by $\ x\ _1 := \max_{1 \leq i \leq m} \sum_{j=1}^n \ x_{ij}\ _1$, where $x \in \ell_1^{m \times n}$ is the matrix (x_{ij}) and each x_{ij} is in ℓ_1 .
ℓ_2	The Banach space of right-sided square summable real sequences with the norm given by $\ x\ _2 := (\sum_{k=0}^{\infty} x^2(k))^{1/2}$.

ℓ_∞	The Banach space of bounded real sequences with the norm given by $\ x\ _\infty := \sup_k x(k) $.
$\hat{x}(\lambda)$	The λ transform of a right-sided real sequence $x = (x(k))_{k=0}^\infty$ defined as $\hat{x}(\lambda) := \sum_{k=0}^\infty x(k)\lambda^k$.
$\mathcal{N}$	The set of all the natural numbers.
P_n	The truncation operator on the space of sequences and is defined by $P_n(x(0) \ x(1) \ \dots) = (x(0) \ x(1) \ \dots \ x(n) \ 0 \ 0 \ \dots)$.
$\mathcal{R}$	The real number system.
$\mathcal{R}^{m \times n}$	The set of m -by- n dimensional matrices.
$\mathcal{R}^n$	The n dimensional Euclidean space.
$W(X^*, X)$	The weak star topology on X^* induced by X .
X^*	The dual space of a normed linear vector space X . For $x^* \in X^*$, $\langle x, x^* \rangle$ denotes the value of the bounded linear functional x^* at $x \in X$ ([1]). For example, it can be verified that $(c_0^{m \times n})^* = \ell_1^{m \times n}$ and for $x \in c_0^{m \times n}$ and $x^* \in \ell_1^{m \times n}$, $\langle x, x^* \rangle = \sum_{i=1}^m \sum_{j=1}^n \sum_{k=0}^\infty x_{ij}(k) x_{ij}^*(k)$.

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ABSTRACT

For many problems, it is very hard or even impossible to obtain analytic solutions. In recent years, powerful numerical tools for solving mathematical programming/optimization problems have been developed. This makes it possible to formulate control design problems as mathematical programming problems and then solve them using numerical optimization techniques. In this thesis, we show that two classes of important robust control design problems can be tackled by employing optimization techniques.

In the first part of the thesis, we present a methodology to address the general multiobjective (GMO) control problem involving the ℓ_1 norm, $\mathcal{H}_2$ norm, $\mathcal{H}_\infty$ norm, time-domain constraint (TDC), and controller structural constraints. We show that the auxiliary problem resulting after imposing a regularizing condition always admits an optimal solution, and suboptimal solutions with performance arbitrarily close to the global optimal cost can be obtained by constructing two sequences of finite dimensional convex optimization problems whose objective values converge to the optimum from below and above. Numerical implementation of the proposed methodology is discussed and several numerical examples are presented to illustrate the effectiveness of the proposed methodology.

In the second part, we consider the integrated parameter and control (IPC) design problem where the system structure parameters enter the state-space representation of the system in a rational manner. This problem is a non-convex infinite dimensional

optimization problem. Converging finite-dimensional sub-optimal problems are constructed and solved via a linear relaxation technique, whereby a global optimal solution to the IPC problem is computed within any given tolerance. A numerical example is provided.

PART I

MULTIOBJECTIVE CONTROL SYNTHESIS

CHAPTER 1 INTRODUCTION

1.1 Motivation

In the last twenty years, designing engineering systems that are insensitive to uncertainties has attracted considerable attention. Various robust control theories (e.g, $\mathcal{H}_\infty$ theory, ℓ_1 theory) have been proposed to deal with the effects of uncertainties. The common practice in these methodologies is to optimize the closed-loop system for a given measure of the system with respect to all the stabilizing controllers. In practice, however, diversity of the uncertainties exerted on engineering systems renders it impossible to evaluate controllers' performance by using a single measure. Thus, in a typical controller synthesis procedure, multiple quantifiers are employed to judge the quality of a controller.

An example where the multiobjective concerns exist naturally is the suspension control for transport vehicles ([2]). In these systems, suspensions are designed to achieve several conflicting goals that can be translated into three norm-based objectives: $\mathcal{H}_2$ minimization to optimize the driver and cargo comfort for stochastic road disturbances; ℓ_1/L_1 optimization to prevent certain variables like control action exceeding specified limits; bounding the $\mathcal{H}_\infty$ norm to deal with the variability in the system parameters and model structure errors. The suspension controller design may be reduced to a search for a suitable tradeoff among the above three norm-based objectives.

To achieve certain desirable aerodynamic characteristics, the wings of the X29 aircraft are designed to be in the forward-swept shape. This renders better maneuverabil-

ity to the aircraft when compared with classical wing design while leaving the aircraft statically unstable ([3]). The control objective for this plant is to design a stabilizing discrete-time controller to minimize ℓ_1 norm of the transfer function from the disturbance w injected at the plant output to the weighted control signal z_1 and the weighted output z_2 while achieving a good tracking performance for step input signals. These objectives can be achieved by solving the following multi-objective optimization problem:

$$\begin{aligned} & \inf_{K \text{ stabilizing}} \left\| \begin{bmatrix} W_1 K S \\ W_2 S \end{bmatrix} \right\|_1 \\ & \text{subject to} \\ & a_{temp}(k) \leq S * stepIn(k) \leq b_{temp}(k), \quad \forall k. \end{aligned}$$

where $stepIn$ denotes a step and a_{temp} and b_{temp} are two prescribed time-domain template constraints (TDCs).

1.2 Mathematical Formulation

Consider the system shown in Figure 1.1, where $G : [w; u] \rightarrow [z; v]$ is the generalized discrete-time linear time-invariant plant and K is the controller. w , z , u , and v are the exogenous input, regulated output, control input, and measured output, respectively. r is a given scalar reference input (such as a step) and s is the time response output.

Let $\hat{R}$ denote the closed-loop transfer matrix from w to z . The set of all the achievable closed-loop maps is given by ([3]):

$$\{\hat{R} = G_{zw} + G_{zu}K(I - G_{yu}K)^{-1}G_{yw} \mid K \text{ stabilizing and structured}\} \quad (1.1)$$

where $G = [G_{zw} \ G_{zu}; G_{yw} \ G_{yu}]$ is the open-loop transfer matrix from $[w; u]$ to $[z; y]$.

To simplify the notations, in the sequel, we use $\hat{R}^i$ ($i = 1, \dots, 6$) to denote the closed-loop transfer matrix from w_i to z_i and $\hat{R}^7$ the transfer function from r to s .

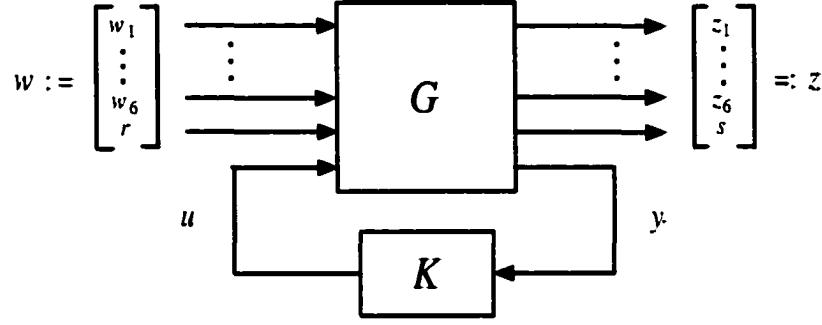


Figure 1.1 Closed-loop system

The GMO problem studied in this paper can be stated as follows: *Given the plant G , constants $c_i > 0$, $i = 1, \dots, 6$, and two sequences $\{a_{temp}(k)\}_{k=0}^{\infty}$ and $\{b_{temp}(k)\}_{k=0}^{\infty}$, solve the following problem,*

$$\begin{aligned}
 & \inf_{K \text{ stabilizing and structured}} \{c_1 \|R^1(K)\|_1 + c_2 \|\hat{R}^2(K)\|_{\mathcal{H}_2}^2 + c_3 \|\hat{R}^3(K)\|_{\mathcal{H}_\infty}\} \\
 & \text{subject to} \\
 & \quad \|R^4(K)\|_1 \leq c_4 \\
 & \quad \|\hat{R}^5(K)\|_{\mathcal{H}_2}^2 \leq c_5 \\
 & \quad \|\hat{R}^6(K)\|_{\mathcal{H}_\infty} \leq c_6 \\
 & \quad a_{temp}(k) \leq s(k) \leq b_{temp}(k), \quad k = 0, 1, 2, \dots
 \end{aligned} \tag{1.2}$$

where $\{s(k)\}_{k=0}^{\infty}$ denotes the time response of the closed-loop system due to the exogenous reference input r with $w_i = 0$, $i = 1, \dots, 6$. Let μ denote the optimal value of the above problem. From now on, we will always assume that problem (1.2) has a nonempty feasible set, which includes the requirement that the optimal cost μ be finite.

The GMO problem defined above represents a large class of multiobjective control problems. Many extensively studied (unstructured) multiobjective problems are special cases of the GMO setup, e.g., $\mathcal{H}_2/\ell_1$ ([4]), ℓ_1/TDC ([5],[6]). Furthermore, for the first time, the $\mathcal{H}_\infty/\ell_1$ problem and $\ell_1/\mathcal{H}_2/\mathcal{H}_\infty$ problem are addressed. The problem

formulation in (1.2) also provides a uniform framework for the performance tradeoff study involving the ℓ_1 , $\mathcal{H}_2$, $\mathcal{H}_\infty$, and TDC. By solving the GMO problem for various combinations of the parameters $c_i (i = 1, \dots, 6)$ and template sequences $\{a_{temp}(k)\}$ and $\{b_{temp}(k)\}$, important information on the limits of system performance can be obtained both qualitatively and quantitatively.

It turns out that, the classes of robust optimal control synthesis problems with structural constraints imposed on the controllers considered in [7] and [8] can be formulated into the proposed framework and solved. It was shown that ([7, 8]) by utilizing a particular class of coprime factorization, the structural constraints imposed on the controller K can be equivalently transformed to the same constraints on the Youla parameter Q . We denote by $\mathcal{S}$ the subspace of stable systems $Q \in \ell_1^{n_u \times n_y}$ that have the required structure. Then, a characterization of all the achievable closed-loop maps can be given as follows:

$$\{R \in \ell_1^{n_z \times n_w} | R = H - U * Q * V \text{ with } Q \in \mathcal{S}\} \quad (1.3)$$

where $H \in \ell_1^{n_z \times n_w}$, $U \in \ell_1^{n_z \times n_u}$, $V \in \ell_1^{n_y \times n_w}$, Q is a free parameter in $\mathcal{S}$ and $'*$ ' denotes the convolution operation. In the sequel, without any loss of generality, we shall always assume that $\hat{U}$ and $\hat{V}$ have full column and row ranks, respectively (see [3]). Also it can be assumed that $\hat{U}$ and $\hat{V}$ are polynomial matrices in λ , i.e. impulse response sequences of U and V are finitely supported. In case that $\hat{U}$ and $\hat{V}$ were rational matrices in λ , doubly-coprime factorizations can be performed on $\hat{U}$ and $\hat{V}$ and the resulting right and left coprime factors of $\hat{U}$ and $\hat{V}$ can be readily incorporated into $\hat{Q}$ ([3]). In the sequel, we will also assume that H has been approximated by a finitely supported impulse response matrix sequence. This assumption is justified by the fact that H is an operator in the $\ell_1^{n_z \times n_w}$ space.

Let H , U and V in the Youla parameterization (1.3) be partitioned into submatrices of compatible dimensions with the exogenous input component w_i and regulated output

component z_i . Then the closed-loop transfer matrix sequences from w_i to z_i can be expressed as $R^i(Q) = H^{ii} - U^i * Q * V^i, i = 1, \dots, 7$.

For the sake of simplicity, and without loss of generality, we shall consider the case when r is a step sequence. Let A_{temp} be defined as:

$$A_{temp} := \begin{pmatrix} 1 & 0 & 0 & \dots \\ 1 & 1 & 0 & \dots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

Then the time response of the closed-loop system due to the reference input r is given by $s = R^7 * r = A_{temp} R^7$.

Based on the Youla parameterization and the discussion above, the problem defined in (1.2) has the following equivalent formulation: *Given a plant P , constants $c_i > 0$, $i = 1, \dots, 6$, and two sequences $\{a_{temp}(k)\}_{k=0}^{\infty}$ and $\{b_{temp}(k)\}_{k=0}^{\infty}$, solve the following problem,*

$$\begin{aligned} \mu = & \inf_{Q \in \mathcal{S}} f(Q) \\ \text{subject to } & \|R^1(Q)\|_1 \leq c_4 \\ & \|R^5(Q)\|_2^2 \leq c_5 \\ & \|\hat{R}^6(Q)\|_{\mathcal{H}_{\infty}} \leq c_6 \\ & a_{temp}(k) \leq [A_{temp} R^7(Q)](k) \leq b_{temp}(k), \quad k = 0, 1, 2, \dots \end{aligned} \tag{1.4}$$

where $f(Q) := c_1 \|R^1(Q)\|_1 + c_2 \|R^2(Q)\|_2^2 + c_3 \|\hat{R}^3(Q)\|_{\mathcal{H}_{\infty}}$, $R^i(Q) = H^{ii} - U^i * Q * V^i, i = 1, \dots, 7$.

1.3 Current Approaches

As indicated in the previous examples, a problem with multiple objectives can be cast as an optimization problem with mixed frequency- and time-domain specifications imposed on the $\mathcal{H}_2$ performance, $\mathcal{H}_{\infty}$ performance, peak-to-peak closed-loop gain, and

transient time response due to exogenous inputs (such as a step). Although it is desirable to have all four types of specifications present in the multiobjective formalism, most current approaches address the problems combining a subset of the objectives listed above. In [10]-[16], various approaches were proposed to compute and improve the upper bounds to the $\mathcal{H}_2/\mathcal{H}_\infty$ combination problem. In [17], an linear matrix inequality (LMI) based approach was presented to compute a sequence of bounds that converge to the optimum from below. This complements the solution to the multiobjective $\mathcal{H}_2/\mathcal{H}_\infty$ control problem by furnishing a stopping criterion for the algorithms developed in [10]-[16].

In the $\ell_1/\mathcal{H}_\infty$ problem, the objective is to minimize the worst case peak output due to persistent disturbances while at the same time satisfying a bound on the $\mathcal{H}_\infty$ norm of a certain given closed-loop transfer matrix. In [3] and [5], linear programming (LP) and duality theory were used to solve this problem by approximating the $\mathcal{H}_\infty$ constraint with a finite set of linear constraints obtained by sampling the unit circle. It has been shown, however, that for a class of problems, this approximation may fail to converge even as the number of sampling points tend to infinity([18]). In [19], the solution to the four-block $\ell_1/\mathcal{H}_\infty$ problem was obtained by solving a finite-dimensional convex optimization problem together with an unconstrained $\mathcal{H}_\infty$ problem. In [20], the existence of an optimal solution to the multi-block $\ell_1/\mathcal{H}_\infty$ problem is established. Moreover, [20] showed that the optimal solution can be approximated arbitrarily closely by real-rational transfer matrices.

For the mixed-norm optimization problems involving $\mathcal{H}_2$ and ℓ_1 objectives, two main lines of approach have been developed to obtain the solution. In [5] and [23], solutions to the $\mathcal{H}_2/\ell_1$ problem were developed by using quadratic programming techniques combined with duality theory. Nevertheless inasmuch as the achievable closed-loop maps are characterized by using zero interpolation, this line of approach will lead to a heavy

computational burden. Also since the inversion of certain rational matrices is required in the recovery of optimal controller from the resulting optimal closed-loop map, these approaches commonly suffer from numerical difficulties given the finite precision accompanying any numerical method. Recently, a new approach was proposed in [24] to deal with the problems involving ℓ_1 optimization. This method, which is referred as the Scaled- Q method, avoids the utilization of zero interpolation to characterize the admissible closed-loop maps. Also it yields the impulse response sequence of the minimizing Youla parameter Q as the optimal solution. This makes straightforward the task of controller recovery. More noticeably, this approach suggests that introducing a norm bound on the Youla parameter in the optimization may lead to a well regularized optimization problem. Motivated by this idea and by appealing to the Banach-Alaoglu theorem, solution to the $\mathcal{H}_2/\ell_1$ problem has been developed in [25] and [4].

Often performance requirements on the transient time response of the closed-loop system to a given test signal (such as a step) are imposed. It is well recognized that standard single-norm optimal control (ℓ_1 , $\mathcal{H}_2$, or $\mathcal{H}_\infty$) strategies cannot handle specifications or constraints on the time response of a closed-loop system exactly. Thus, there exists a need to consider the time response specifications explicitly in the multi-objective problem setup. The multiobjective problem of minimizing the $\mathcal{H}_\infty$ norm with finite horizon TDC was solved in [26]-[28]. Solutions were obtained in [29] to the case when the template constraints were imposed over an infinite horizon. In [30], the problem of $\mathcal{H}_2$ minimization with constraints on the time-domain response of a closed-loop transfer function was studied. The TDC was first translated into an ℓ_∞ bound on the closed-loop map of interest, and then the problem was solved by solving a sequence of finite-dimensional quadratic programming problems. In [31], an algorithm was proposed to explicitly obtain the state feedback control law to minimize a quadratic performance criterion with TDC on inputs quadratic performance criterion with TDC on inputs and

states. For both the finite horizon and the infinite horizon problems, the control laws were shown to be piecewise linear and continuous. In [5] and [6], the problem of ℓ_1 optimization with TDC was addressed by a method which needed zero interpolation.

CHAPTER 2 PROPOSED SOLUTION

In this Chapter, we present a solution([33]) to the problem (1.2). The approach we pursue here evolves from the solution to the $\mathcal{H}_2/\ell_1$ problem presented in [4] and [25], where the idea of introducing Youla parameter Q as the optimizing variables was used. This has similarity to the Q -Parameter design mentioned in [34]. To accommodate the inclusion of $\mathcal{H}_\infty$ norm objectives in the GMO problem formalism, we make use of the LMI relations proposed in [12] and [35]. For special cases of the GMO problem, we present simplified solutions whose computation does not call for the LMI tools and therefore the computational burden is significantly reduced.

2.1 An Auxiliary Problem

In the general case, (1.4) is a difficult problem to solve. To facilitate the solution of this problem, we define an auxiliary problem closely related to it. The auxiliary GMO problem statement is: *Given constants $\gamma > 0$, $c_i > 0$, $i = 1, \dots, 6$, and two sequences*

$\{a_{temp}(k)\}_{k=0}^{\infty}$ and $\{b_{temp}(k)\}_{k=0}^{\infty}$, solve the following problem,

$$\begin{aligned}
\nu = & \inf_{Q \in \ell_1^{n_u \times n_v}} f(Q) \\
\text{subject to } & \|Q\|_1 \leq \gamma \\
& \|R^4(Q)\|_1 \leq c_4 \\
& \|R^5(Q)\|_2^2 \leq c_5 \\
& \|\hat{R}^6(Q)\|_{\mathcal{H}_{\infty}} \leq c_6 \\
& a_{temp}(k) \leq [A_{temp}R^T(Q)](k) \leq b_{temp}(k), \quad k = 0, 1, 2, \dots
\end{aligned} \tag{2.1}$$

Note there is an extra one norm bound on the Youla parameter Q in the auxiliary problem compared with the original GMO problem (1.4). As will be seen later, this extra constraint $\|Q\|_1 \leq \gamma$ plays an essential role in obtaining solution to (1.4). Also, introducing Q as an optimization variable facilitates the computation of the optimal controller. This avoids the numerical difficulties involved with zero interpolation methods.

2.2 Relationship between the GMO Problem and the Auxiliary Problem

In the problem formulation of (1.2), Q needs to satisfy the constraint $\|R^4(Q)\|_1 = \|H^{44} - U^4 * Q * V^4\|_1 \leq c_4$. Suppose $\hat{U}^4$ and $\hat{V}^4$ have full normal column and row rank and have no zeros on the unit circle. Then U^4 and V^4 are left- and right-invertible in ℓ_1 and it follows that $\|Q\|_1 \leq \|(U^4)^{-l}\|_1 (\|H^{44}\|_1 + c_4) \|(V^4)^{-r}\|_1 := \beta$, where $(U^4)^{-l}$ and $(V^4)^{-r}$ denote the left and right inverse of U^4 and V^4 , respectively. Consequently if we choose $\gamma \geq \beta$ in the auxiliary problem, the constraint $\|Q\|_1 \leq \gamma$ is redundant in GMO problem and we get $\nu = \mu$. In the case where $\hat{U}^4$ or $\hat{V}^4$ has zeros on the unit circle, there is a possibility that the original GMO problem does not admit an optimal solution and the one norm of the optimization variable Q can not be restricted to any bounded set.

Thus, from a computational point of view, it would be desirable to impose a reasonable bound on $\|Q\|_1$ in the optimization for this case as well.

In what follows, we shall focus our attention on the auxiliary problem. In proving the main results of the paper, we make the following assumption on the TDC.

Assumption *For all k , $a_{temp}(k) < b_{temp}(k)$. Furthermore, there exists N_1, N_2 so that $a_{temp}(k) = a_{temp}(N_1)$ for all $k \geq N_1$ and $b_{temp}(k) = b_{temp}(N_2)$ for all $k \geq N_2$.*

2.3 Existence of an Optimal Solution and Converging Lower Bounds

In this subsection, we develop a sequence of finite dimensional convex optimization problems whose objective values converges to ν from below. We will also prove the existence of an optimal solution to (2.1). Define

$$T_i(Q) := \begin{pmatrix} R^i(0) & 0 & 0 & \cdots & \cdots \\ R^i(1) & R^i(0) & 0 & \cdots & \cdots \\ R^i(2) & R^i(1) & R^i(0) & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

$$T_{i,k}(Q) := \begin{pmatrix} R^i(0) & 0 & \cdots & 0 \\ R^i(1) & R^i(0) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ R^i(k) & \cdots & R^i(1) & R^i(0) \end{pmatrix}.$$

It is a standard result that $\|\hat{R}^i(Q)\|_{\mathcal{H}_\infty} = \|T_i(Q)\| := \sup_k \sigma_{max}(T_{i,k}(Q)) = \sup_k \|T_{i,k}(Q)\|$, where $\|\cdot\|$ denotes the matrix spectral norm. Furthermore, from standard results in linear algebra (e.g., Theorem 4.3.8. in [36] or Chapter 2 in [37]), we have $\|T_{i,k}(Q)\| \leq \|T_{i,k+1}(Q)\| \leq \|T_i(Q)\|$, for all k .

Based on the above discussion, we define a candidate lower bound of ν as

$$\begin{aligned}
\nu_n &:= \inf_{Q \in \mathcal{S}} f_n(Q) \\
\text{subject to } & \|Q\|_1 \leq \gamma \\
& \|P_n(R^4(Q))\|_1 \leq c_4 \\
& \|P_n(R^5(Q))\|_2^2 \leq c_5 \\
& \|T_{6,n}(Q)\| \leq c_6 \\
& a_{temp}(k) \leq [A_{temp}R^T(Q)](k) \leq b_{temp}(k), \quad k = 0, 1, \dots, n.
\end{aligned} \tag{2.2}$$

where $f_n(Q) := c_1 \|P_n(R^1(Q))\|_1 + c_2 \|P_n(R^2(Q))\|_2^2 + c_3 \|T_{3,n}(Q)\|$. Since only the parameters of $Q(0), \dots, Q(n)$ enter into the optimization, problem (2.2) is a finite dimensional convex programming problem. Thus, it always admits an FIR optimal solution on the nonempty compact feasible set.

The following lemma is an immediate consequence of the above definition.

Lemma 2.3.1 *For all n , $\nu_n \leq \nu_{n+1} \leq \nu$.*

Proof: For any $Q \in \mathcal{S}$, it is true that

$$\begin{aligned}
\|P_n(R^i(Q))\|_1 &\leq \|P_{n+1}(R^i(Q))\|_1, \quad (i = 1, 4) \\
\|P_n(R^i(Q))\|_2^2 &\leq \|P_{n+1}(R^i(Q))\|_2^2, \quad (i = 2, 5) \\
\|T_{i,n}(Q)\| &\leq \|T_{i,n+1}(Q)\|, \quad (i = 3, 6).
\end{aligned}$$

Now suppose for some n , $\nu_n > \nu_{n+1}$. By the definition of ν_{n+1} , there is some Q in $\mathcal{S}$ such that

$$\begin{aligned}
\nu_{n+1} &\leq f_{n+1}(Q) < \nu_n, & \|Q\|_1 &\leq \gamma \\
\|P_{n+1}(R^4(Q))\|_1 &\leq c_4 & \|P_{n+1}(R^5(Q))\|_2^2 &\leq c_5 & \|T_{6,n+1}(Q)\| &\leq c_6 \\
a_{temp}(k) &\leq [A_{temp}R^T(Q)](k) \leq b_{temp}(k), \quad k = 0, 1, \dots, n+1.
\end{aligned}$$

Then Q also belongs to the feasible set of ν_n and it follows that $f_{n+1}(Q) \geq \nu_n$, which is a contradiction. Similarly, it can be shown that for all n , $\nu_n \leq \nu$. ■

In proving the existence of an optimal solution to the auxiliary problem and the convergence of the lower bounds ν_n to ν , we use the following result from linear operator theory:

Theorem 2.3.1 (Banach-Alaoglu) *Let $(X, \|\cdot\|_X)$ be a normed vector space with X^* as its dual. The set*

$$B^* := \{x^* \in X^* : \|x^*\| \leq M\}$$

is compact in the weak-star topology for any $M \in \mathcal{R}$.

Now we are ready to present the main result of this section.

Theorem 2.3.2 *There is an optimal solution Q^0 in $\ell_1^{n_u \times n_v}$ to problem (2.1). Moreover, $\nu_n \rightarrow \nu$.*

Proof: Suppose $Q_n \in \mathcal{S}$ is a finitely supported optimal solution to (2.2). Note that $\|Q_n\|_1 \leq \gamma$ for any positive integer n and Banach-Alaoglu theorem implies that $B_\gamma := \{Q \in \ell_1^{n_u \times n_v} : \|Q\|_1 \leq \gamma\}$ is weak-star compact. Thus, there exists a subsequence $\{Q_{n_m}\}$ of $\{Q_n\}$ and Q^0 in $\ell_1^{n_u \times n_v}$ such that $(Q_{n_m})_{ij} \rightarrow (Q^0)_{ij}$ ($i = 1, \dots, n_u, j = 1, \dots, n_v$) in the $W(c_0^*, c_0)$ topology. It follows that for all t , $Q_{n_m}(t) \rightarrow Q^0(t)$ and for all n , $P_n(R(Q_{n_m})) \rightarrow P_n(R(Q^0))$ and $T_{i,n}(Q_{n_m}) \rightarrow T_{i,n}(Q^0)$ ($i = 3, 6$) as $m \rightarrow \infty$. Moreover, suppose without loss of generality that, Q_{n_m} in $\mathcal{S}$ is required to be such that $(Q_{n_m})_{ij} = 0$. Then this is equivalent to require that $(Q_{n_m})_{ij}(t) = 0, \forall t$ and so it follows from the above arguments that for all t , $Q_{ij}^0(t) = 0$, that is, $Q^0 \in \mathcal{S}$.

For any $n > 0$ and for any $n_m > n$,

$$f_n(Q_{n_m}) \leq f_{n_m}(Q_{n_m}) = \nu_{n_m} \leq \nu.$$

By letting $m \rightarrow \infty$, we get

$$f_n(Q^0) \leq \nu, \forall n.$$

Since n is arbitrary, it follows that

$$f(Q^0) \leq \nu.$$

Similar arguments show that

$$\|Q^0\|_1 \leq \gamma, \quad \|R^1(Q^0)\|_1 \leq c_4, \quad \|R^5(Q^0)\|_2^2 \leq c_5.$$

Furthermore, for any given $k > 0$ and for any $n_m \geq k$,

$$\|T_{6,k}(Q_{n_m})\| \leq \|T_{6,n_m}(Q_{n_m})\|.$$

Recall the fact that $T_{6,k}(Q_{n_m})$ is a function of $Q_{n_m}(0), \dots, Q_{n_m}(k)$ only. By letting $m \rightarrow \infty$, we have

$$\|T_{6,k}(Q^0)\| \leq c_6, \quad \forall k.$$

Since k is arbitrary, it follows that

$$\|T_6(Q^0)\| := \sup_k \|T_{6,k}(Q^0)\| \leq c_6.$$

Finally, for any given $k > 0$, there exists some $n_m > k$ so that

$$a_{temp}(k) \leq [A_{temp}R^7(Q_{n_m})](k) \leq b_{temp}(k).$$

Then for all $l \geq m$, we have

$$a_{temp}(k) \leq [A_{temp}R^7(Q_{n_l})](k) \leq b_{temp}(k).$$

By letting l tend to infinity, it follows that

$$a_{temp}(k) \leq [A_{temp}R^7(Q^0)](k) \leq b_{temp}(k), \quad \forall k.$$

Thus, Q^0 is an optimal solution to problem (2.1).

To prove that $\nu_n \rightarrow \nu$, we note that for all $n > 0$ and $n_m > n$,

$$f_n(Q_{n_m}) \leq f_{n_m}(Q_{n_m}) = \nu_{n_m}.$$

Taking the limit as m tends to infinity we have

$$f_n(Q^0) \leq \lim_{m \rightarrow \infty} \nu_{nm}, \text{ for all } n > 0.$$

It follows that

$$f(Q^0) \leq \lim_{m \rightarrow \infty} \nu_{nm}.$$

Thus, we have shown that $\lim_{m \rightarrow \infty} \nu_{nm} = \nu$. Since ν_n is a monotonically increasing sequence bounded above by ν , it follows that $\nu_n \rightarrow \nu$. ■

2.4 Converging Upper Bounds

In the last section, we have shown that ν_n provides a lower bound for ν and that the sequence $\{\nu_n\}$ converges monotonically to ν . However, it is clear that ν_n itself does not provide any information on its distance to the optimal cost ν . This motivates the computation of an upper bound of ν . To this effect, we shall develop a sequence of finite dimensional convex optimization problems whose objective values converge to ν monotonically from above. By combining these upper bounds with the lower bounds derived in the last subsection, we obtain an effective method to synthesize suboptimal controllers with performance within any prescribed tolerance of the optimal.

Let ν^n be defined by

$$\begin{aligned} \nu^n := & \inf_{Q \in \mathcal{S}} f(Q) \\ \text{subject to } & \|Q\|_1 \leq \gamma \\ & \|R^4(Q)\|_1 \leq c_4 \\ & \|R^5(Q)\|_2^2 \leq c_5 \\ & \|\hat{R}^6(Q)\|_{\mathcal{H}_\infty} \leq c_6 \\ & a_{temp}(k) \leq [A_{temp} R^7(Q)](k) \leq b_{temp}(k), \quad k = 0, 1, 2, \dots \\ & Q(k) = 0 \text{ if } k > n. \end{aligned} \tag{2.3}$$

The numerical solution of this problem amounts to solving a convex programming problem involving only $Q(0), \dots, Q(n)$. It is clear that since H , U and V are all finitely supported, the time response $s = A_{temp}R^7(Q)$ would be a constant after some finite time instant $N \geq n$ and (2.3) is a finite dimensional optimization problem.

Lemma 2.4.1 *Given any Q in $\ell_1^{n_u \times n_v}$ and positive real number δ , there exists some N so that $n \geq N$ implies*

$$\begin{aligned} & | \|R^4(P_n(Q))\|_1 - \|R^4(Q)\|_1 | < \delta \\ & | \|R^5(P_n(Q))\|_2^2 - \|R^5(Q)\|_2^2 | < \delta \\ & | \|\hat{R}^6(P_n(Q))\|_{\mathcal{H}_\infty} - \|\hat{R}^6(Q)\|_{\mathcal{H}_\infty} | < \delta \\ & | [A_{temp}R^7(P_n(Q))](k) - [A_{temp}R^7(Q)](k) | < \delta, \forall k. \end{aligned}$$

Proof: It is clear that

$$\begin{aligned} & \|R^4((I - P_n)(Q))\|_1 \\ &= \|U^4 * ((I - P_n)(Q)) * V^4\|_1 \\ &\leq \|U^4\|_1 \|(I - P_n)(Q)\|_1 \|V^4\|_1 \\ & \quad \|A_{temp}R^7((I - P_n)(Q))\|_\infty \\ &= \|R^7((I - P_n)(Q)) * r\|_\infty \\ &\leq \|R^7((I - P_n)(Q))\|_1 \\ &\leq \|U^7\|_1 \|(I - P_n)(Q)\|_1 \|V^7\|_1. \end{aligned}$$

Since ℓ_1 is a proper subspace of ℓ_2 , we infer by Hölder inequality that

$$\|R^5((I - P_n)(Q))\|_2 \leq \|U^5\|_1^{\frac{1}{2}} \|(I - P_n)(Q)\|_1 \|V^5\|_1.$$

Moreover, for any given $x \in \ell_1^{n_u \times n_v}$, $\|\hat{x}\|_{\mathcal{H}_\infty} \leq \sqrt{n_u} \|x\|_1$ and it follows that

$$\|\hat{R}^6((I - P_n)(Q))\|_{\mathcal{H}_\infty} \leq \sqrt{n_u} \|U^6\|_1 \|(I - P_n)(Q)\|_1 \|V^6\|_1.$$

Since $Q \in \ell_1^{n_u \times n_v}$, $\|(I - P_n)(Q)\|_1$ can be made arbitrarily small by letting n large enough and the conclusion follows immediately from the above four inequalities. ■

Define

$$C := \{ (\gamma, c_4, c_5, c_6, a_{temp}(0), \dots, a_{temp}(N_1), b_{temp}(0), \dots, b_{temp}(N_2)) \in \mathcal{R}^{6+N_1+N_2} \}.$$

Then there exists $Q \in S$ so that

$$\begin{aligned} \|Q\|_1 &\leq \gamma \\ \|R^4(Q)\|_1 &\leq c_4 \\ \|R^5(Q)\|_2^2 &\leq c_5 \\ \|\hat{R}^6(Q)\|_{\mathcal{H}_\infty} &\leq c_6 \\ a_{temp}(k) &\leq [A_{temp}R^7(Q)](k) \leq b_{temp}(k), \forall k. \end{aligned}$$

Lemma 2.4.2 C is a convex set.

Proof: Let $(\gamma, c_4, c_5, c_6, a_{temp}, b_{temp}), (\bar{\gamma}, \bar{c}_4, \bar{c}_5, \bar{c}_6, \bar{a}_{temp}, \bar{b}_{temp}) \in S$, then there exist Q and $\bar{Q}$ so that

$$\begin{aligned} \|Q\|_1 &\leq \gamma, \|\bar{Q}\|_1 \leq \bar{\gamma}, \|R^4(Q)\|_1 \leq c_4, \|R^4(\bar{Q})\|_1 \leq \bar{c}_4 \\ \|R^5(Q)\|_2^2 &\leq c_5, \|R^5(\bar{Q})\|_2^2 \leq \bar{c}_5, \|\hat{R}^6(Q)\|_{\mathcal{H}_\infty} \leq c_6, \|\hat{R}^6(\bar{Q})\|_{\mathcal{H}_\infty} \leq \bar{c}_6 \\ a_{temp}(k) &\leq [A_{temp}R^7(Q)](k) \leq b_{temp}(k) \\ \bar{a}_{temp}(k) &\leq [A_{temp}R^7(\bar{Q})](k) \leq \bar{b}_{temp}(k), \forall k. \end{aligned}$$

Then for any $\lambda \in [0, 1]$,

$$\begin{aligned} \|\lambda Q + (1 - \lambda)\bar{Q}\|_1 &\leq \lambda\gamma + (1 - \lambda)\bar{\gamma} \\ \|R^4(\lambda Q + (1 - \lambda)\bar{Q})\|_1 &\leq \lambda c_4 + (1 - \lambda)\bar{c}_4 \end{aligned}$$

$$\begin{aligned}
& \|R^5(\lambda Q + (1 - \lambda)\bar{Q})\|_2^2 \\
& \leq (\lambda\|R^5(Q)\|_2 + (1 - \lambda)\|R^5(\bar{Q})\|_2)^2 \\
& = \lambda(\lambda - 1)(\|R^5(Q)\|_2 - \|R^5(\bar{Q})\|_2)^2 + \lambda\|R^5(Q)\|_2^2 + (1 - \lambda)\|R^5(\bar{Q})\|_2^2 \\
& \leq \lambda c_5 + (1 - \lambda)\bar{c}_5 \\
& \quad \|\hat{R}^6(\lambda Q + (1 - \lambda)\bar{Q})\|_{\mathcal{H}_\infty} \\
& \leq \lambda c_6 + (1 - \lambda)\bar{c}_6 \\
& \quad \lambda a_{temp}(k) + (1 - \lambda)\bar{a}_{temp}(k) \\
& \leq A_{temp}[R^7(\lambda Q + (1 - \lambda)\bar{Q})](k) \\
& \leq \lambda b_{temp}(k) + (1 - \lambda)\bar{b}_{temp}(k), \quad \forall k.
\end{aligned}$$

Let $Q^\lambda := \lambda Q + (1 - \lambda)\bar{Q}$, then $Q^\lambda \in \ell_1^{n_u \times n_v}$ and $\lambda(\gamma, c_4, c_5, c_6, a_{temp}, b_{temp}) + (1 - \lambda)(\bar{\gamma}, \bar{c}_4, \bar{c}_5, \bar{c}_6, \bar{a}_{temp}, \bar{b}_{temp}) \in C$. ■

Lemma 2.4.3 ([22]) *Let $f : \Omega \rightarrow \mathcal{R}$ be a real valued convex function on a convex set Ω of a vector space X . Let G be a convex mapping from Ω into $\mathcal{R}^n$. For any z in $\mathcal{R}^n$, define*

$$w(z) := \inf\{f(x) : x \in \Omega, G(x) \leq z\}.$$

If C is any convex subset of $\mathcal{R}^n$ such that for all z in C , $w(z)$ is in $\mathcal{R}$, then w is a continuous function of z in the interior of C .

Lemma 2.4.4 *ν is a continuous function with respect to $(\gamma, c_4, c_5, c_6, a_{temp}, b_{temp})$ in the interior of C .*

Proof: The conclusion follows immediately from Lemma 2.4.2 and Lemma 2.4.3. ■

In what follows, we shall assume $(\gamma, c_4, c_5, c_6, a_{temp}, b_{temp})$ lies in the interior of C .

Theorem 2.4.1 *$\{\nu^n\}$ forms a monotonically decreasing sequence of upper bounds of ν . Furthermore,*

$$\nu^n \rightarrow \nu, \text{ as } n \rightarrow \infty.$$

Proof: Clearly, $\nu^n \geq \nu^{n+1}$ since any Q in $\mathcal{S}$ which belongs to the feasible set of ν^n will also be feasible for problem ν^{n+1} . For same reason, we have $\nu^n \geq \nu$ for all n . Thus, $\{\nu^n\}$ comes into being a decreasing sequence of real numbers bounded below by ν .

For notational simplicity, in what follows, we shall omit the symbol γ in ν^n and ν . For any given $\epsilon > 0$, the continuity of ν implies that there exists $\delta > 0$ such that

$$\nu(c_4 - \delta, c_5 - \delta, c_6 - \delta, a_{temp} + \delta, b_{temp} - \delta) - \nu(c_4, c_5, c_6, a_{temp}, b_{temp}) < \epsilon/4$$

where $a_{temp} + \delta := \{a_{temp}(k) + \delta\}_{k=0}^{\infty}$ and $b_{temp} - \delta := \{b_{temp}(k) - \delta\}_{k=0}^{\infty}$. Also, there exists some Q^δ such that

$$\begin{aligned} f(Q^\delta) - \nu(c_4 - \delta, c_5 - \delta, c_6 - \delta, a_{temp} + \delta, b_{temp} - \delta) &< \epsilon/4 \\ \|R^4(Q^\delta)\|_1 &\leq c_4 - \delta \\ \|R^5(Q^\delta)\|_2^2 &\leq c_5 - \delta \\ \|\hat{R}^6(Q^\delta)\|_{\mathcal{H}_\infty} &\leq c_6 - \delta \\ a_{temp}(k) + \delta &\leq [A_{temp}R^7(Q^\delta)](k) \leq b_{temp}(k) - \delta, \forall k \\ \|Q^\delta\|_1 &\leq \gamma. \end{aligned}$$

By Lemma 2.4.1, there exists some positive integer N large enough so that $n \geq N$ implies

$$\begin{aligned} f(P_n(Q^\delta)) - f(Q^\delta) &< \epsilon/2 \\ \|R^4(P_n(Q^\delta))\|_1 - \|R^4(Q^\delta)\|_1 &< \delta/2 \\ \|R^5(P_n(Q^\delta))\|_2^2 - \|R^5(Q^\delta)\|_2^2 &< \delta/2 \\ \|\hat{R}^6(P_n(Q^\delta))\|_{\mathcal{H}_\infty} - \|\hat{R}^6(Q^\delta)\|_{\mathcal{H}_\infty} &< \delta/2 \\ \left| [A_{temp}R^7(P_n(Q^\delta))](k) - [A_{temp}R^7(Q^\delta)](k) \right| &< \delta/2, \forall k. \end{aligned}$$

It follows from above that for all $n \geq N$,

$$\begin{aligned}
f(P_n(Q^\delta)) &< \nu(c_4, c_5, c_6, a_{temp}, b_{temp}) + \epsilon \\
\|R^4(P_n(Q^\delta))\|_1 &< c_4 \\
\|R^5(P_n(Q^\delta))\|_2^2 &< c_5 \\
\|\hat{R}^6(P_n(Q^\delta))\|_{\mathcal{H}_\infty} &< c_6 \\
a_{temp}(k) &\leq [A_{temp}R^7(P_n(Q^\delta))](k) \leq b_{temp}(k), \forall k \\
\|P_n(Q^\delta)\|_1 &\leq \gamma.
\end{aligned}$$

Thus, $P_n(Q^\delta)$ satisfies all the constraints of problem $\nu^n(c_4, c_5, c_6, a_{temp}, b_{temp})$ and it follows that for all $n \geq N$,

$$\nu^n(c_4, c_5, c_6, a_{temp}, b_{temp}) - \nu(c_4, c_5, c_6, a_{temp}, b_{temp}) < \epsilon.$$

This proves the theorem. ■

After establishing the convergence of ν_n and ν^n to ν , we now briefly address the issue of constructing suboptimal controllers from the optimizing Youla parameter Q . For any prescribed performance tolerance $\delta > 0$, the optimizing process can be stopped once for some s , $|\nu_s - \nu^s| \leq \delta$. The minimizing variable Q^s to the upper bound ν^s can then be used to recover a suboptimal controller which achieves the objective value ν^s . In some cases, the supported length of the optimizer Q^s (i.e. s) will be rather large and this would lead to a suboptimal controller with undesirable high order. Notice that there still is no known results available on the problem of model reduction with structure preservation and this problem still warrants further research and investigation. However, in almost all the numerical examples encountered by the authors, structured controllers with acceptable low orders can be easily obtained by using some well-established approximation techniques (e.g. the Hankel SVD method by S. Kung).

2.5 Uniqueness and Convergence of the Optimal Solution

Having established the existence of an optimal solution Q^0 to problem (2.1) and the convergence of ν^n to ν , we now present results addressing the uniqueness and convergence properties of the suboptimal and optimal solutions. The proofs for these results are given in the appendix.

Lemma 2.5.1 ([22]) *Let Ω be a convex subset of a Banach space X and $f : \Omega \rightarrow \mathbb{R}$ be strictly convex. If f achieves its minimum on Ω then the minimizer is unique.*

Theorem 2.5.1 *Suppose $\hat{U}^2$ and $\hat{V}^2$ have full column and row rank on the unit circle respectively. Let Q^n denote an optimal solution to ν^n . Let Q^0 denote an optimal solution to ν . Let $R^n := H - U * Q^n * V$, $n = 0, 1, \dots$. Then the following is true:*

(1) R^n ($n = 0, 1, \dots$) is unique.

(2) Q^n ($n = 0, 1, \dots$) is unique.

Proof: (1) Define $A_{feas}^0 := \{R \in \ell_1^{n_z \times n_w} : \mid \text{there exists } Q \in \ell_1^{n_u \times n_v} \text{ so that}$

$$R = H - U * Q * V$$

$$\|Q\|_1 \leq \gamma$$

$$\|R^4(Q)\|_1 \leq c_4$$

$$\|R^5(Q)\|_2^2 \leq c_5$$

$$\|\hat{R}^6(Q)\|_{\mathcal{H}_\infty} \leq c_6$$

$$a_{temp}(k) \leq [A_{temp} R^7(Q)](k) \leq b_{temp}(k), \forall k\}.$$

Then A_{feas}^0 is a convex set. Problem (2.1) is equivalent to

$$\nu = \inf_{R \in A_{feas}^0} c_1 \|R^1\|_1 + c_2 \|R^2\|_2^2 + c_3 \|\hat{R}^3\|_{\mathcal{H}_\infty}. \quad (2.4)$$

Furthermore, it is clear that the relationship between $\hat{R}^i$ and $\hat{R}$ can be expressed as

$$\hat{R}^i = \hat{E}_i \hat{R} \hat{F}_i$$

where

$$\hat{E}_i = [0_{n_{z_i} \times n_{z_1}} \cdots I_{n_{z_i}} \cdots 0_{n_{z_i} \times n_{z_7}}] \in \mathcal{R}^{n_{z_i} \times n_z}$$

$$\hat{F}_i = [0_{n_{w_1} \times n_{w_i}} \cdots I_{n_{w_i}} \cdots 0_{n_{w_7} \times n_{w_i}}]^T \in \mathcal{R}^{n_w \times n_{w_i}}, i = 1, \dots, 7.$$

Also, the $\hat{E}_i = E_i$ and $\hat{F}_i = F_i$. Thus, problem (2.4) can be reformulated as

$$\nu = \inf_{R \in \mathcal{A}_{feas}^0} g(R) \quad (2.5)$$

where

$$\begin{aligned} g(R) &:= c_1 \|E_1 * R * F_1\|_1 + c_2 \|E_2 * R * F_2\|_2^2 + c_3 \|\hat{E}_3 \hat{R} \hat{F}_3\|_{\mathcal{H}_\infty} \\ &= c_1 \|R^1\|_1 + c_2 \|R^2\|_2^2 + c_3 \|\hat{R}^3\|_{\mathcal{H}_\infty}. \end{aligned}$$

We claim that $g(R)$ is a strict convex function of R given the assumption $c_2 > 0$. To see this, choose $R, S \in \mathcal{A}_{feas}^0$ such that $R \neq S$. Then it follows from the invertibility of $\hat{U}^2$ and $\hat{V}^2$ that $R^2 \neq S^2$. Then for any $\alpha \in (0, 1)$,

$$\begin{aligned} g(\alpha R + (1 - \alpha)S) &= c_1 \|E_1 * (\alpha R) * F_1 + E_1 * ((1 - \alpha)S) * F_1\|_1 \\ &\quad + c_2 \|E_2 * (\alpha R) * F_2 + E_2 * ((1 - \alpha)S) * F_2\|_2^2 \\ &\quad + c_3 \|\hat{E}_3(\alpha \hat{R}^3) \hat{F}_3 + \hat{E}_3((1 - \alpha)\hat{S}) \hat{F}_3\|_{\mathcal{H}_\infty} \\ &= c_1 \|\alpha R^1 + (1 - \alpha)S^1\|_1 + c_2 \|\alpha R^2 + (1 - \alpha)S^2\|_2^2 \\ &\quad + c_3 \|\alpha \hat{R}^3 + (1 - \alpha)\hat{S}^3\|_{\mathcal{H}_\infty} \\ &< c_1 \alpha \|R^1\|_1 + c_1 (1 - \alpha) \|S^1\|_1 + c_2 \alpha \|R^2\|_2^2 + c_2 (1 - \alpha) \|S^2\|_2^2 \\ &\quad + c_3 \alpha \|\hat{R}^3\|_{\mathcal{H}_\infty} + c_3 (1 - \alpha) \|\hat{S}^3\|_{\mathcal{H}_\infty} \\ &= \alpha g(R) + (1 - \alpha)g(S) \end{aligned}$$

where the strict convexity of $\|\cdot\|_2^2$ and the convexity of $\|\cdot\|_1$ and $\|\cdot\|_{\mathcal{H}_\infty}$ are employed to justify the strict inequality sign. This proves that $g(\cdot)$ is strictly convex on $\mathcal{A}_{feas}^0$. Then by Lemma 2.5.1, R^0 , the minimizing closed-loop map to problem (2.4), is unique.

Similar arguments as above show that R^n ($n = 1, 2, \dots$) is unique.

(2) The uniqueness of Q^n ($n = 0, 1, \dots$) follows immediately from (1) and the invertibility of $\hat{U}^2$ and $\hat{V}^2$. ■

One direct consequence of Theorem 2.5.1 is that Q^0 is the weak-star convergent limit of a subsequence of $\{Q^n\}$.

Lemma 2.5.2 *Suppose $\hat{U}^2$ and $\hat{V}^2$ have full column and row rank on the unit circle respectively. Then there exists a subsequence $\{Q^{n_m}\}$ of $\{Q^n\}$ such that $(Q^{n_m})_{ij} \rightarrow (Q^0)_{ij}$ ($i = 1, \dots, n_u$, $j = 1, \dots, n_v$) in the $W(c_0^*, c_0)$ topology.*

Proof: Since $\{Q^n\}$ is uniformly bounded by γ in ℓ_1 norm sense, Banach-Alaoglu Theorem implies that there is a subsequence $\{Q^{n_m}\}$ of $\{Q^n\}$ and some $\bar{Q}^0 \in \ell_1^{n_u \times n_v}$ such that $(Q^{n_m})_{ij} \rightarrow (\bar{Q}^0)_{ij}$ ($i = 1, \dots, n_u$, $j = 1, \dots, n_v$) in the $W(c_0^*, c_0)$ topology. So it follows that $Q^{n_m}(t) \rightarrow \bar{Q}^0(t)$ for all t and for all n , $P_n(R(Q^{n_m})) \rightarrow P_n(R(Q^0))$ and $T_{i,n}(Q^{n_m}) \rightarrow T_{i,n}(Q^0)$ ($i = 3, 6$) as $m \rightarrow \infty$.

Now it is clear that for any $n > 0$ and any $n_m > n$,

$$\begin{aligned} & c_1 \|P_n(R^1(Q^{n_m}))\|_1 + c_2 \|P_n(R^2(Q^{n_m}))\|_2^2 + c_3 \|T_{3,n}(Q^{n_m})\| \\ & \leq c_1 \|R^1(Q^{n_m})\|_1 + c_2 \|R^2(Q^{n_m})\|_2^2 + c_3 \|\hat{R}^3(Q^{n_m})\|_{\mathcal{H}_\infty} \\ & =: \nu^{n_m} \end{aligned}$$

By letting $m \rightarrow \infty$, we have

$$c_1 \|P_n(R^1(\bar{Q}^0))\|_1 + c_2 \|P_n(R^2(\bar{Q}^0))\|_2^2 + c_3 \|T_{3,n}(\bar{Q}^0)\| \leq \lim_{m \rightarrow \infty} \nu^{n_m} = \nu, \quad \forall n.$$

So it follows that

$$c_1 \|R^1(\bar{Q}^0)\|_1 + c_2 \|R^2(\bar{Q}^0)\|_2^2 + c_3 \|\hat{R}^3(\bar{Q}^0)\|_{\mathcal{H}_\infty} \leq \nu.$$

Furthermore, by exactly the same argument as in the proof of Theorem 2.3.2, we can verify that $\bar{Q}^0$ satisfies all the constraints of problem (2.1). Thus, $\bar{Q}^0$ is an optimal solution to ν and by the uniqueness of the optimal solution of ν , we have $\bar{Q}^0 = Q^0$. ■

If no $\mathcal{H}_\infty$ term is present in the objective function of the GMO problem, the conclusion of Theorem 2.5.1 can be made stronger. More explicitly, suppose $c_3 = 0$ in the GMO problem setup (2.1), i.e, $f(Q) = c_1 \|R^1(Q)\|_1 + c_2 \|R^2(Q)\|_2^2$. It can be easily seen that the conclusions established in Theorem 2.3.2 and Theorem 2.4.1 hold.

Lemma 2.5.3 *Let $f : (R^1, R^2) \rightarrow \mathcal{R}$ (where R^1, R^2 are matrices consisting of elements in ℓ_1) be defined by:*

$$f(R^1, R^2) := c_1 \|R^1\|_1 + c_2 \|R^2\|_2^2.$$

Let $\{(R^{1,k}, R^{2,k})\}$ be a sequence such that

$$(R^{1,k}(t), R^{2,k}(t)) \rightarrow (R^{1,o}(t), R^{2,o}(t)) \text{ for all } t$$

and

$$f(R^{1,k}, R^{2,k}) \leq f(R^{1,o}, R^{2,o}) \text{ for all } k. \quad (2.6)$$

Let $\|R^{1,o}\|_1 = \|(R^{1,o})_p\|_1$ where $(R^{1,o})_p$ represents the p^{th} row of $R^{1,o}$. Then

$$c_1 \|(R^{1,k})_p - (R^{1,o})_p\|_1 + c_2 \|R^{2,k} - R^{2,o}\|_2^2 \rightarrow 0 \text{ as } k \rightarrow \infty.$$

The same conclusion holds if condition (2.6) is replaced with the following condition:

$$f(R^{1,k}, R^{2,k}) \rightarrow f(R^{1,o}, R^{2,o}). \quad (2.7)$$

Proof: We prove for the case when $c_2 > 0$ and when condition (2.6) is true. We leave the rest of the proof to the reader. For notational convenience we will denote (R^1, R^2) by R . Also we define

$$g((R^1, R^2)) := c_1 \|(R^1)_p\|_1 + c_2 \|R^2\|_2^2.$$

It is clear that

$$g((R^{1,k}, R^{2,k})) \leq g((R^{1,o}, R^{2,o})) \text{ for all } k.$$

We claim that

$$g(R^k) \rightarrow g(R^o) \text{ as } k \rightarrow \infty. \quad (2.8)$$

Suppose not, then there exists a subsequence $\{R^{k_s}\}$ of $\{R^k\}$ and an $\epsilon_1 > 0$ such that

$$g(R^o) - g(R^{k_s}) > \epsilon_1 \text{ for all } s. \quad (2.9)$$

Choose m such that $g((I - P_m)R^o) \leq \epsilon_1/2$. Thus

$$g(P_m R^o) + \frac{\epsilon_1}{2} - g(R^{k_s}) > g(R^o) - g(R^{k_s}) > \epsilon_1,$$

which implies that

$$g(P_m R^o) - g(P_m R^{k_s}) \geq g(P_m R^o) - g(R^{k_s}) > \epsilon_1/2,$$

But we know that R^{k_s} converges to R^o pointwise and therefore $g(P_m R^{k_s}) \rightarrow g(P_m R^o)$.

Thus we have reached a contradiction to our supposition which proves (2.8).

Given $\epsilon > 0$ choose n such that

$$\begin{aligned} \sum_{(p,q)} \|(I - P_n)R_{pq}^{2,o}\|_2 &\leq \epsilon/(8Mc_2) \\ g((I - P_n)R^o) &\leq \epsilon/8. \end{aligned}$$

where M is an upper bound on $\sum_{(p,q)} \|R_{pq}^{22,k}\|_2$, which exists because $g(R^k) \leq f(R^o)$.

As $g(R^k)$ converges to $g(R^o)$ and $R^k(t)$ converges to $R^o(t)$ for all t it follows that $g((I - P_n)R^k)$ converges to $g((I - P_n)R^o)$. Thus there exists an integer K_1 such that $k > K_1$ implies that $g((I - P_n)R^k) \leq g((I - P_n)R^o) + \epsilon/4$.

As $R^k(t)$ converges to $R^o(t)$ for all t it also follows that $g(P_n(R^k - R^o))$ converges to zero. Thus we can choose an integer K_2 such that if $k > K_2$ then $g(P_n(R^k - R^o)) \leq \epsilon/4$.

Thus for any $k > \max\{K_1, K_2\}$ we have

$$\begin{aligned} g(R^k - R^o) &= g(P_n(R^k - R^o)) + g((I - P_n)(R^k - R^o)) \\ &\leq g(P_n(R^k - R^o)) + g((I - P_n)R^k) + g((I - P_n)R^o) \\ &\quad + 2c_2 \sum_{(p,q)} \sum_{t=n+1}^{\infty} |R_{pq}^{22,k}(t)| |R_{pq}^{22,o}(t)| \\ &\leq g(P_n(R^k - R^o)) + g((I - P_n)R^k) + g((I - P_n)R^o) \\ &\quad + 2c_2 \sum_{(p,q)} \|(I - P_n)R_{pq}^k\|_2 \|(I - P_n)R_{pq}^o\|_2 \\ &\leq \frac{\epsilon}{4} + 2g((I - P_n)R^o) + \frac{\epsilon}{4} + 2c_2 M \sum_{(p,q)} \|(I - P_n)R_{pq}^o\|_2 \\ &\leq \epsilon. \end{aligned}$$

This proves the lemma. ■

Theorem 2.5.2 *Suppose $\hat{U}^2$ and $\hat{V}^2$ have full column and row rank on the unit circle respectively. Let Q^n denote an optimal solution to $\nu^n(c_3 = 0)$ and let $R^n := H - U * Q^n * V$, $R^{i,n} := H^{ii} - U^i * Q^n * V^i$, $i = 1, \dots, 7$, $n = 0, 1, \dots$. Then the following is true:*

- (1) R^n ($n = 0, 1, \dots$) is unique.
- (2) Q^n ($n = 0, 1, \dots$) is unique.
- (3) $\|R^{2,n} - R^{2,0}\|_2 \rightarrow 0$, as $n \rightarrow \infty$.

Proof: The proof for (1) and (2) can be carried out in the exactly the same way as the proof for (1) and (2) in Theorem 2.5.1 and will not be repeated here.

We prove (3) by using contradiction. Suppose the sequence $\{\|R^{2,n} - R^{2,0}\|_2\}_{n=1}^\infty$ doesn't converge to zero. Then there exists a subsequence $\{R^{n_m}\}$ of R^n and an $\epsilon > 0$ such that

$$\|R^{2,n_m} - R^{2,0}\|_2 \geq \epsilon, \quad \forall m. \quad (2.10)$$

Then by using the same argument as in the proof of Lemma 2.5.2, we can prove that a subsequence $\{Q^{n_{m_k}}\}$ of $\{Q^{n_m}\}$ converges pointwise to the optimal solution Q^0 of problem ν (with $c_3 = 0$). Furthermore, since $Q^{n_{m_k}}$ is the optimal solution to problem $\nu^{n_{m_k}}$, whose limit converges to ν as $k \rightarrow \infty$, we have

$$\nu^{n_{m_k}} = c_1 \|R^{1,n_{m_k}}\|_1 + c_2 \|R^{2,n_{m_k}}\|_2^2 \rightarrow c_1 \|R^{1,0}\|_1 + c_2 \|R^{2,0}\|_2^2 = \nu, \quad \text{as } k \rightarrow \infty.$$

Thus, the assumptions of Lemma 2.5.3 are satisfied and it follows that

$$c_1 \|R^{1,n_{m_k}} - R^{1,0}\|_1 + c_2 \|R^{2,n_{m_k}} - R^{2,0}\|_2^2 \rightarrow 0 \quad \text{as } k \rightarrow \infty$$

which is a contradiction to inequality (2.10). ■

It should be remarked that the lower bound version of Theorem 2.5.2 also holds and the proof can be carried out in exactly the same manner as that for Theorem 2.5.2.

As a concluding remark for this section, we want to point out that the GMO control design framework we have developed here is flexible. Given any finite numbers of $\ell_1/\mathcal{H}_2/\mathcal{H}_\infty$ norm objectives and TDCs, they can be directly stacked into the GMO problem formalism and the theoretical and numerical schemes established in this and the previous section can be extended in a straightforward manner to obtain the solution.

CHAPTER 3 NUMERICAL IMPLEMENTATION AND SIMULATIONS

3.1 LMI Formulations

In the section, we introduce some standard results in linear matrix inequalities (LMIs) and semidefinite programming (SDP), and show how these results can be applied to transform problems (2.2) and (2.3) into solvable SDP forms.

In what follows, if $L(x)$ is a n -by- n symmetric matrix, the (strict) inequality sign in $(L(x) > 0) \ L(x) \geq 0$ means that $L(x)$ is positive (definite) semidefinite, i.e., $(y^T L(x) y > 0 \text{ for all } y \neq 0 \text{ and } y \in \mathcal{R}^n) \ y^T L(x) y \geq 0 \text{ for all } y \in \mathcal{R}^n$. In the case that $L(x)$ is a vector in $\mathcal{R}^n$, the (strict) inequality sign in $(L(x) > 0) \ L(x) \geq 0$ means that $L(x)$ is componentwise (positive) nonnegative, i.e., $([L(x)](i) \geq 0) \ [L(x)](i) \geq 0 \text{ for } i = 1, \dots, n$.

Lemma 3.1.1 ([35], **matrix norm bound**) *Given a matrix $A(x) = A_0 + x_1 A_1 + \dots + x_k A_k \in \mathcal{R}^{p \times q}$. (Here A_i need not to be symmetric.) Let $\|A(x)\|$ denotes the spectral norm (maximum singular value) of $A(x)$. We have*

$$\|A(x)\| \leq t \tag{3.1}$$

if and only if the following LMI in x is feasible

$$\begin{bmatrix} tI & A(x) \\ A(x)^T & tI \end{bmatrix} \geq 0. \tag{3.2}$$

Lemma 3.1.2 ([35], LP to SDP) *Given coefficients A and b for a linear program (LP). We have*

$$Ax + b \geq 0 \quad (3.3)$$

if and only if the following LMI in x is feasible

$$F(x) := \mathbf{diag}(Ax + b) \geq 0, \quad (3.4)$$

where

$$\begin{aligned} F &= F_0 + \sum_{i=1}^m x_i F_i & F_0 &= \mathbf{diag}(b) & F_i &= \mathbf{diag}(a_i), i = 1, \dots, m \\ A &= [a_1 \dots a_m] \in \mathcal{R}^{n \times m}. \end{aligned} \quad (3.5)$$

Lemma 3.1.3 ([35], QCQP to SDP) *Given coefficients A, b, c , and d for a general quadratically constraint quadratic program (QCQP). We have*

$$f(x) := (Ax + b)^T (Ax + b) - c^T x - d \leq 0 \quad (3.6)$$

if and only if the following LMI in x is feasible

$$\begin{bmatrix} I & Ax + b \\ (Ax + b)^T & c^T x + d \end{bmatrix} \geq 0. \quad (3.7)$$

Lemma 3.1.4 ([12], $\mathcal{H}_2$ norm bound) *Given $G = D + C(zI - A)^{-1}B$. We have*

$$\|G\|_2^2 \leq \alpha \quad (3.8)$$

A asymptotically stable

if and only if the following LMI in X and S is feasible:

$$\begin{aligned} & \begin{bmatrix} A^T X A - X & A^T X B \\ B^T X A & B^T X B - I \end{bmatrix} < 0 \\ & \begin{bmatrix} X & 0 & C^T \\ 0 & I & D^T \\ C & D & S \end{bmatrix} > 0 \\ & \text{Tr}(S) - \alpha < 0 \\ & X > 0. \end{aligned} \quad (3.9)$$

Lemma 3.1.5 ([12], Bounded Real lemma) *Given $G = D + C(zI - A)^{-1}B$. We have*

$$\begin{aligned} \|G\|_{\mathcal{H}_\infty} &\leq \beta \\ A &\text{ asymptotically stable} \end{aligned} \quad (3.10)$$

if and only if the following LMI in X and S is feasible:

$$\begin{aligned} &\begin{bmatrix} A^T X A - X & A^T X B & C^T \\ B^T X A & B^T X B - \beta I & D^T \\ C & D & -\beta I \end{bmatrix} < 0 \\ &X > 0. \end{aligned} \quad (3.11)$$

For ease of exposition, let us rewrite the definition of ν_n in the following equivalent form:

$$\inf_{Q \in \ell_1^{n_u \times n_v}} c_1 t_1 + c_2 t_2 + c_3 t_3$$

subject to

$$\begin{aligned} \|P_n(R^1(Q))\|_1 &\leq t_1 \\ \|P_n(R^2(Q))\|_2^2 &\leq t_2 \\ \|T_{3,n}(Q)\| &\leq t_3 \\ \|Q\|_1 &\leq \gamma \\ \|P_n(R^4(Q))\|_1 &\leq c_4 \\ \|P_n(R^5(Q))\|_2^2 &\leq c_5 \\ \|T_{6,n}(Q)\| &\leq c_6 \\ a_{temp}(k) &\leq [A_{temp} R^7(Q)](k) \leq b_{temp}(k), \quad k = 0, 1, \dots, n. \end{aligned} \quad (3.12)$$

It is clear that the above problem is a finite dimensional convex optimization program involving only $Q(0), \dots, Q(n)$. It is also clear that the ℓ_1 norm constraints and template constraints are in the form of (3.3) and can be transformed into SDP constraints immediately by using Lemma 3.1.2. Moreover, the equivalent SDP forms for the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ constraints can be obtained by appealing to Lemma 3.1.1 and Lemma 3.1.3 respectively. Note if for some particular reason, $\|R^i(Q)\|_2$ ($i = 2, 5$) is desired to be

considered in the GMO setup instead of $\|R^i(Q)\|_2^2$, the equivalent SDP constraint form for $\|R^i(Q)\|_2 \leq t$ can be established by using the following Lemma:

Lemma 3.1.6 ($\mathcal{H}_2$ to SDP) *Given coefficients A, b, c for a quadratically constraint program. We have*

$$f(x) := (Ax + b)^T(Ax + b) - (c^T x)^2 \leq 0 \quad (3.13)$$

if and only if the following LMI in x is feasible

$$\begin{bmatrix} c^T x & Ax + b \\ (Ax + b)^T & c^T x \end{bmatrix} \geq 0. \quad (3.14)$$

Proof: The proof is just a simple application of Schur complements and thus is omitted here. ■

The equivalent reformulation of ν^n is given by

$$\begin{aligned} & \inf_{Q \in \ell_1^{n_u \times n_v}} c_1 t_1 + c_2 t_2 + c_3 t_3 \\ & \text{subject to} \\ & \|R^1(Q)\|_1 \leq t_1 \\ & \|R^2(Q)\|_2^2 \leq t_2 \\ & \|\hat{R}^3(Q)\|_{\mathcal{H}_\infty} \leq t_3 \\ & \|Q\|_1 \leq \gamma \\ & \|R^4(Q)\|_1 \leq c_4 \\ & \|R^5(Q)\|_2^2 \leq c_5 \\ & \|\hat{R}^6(Q)\|_{\mathcal{H}_\infty} \leq c_6 \\ & a_{temp}(k) \leq [A_{temp} R^7(Q)](k) \leq b_{temp}(k), \quad k = 0, 1, 2, \dots \\ & Q(k) = 0 \text{ if } k > n. \end{aligned} \quad (3.15)$$

Given the discussion above, we shall only focus on the equivalent SDP formulation of the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norm constraints in (3.15). Actually we can still deal with the $\mathcal{H}_2$ norm constraints by appealing to Lemma 3.1.3 (or Lemma 3.1.6). However, Lemma 3.1.4

and Lemma 3.1.5 provide us with a more uniform SDP (LMI) formulation for the two norm constraints. Note in particular that similar techniques as in [12] are needed here to ascertain the finite dimensionality of (3.15). By using an alternative state space description obtained from the Youla parametrization via system Kronecker products, only the parameters of $Q(0), \dots, Q(n)$ will be involved into the optimization and it follows that (3.15) is a finite dimensional convex optimization problem. Since the concrete form of the equivalent SDP form for ν_n and ν^n may vary with the change of the particular SDP solver used, we will not make further comments here.

It should be noted that in the case where there is no $\mathcal{H}_\infty$ norm involved in the problem setup (i.e. $c_3 = 0$, $c_6 = \infty$) and no $\mathcal{H}_2$ norm constraint imposed on the closed-loop system ($c_5 = 0$), the corresponding GMO problem can be solved in a less computationally expensive manner without appealing to the LMI mechanism. More explicitly, quadratic programming techniques can be applied to obtain the solutions with high efficiency and precision. For the case where neither $\mathcal{H}_2$ norm nor $\mathcal{H}_\infty$ norm involved in the problem setup, the GMO problem is simply the ℓ_1 minimization with TDC and ℓ_1 constraint and it can be efficiently solved by using linear programming techniques.

As a concluding remark for this section, we want to point out that the GMO control design framework we have developed here has a flexible structure. More explicitly, given any finite numbers of $\ell_1/\mathcal{H}_2/\mathcal{H}_\infty$ norm objectives and TDCs, they can be directly stacked into the GMO problem formalism and the theoretical and numerical schemes established in this and the previous section can be extended in a straightforward manner to obtain the solution.

3.2 GMO v1.0 Package

It is clear from the definitions (2.2) and (2.3) that only the parameters of $Q(0), \dots, Q(n)$ enter into the optimization of ν_n and ν^n . Thus (2.2) and (2.3) are actually two finite dimensional convex programming problems and by appealing to the LMI formulas proposed in [35] and [12], they can be readily transformed into solvable SDP forms and be effectively solved by using some well-developed SDP techniques. In the case where there is no $\mathcal{H}_\infty$ norm involved in the problem setup (i.e. $c_3 = 0, c_6 = \infty$) and no $\mathcal{H}_2$ norm constraint imposed on the closed-loop system ($c_5 = 0$), the corresponding GMO problem can be solved in a less computationally expensive manner by using quadratic programming techniques. For the case where neither $\mathcal{H}_2$ norm nor $\mathcal{H}_\infty$ norm is involved in the problem setup, the GMO problem can be efficiently solved by using LP techniques.

It should be noted that the GMO control design framework we have developed here is flexible; given any finite numbers of $\ell_1/\mathcal{H}_2/\mathcal{H}_\infty$ norm objectives and TDCs, they can be directly stacked into the GMO problem formalism and the theoretical and numerical schemes established in this and the previous sections can be extended in a straightforward manner to obtain the solution.

A Matlab based subroutine package([39]), GMO 1.0, has been accomplished by the authors to implement the proposed algorithm for synthesizing (sub-)optimal controllers for the general multiobjective (GMO) control problem involving ℓ_1 norm, $\mathcal{H}_2$ norm, $\mathcal{H}_\infty$ norm, time-domain constraint (TDC), and controller structure constraints. By using this package, several multiobjective design problems from the literature have been solved (see [39]) to demonstrate the effectiveness of the proposed framework and the software.

3.3 Illustrative Examples

In this section, we present several examples from the literature to illustrate how to use GMO 1.0 package to compute multiobjective (sub-)optimal controllers. All the simulation results shown here were obtained by using GMO 1.0 package on a PII-350/312MB/Win2000 personal computer system under Matlab 5.3 environment. For ease of understanding the notations to be used in this section, we have placed in the appendix a simplified version of the user manual for GMO 1.0 package.

3.3.1 An $\ell_1/\mathcal{H}_\infty$ example

Consider the example of the $\ell_1/\mathcal{H}_\infty$ multi-block problem addressed in [19]. The problem setup is as follows:

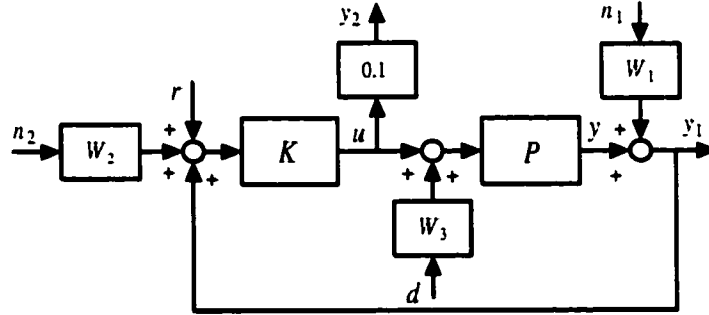


Figure 3.1 Block diagram of the $\ell_1/\mathcal{H}_\infty$ example

$$\begin{aligned} P(\lambda) &= \frac{\lambda(5 - 10\lambda)}{(1 - 10\lambda)(1 - 0.5\lambda)} & W_1(\lambda) &= \frac{0.4\lambda}{1 - 0.6\lambda}, \\ W_2(\lambda) &= \frac{1 - 0.75\lambda}{1 - 0.25\lambda}, & W_3(\lambda) &= \frac{0.02\lambda}{1 - 0.2\lambda}. \end{aligned} \quad (3.16)$$

The optimization problem of interest is $\{\min \|\Phi\|_1 : \|\Psi\|_\infty \leq 37\}$, where Φ is the transfer matrix from $w_1^T := [n_1 \ n_2]$ to $z_1^T := [y_1 \ y_2]$ and Ψ is the transfer matrix from $w_2^T := [r \ d]$ to $z_2^T := [y \ u]$ respectively. With an FIR length of three (lenqind=3), the GMO routines yield a pair of lower and upper bounds [72.5960, 73.0380] with

$\|\Phi\|_1 = 72.8220$ achieved by a 9th order unstable suboptimal controller. The $\mathcal{H}_\infty$ performance of the Ψ subsystem is 36.9583. These results coincide with those obtained in [19], where the optimal ℓ_1 performance is computed to be 72.6418 achieved by a 14th order optimal controller.

3.3.2 Active suspension control

The active suspension control design for transport vehicles ([2], [40], and [41]) aims to handle the following conflicting goals:

1. low levels of acceleration for the comfort of drive and cargo (isolation goal),
2. bounded suspension deflection (connection goal),
3. bounded tire deflection (connection goal).

Besides the above requirements, in practical implementations, the actuation system of the suspension control system should also be limited not to require and dissipate too much power so as to avoid the cooling problems and to make the system more efficient. Besides average power, the peak value of the actuator force generated by the controller should also be limited to avoid large equipment costs.

The problem proposed above can be formulated into a multiobjective control design problem involving ℓ_1 and $\mathcal{H}_2$ norms. Consider the following two DOF (degree-of-freedom) rear suspension system model ([40]):

$$\begin{aligned} m_2 \ddot{q}_2 + b_2(\dot{q}_2 - \dot{q}_1) + k_2(q_2 - q_1) &= F \\ m_1 \ddot{q}_1 + b_2(\dot{q}_1 - \dot{q}_2) + k_2(q_1 - q_2) + b_1(\dot{q}_1 - \dot{q}_0) + k_1(q_1 - q_0) &= -F. \end{aligned} \tag{3.17}$$

In the above equations, $m_1 = 1.5e3kg$ denotes the mass of tires, wheels, and real axle. $m_2 = 5.75e3kg$ denotes the sum of mass of the chassis and a half-loaded semitrailer. $b_1 = 1.7e3N/m$ and $b_2 = 5e3N/m$ represent the tire and suspension damping coefficients while $k_1 = 5e6N/m$ and $k_2 = 5e5N/m$ denote the tire and suspension stiffness

respectively. q_0 , q_1 , and q_2 are road level, suspension displacement, and semitrailer displacement respectively.

By a suitable choice of states, the above model can be transformed into a fourth order state space model. According to the required performance specifications, after incorporating suitable weights, the system can be transformed into the following open-loop generalized plant setup that is amenable to control design purpose:



Figure 3.2 Open-loop system of the suspension control example

where we choose the exogenous input to be $w := q_0$, control input to be $u := F$, measured output to be $y^T := [\ddot{q}_2 \ q_2 - q_1]$, and the controlled output to be $z^T := [\ddot{q}_2 \ q_2 - q_1 \ q_1 - q_0 \ F]$. Note that in this setup, w denotes the road surface level, and z consists of the vertical acceleration $\ddot{q}_2$, suspension deflection $q_2 - q_1$, tire deflection or dynamic tire force $q_1 - q_0$, and actuator force F .

Assume that $w = q_0$ denotes a given deterministic-and-stochastic mixed road profile with a known l_∞ bound and a (spatial) power spectral density (incorporated into the generalized plant as a weight already), then according to the arguments above, the following control design would be of significant interest to designers:

$$\begin{aligned}
 & \inf_{Q \in \ell_1^{n_u \times n_v}} \|R_{z_1 w}(Q)\|_2^2 \\
 & \text{subject to} \\
 & \|R_{z_2 w}\|_1 \leq c_2 \\
 & \|R_{z_3 w}\|_1 \leq c_3 \\
 & \|R_{z_4 w}\|_1 \leq c_4
 \end{aligned} \tag{3.18}$$

where $c_i (i = 2, 3, 4)$ are certain parameters to be chosen. Note here we choose to minimize the $\mathcal{H}_2$ norm to address the comfort performance requirement of the driver and cargo. To ascertain the achievable ranges for c_i , we can first carry out the following study of performance limits:

$$c_i^0 := \inf_{Q \in \ell_1^{n_u \times n_v}} \|R_{z_i w}(Q)\|_p \quad (3.19)$$

where $\|\cdot\|_p$ denotes $\mathcal{H}_2$ norm for $i = 1$ and ℓ_1 norm for $i = 2, 3, 4$. This set of problem can be readily solved by using GMO routine (for more details, please refer to the GMO 1.0 user manual in the Appendix) and the design yields $c_2^0 = 0.1701$, $c_3^0 = 0.1660$, and $c_4^0 = 0.0082$ and the best achievable $\mathcal{H}_2$ norm of $R_{z_1 w}$ is 0.0665 while the other three channels achieve ℓ_1 performance of 0.6733, 0.7296, and 0.0419 respectively. According to the minimum achievable ℓ_1 performance obtained above, $c_i (i = 2, 3, 4)$ are chosen to be 0.6, 0.6, and 0.1 in (3.18).

The final resulting $\mathcal{H}_2$ performance of (3.18) is 0.0729 with ℓ_1 performance of 0.6026, 0.6037 and 0.0333 in other three channels. It is clear from this example that GMO routine has a flexible structure and various control system design demands can be easily captured in its framework.

3.3.3 Optimal Control Design for a 3-nodal ABR Network

Consider the schematic in Figure 3.3, that depicts a network of three nodes. The purpose of the model is to study various aspects of coordination control between various nodes and its relation to the information structure. An associated application is congestion control in the case of an available bit rate (ABR) communication network ([42]).

In Figure 3.3, r_1, r_2 and r_3 denote the flow rates from data sources into network nodes 1, 2, 3 respectively. r_{12} denotes the rate of flow from node 1 to node 2 and r_{23}

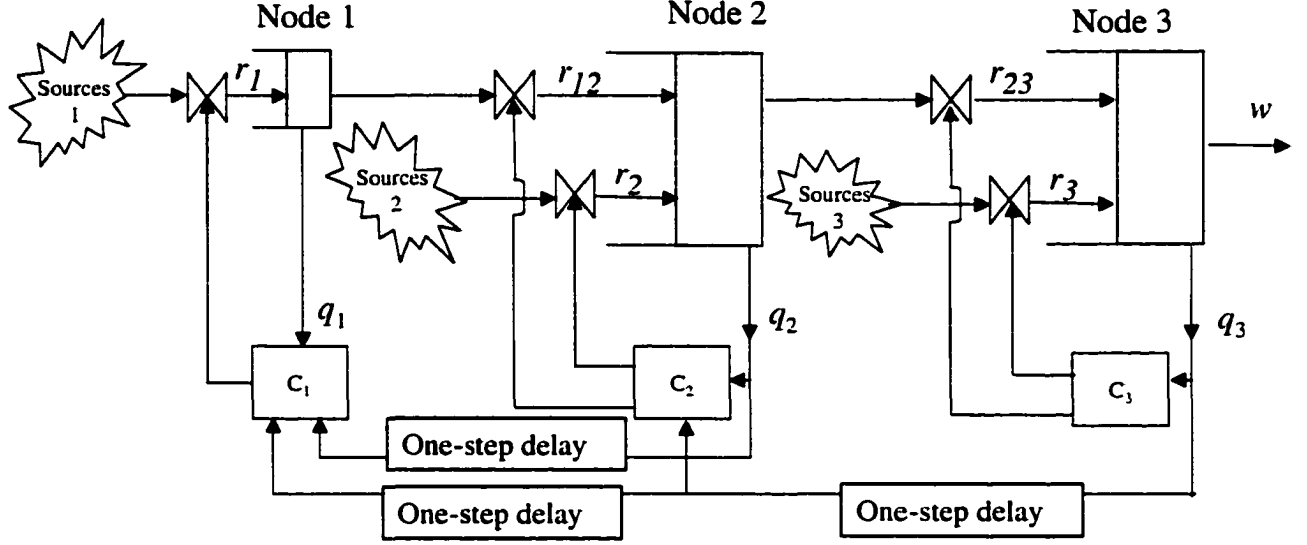


Figure 3.3 3-nodal ABR Network

denotes the rate of flow from node 2 to node 3. w represents the total capacity available for the three data sources. q_i denotes the buffer length at the i_{th} node. The network exerts control over the network traffic by assigning the rate for each data source. In particular, there are three (nodal) subcontrollers C_1, C_2, C_3 that dictate respectively r_1 , (r_{12}, r_2) , and (r_{23}, r_3) . Moreover, there is a one-step delay in passing nodal information (q_i) from one nodal subcontroller C_i to its preceding one C_{i-1} , while each C_i does not receive information from the previous nodes C_{i+1} . The goals are to prevent the node buffers from overflowing so as to avoid possible data loss ('stabilization goal'), and to optimally utilize the available transfer capacity w such that the sum of the data rates $r_i (i = 1, 2, 3)$ matches w as closely as possible ('optimality goal').

For this system, the exogenous input signal is identified as the available capacity w . The control input, and measured output signals are identified respectively as:

$$\bullet u = [r_1 \ r_{12} \ r_2 \ r_{23} \ r_3]^T$$

- $y = [q_1 \ q_2 \ q_3]^T$.

The goal of the congestion control for the above network can be captured by adopting the following signal identification for the regulated output:

- $z = [q_1 \ q_2 \ q_3 \ r_1 - w \cdot a_1 \ r_2 - w \cdot a_2 \ r_3 - w \cdot a_3]^T$.

where a_i is a prescribed constant representing the ratio of available resource assigned to i_{th} source.

Suppose also that steps are the typical exogenous input signals w we would like to optimally track. Then, we can impose TDCs on $z_i (i = 4, 5, 6)$ such that the step response of $z_i (i = 4, 5, 6)$ is forced to stay within a prescribed envelope. In the sequel we consider the coordination of the network operation around a desired equilibrium point where the queues at the nodes and the traffic rates are at a desired non-zero, positive level. The linearized fluid model nodal dynamics that we adopt are given by:

- Node 1: $q_1(k+1) = q_1(k) + r_1(k) - r_{12}(k)$
- Node 2: $q_2(k+1) = q_2(k) + r_2(k) + r_{12}(k) - r_{23}(k)$
- Node 3: $q_3(k+1) = q_3(k) + r_3(k) + r_{23}(k) - w(k)$

There are three local controllers corresponding to the three nodes such that the controllers are required to satisfy the following structural constraints:

- $C_1: r_1 = f_1(q_1, \lambda q_2, \lambda^2 q_3)$
- $C_2: \begin{cases} r_{12} = f_{12}(q_2, \lambda q_3) \\ r_2 = f_2(q_2, \lambda q_3) \end{cases}$
- $C_3: \begin{cases} r_{23} = f_{23}(q_3) \\ r_3 = f_3(q_3) \end{cases}$

where the various f_i and f_{ij} are (causal) linear operators and λ is interpreted as the one step delay operator.

Clearly, the plant G_{22} and the controller K are upper triangular operators of the following form:

$$G_{22} := \begin{bmatrix} * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & 0 & * & * \end{bmatrix}$$

$$K := \begin{bmatrix} * & \lambda* & \lambda^2* \\ 0 & * & \lambda* \\ 0 & * & \lambda* \\ 0 & 0 & * \\ 0 & 0 & * \end{bmatrix} \quad (3.20)$$

In this example we provide a tradeoff study between ℓ_1 and $\mathcal{H}_2$ performance of the closed-loop system by solving the following multiobjective problem:

$$\nu := \inf \quad c_1 \|R(K)\|_1 + c_2 \|R(K)\|_2^2$$

subject to

K is stabilizing

K satisfies structural and delay constraints (3.20)

$z_i (i = 4, 5, 6)$ satisfies prescribed TDCs.

where c_1 and c_2 are prescribed weighting constants. Following the framework established in [7] and [8], we now detail the procedure of how the upper block triangular structural constraints on K as specified in (3.20) are transformed to the same structural constraints on Q .

The state-space description of G_{22} is given by

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] = \left[\begin{array}{ccc|ccc} A_1 & 0 & 0 & B_1 & B_{12} & 0 \\ 0 & A_2 & 0 & 0 & B_2 & B_{23} \\ 0 & 0 & A_3 & 0 & 0 & B_3 \\ \hline C_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_3 & 0 & 0 & 0 \end{array} \right]$$

where

$$A_1 = A_2 = A_3 = 1, C_1 = C_2 = C_3 = 1$$

$$B_1 = 1, B_{12} = [-1 \ 0], B_2 = [1 \ 1], B_{23} = [-1 \ 0], B_3 = [1 \ 1].$$

The state feedback and observer gain matrices F and L for G_{22} are chosen to be

$$F = \begin{bmatrix} F_1 & 0 & 0 \\ 0 & F_2 & 0 \\ 0 & 0 & F_3 \end{bmatrix}, L = \begin{bmatrix} L_1 & 0 & 0 \\ 0 & L_2 & 0 \\ 0 & 0 & L_3 \end{bmatrix},$$

where

$$F_1 = -0.90, F_2 = F_3 = [0 \ -0.9]^T, L_1 = L_2 = L_3 = -0.90.$$

This choice of F and L guarantees that $A + BF$ and $A + LC$ are stable matrices. The first four doubly-coprime factors of the plant ([3]) are given by:

$$Y_r := \left[\begin{array}{c|c|c} \frac{-0.81\lambda}{1-0.1\lambda} & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & \frac{-0.81\lambda}{1-0.1\lambda} & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & \frac{-0.81\lambda}{1-0.1\lambda} \end{array} \right], D_r := \left[\begin{array}{c|c|c|c|c} \frac{1+0.8\lambda}{1-0.1\lambda} & \frac{-0.9\lambda}{1-0.1\lambda} & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{0.9\lambda}{1-0.1\lambda} & \frac{1+0.8\lambda}{1-0.1\lambda} & \frac{-0.9\lambda}{1-0.1\lambda} & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{0.9\lambda}{1-0.1\lambda} & \frac{1+0.8\lambda}{1-0.1\lambda} \end{array} \right]$$

$$X_r := \begin{bmatrix} \frac{1+0.8\lambda}{1-0.1\lambda} & 0 & 0 \\ 0 & \frac{1+0.8\lambda}{1-0.1\lambda} & 0 \\ 0 & 0 & \frac{1+0.8\lambda}{1-0.1\lambda} \end{bmatrix}, \quad N_r := \begin{bmatrix} \frac{-\lambda}{1-0.1\lambda} & \frac{\lambda}{1-0.1\lambda} & 0 & 0 & 0 \\ 0 & \frac{-\lambda}{1-0.1\lambda} & \frac{-\lambda}{1-0.1\lambda} & \frac{\lambda}{1-0.1\lambda} & 0 \\ 0 & 0 & 0 & \frac{\lambda}{1-0.1\lambda} & \frac{-\lambda}{1-0.1\lambda} \end{bmatrix}.$$

We define a class of transfer matrices as

$$\mathcal{T} := \left\{ T = \begin{bmatrix} T_{11} & \lambda T_{12} & \cdots & \cdots & \lambda^{n-1} T_{1n} \\ & T_{22} & \lambda T_{23} & \cdots & \\ & & T_{33} & \cdots & \\ & & & \ddots & \\ & & & & T_{nn} \end{bmatrix} \right\},$$

where T_{ij} 's are matrices of compatible dimensions. We can see clearly that the set $\mathcal{T}$ is closed under the addition, subtraction, and multiplication for any two elements of $\mathcal{T}$ that have compatible dimensions. Moreover, using the adjoint formula, it is easy to see that the inverse of any nonsingular element of $\mathcal{T}$ also belongs to $\mathcal{T}$. Noting that Y_r , D_r , X_r and N_r are elements of $\mathcal{T}$ and that ([3])

$$K = (Y_r - D_r Q)(X_r - N_r Q)^{-1}, \quad Q = (KN_r - M_r)^{-1}(KX_r - Y_r),$$

we infer that Q admits the structure described by (11), if and only if K admits the same structure.

Therefore we conclude that for this example, the structural constraints on the controller K transform to the same constraints on Q . Hence we equivalently formulate problem ν as:

$$\nu := \inf \quad c_1 \|R(Q)\|_1 + c_2 \|R(Q)\|_2^2$$

subject to

Q is stable

$Q \in \mathcal{T}$

$z_i (i = 4, 5, 6)$ satisfies prescribed TDCs.

For simplicity, the fairness index a_i is taken to be $a_1 = a_2 = a_3 = 1/3$ and the upper bounds of $\|Q\|_1$ are chosen to be $\gamma = 100$. For a given increasing sequence of nonnegative ratios of c_2/c_1 (7 points), the auxiliary problem of ν was solved by using GMO 1.0 package and the optimal Youla parameters and the values of $\|R(Q)\|_1$ and $\|R(Q)\|_2^2$ were obtained. For all pairs of c_1 and c_2 , the ℓ_1 norms of the optimal Q 's are far less than γ (typically $\|Q\|_1 \leq 1$). This shows that the extra ℓ_1 norm constraint on Q is inactive and problem ν and its auxiliary problem admit the same optimal cost.

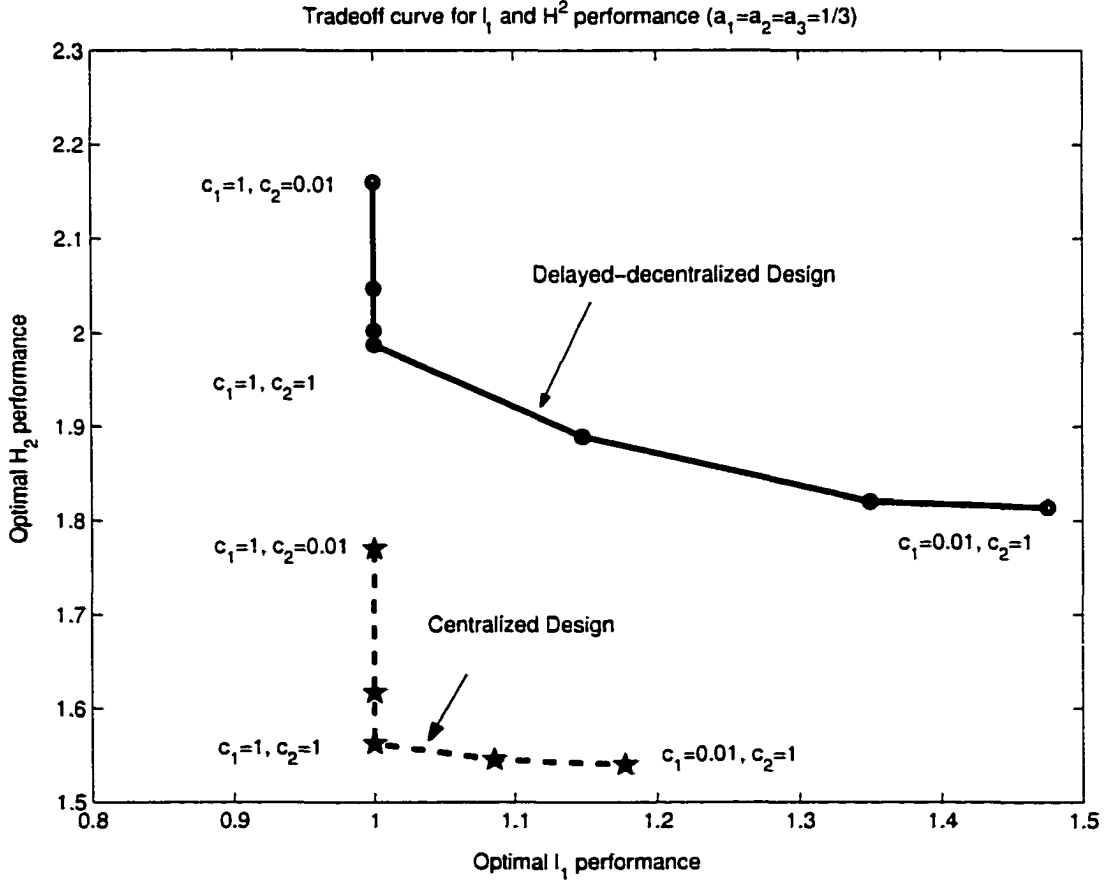


Figure 3.4 Tradeoff Curve between ℓ_1 and $\mathcal{H}_2$ performance

The plots of $\|R(Q)\|_1$ versus $\|R(Q)\|_2^2$ are shown in Figure 3.4, where the dashed curve denotes the cases of centralized design with no information transfer delay while

the solid curve denotes the cases where there exists transfer delay in the feedback path, as illustrated in Figure 3.3. From these two curves, important information on the tradeoff among system performance specifications are obtained. For example, it is clear that each of these two curves denote exactly the boundary between achievable and unachievable performance specifications of the closed-loop system. The region above the curve denotes the performance requirement that can be achieved by some stabilizing controller while the region below the curve represents the specifications that cannot be obtained by any stabilizing controller. Moreover, it can be concluded that for this example, the structure constraints imposed on the stabilizing controllers as specified in (3.20) induce a significant loss of the closed-loop system performance.

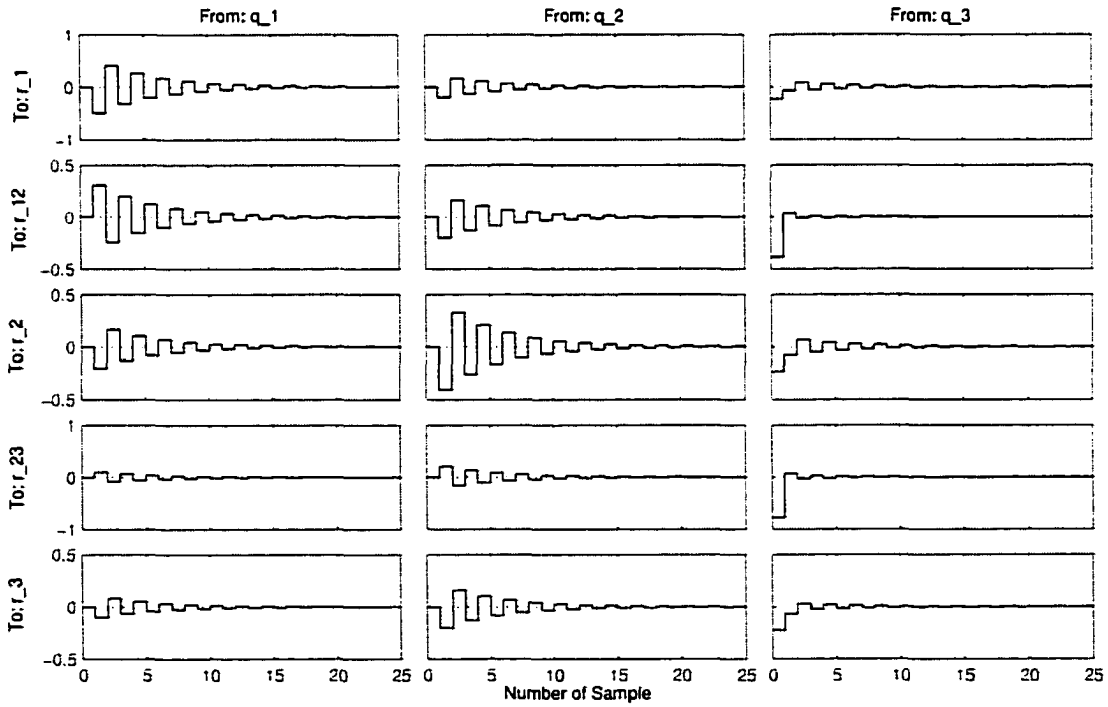


Figure 3.5 Impulse Response of Centralized Controller

The impulse responses of the centralized sub-optimal controller (case $c_1 = c_2 = 1$, performance tolerance $\delta = 0.01$), and the decentralized, delayed sub-optimal controller

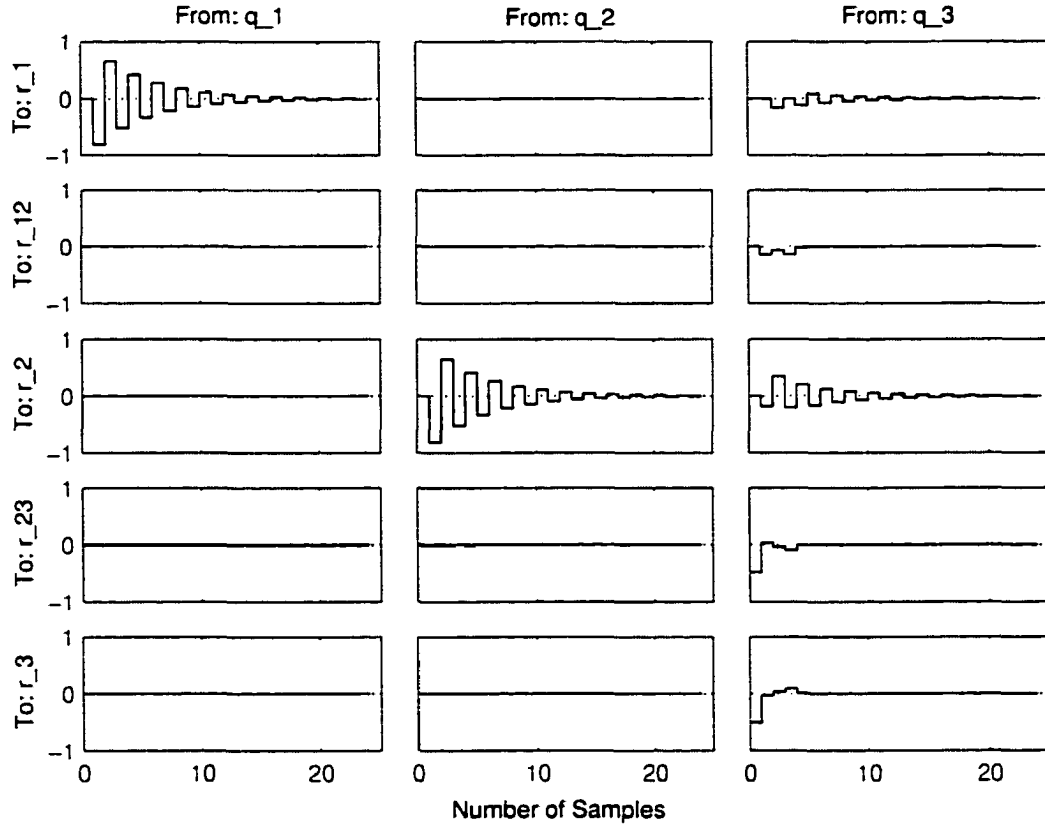


Figure 3.6 Impulse Response of Decentralized, Delayed Controller

(case $c_1 = c_2 = 1$, performance tolerance $\delta = 0.01$) are plotted in Figure 3.5 and Figure 3.6, respectively. From the last figure, it can be clearly observed that the structural constraints imposed on the stabilizing controller are satisfied. That is, the controller admits the upper block triangular structure specified in (3.20) while the centralized controller does not admit such a structure. The order of the Youla parameter Q is 3 and the order of the corresponding decentralized, delayed sub-optimal controller is 6. In Figure 3.7, the step response of the closed-loop system with decentralized, delayed controller is plotted, where the dash-dotted lines denote the TDC envelops imposed on the step responses of $z_i (i = 4, 5, 6)$. It is clear from the response plots that the time response of $z_i (i = 4, 5, 6)$ satisfies the requirement of zero steady value, which implies

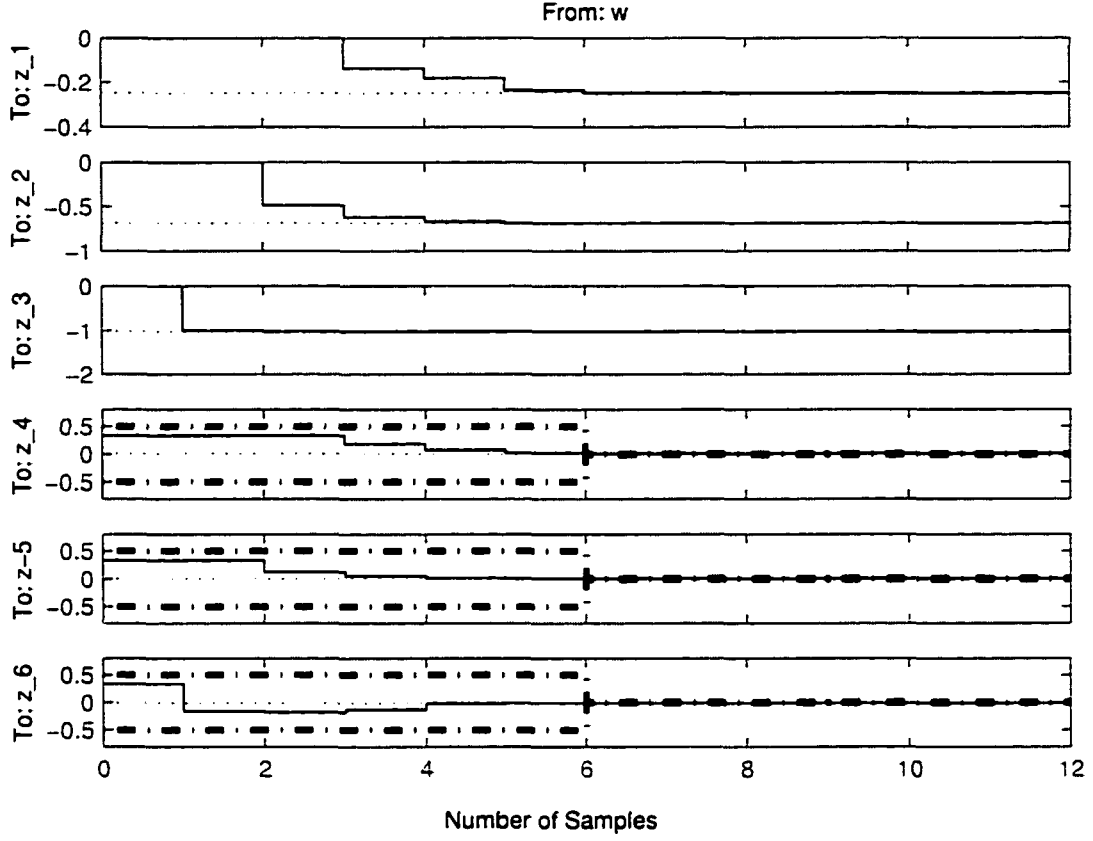


Figure 3.7 Step Response of Closed-loop system with Decentralized, Delayed Controller

that the optimality goal of the congestion control mechanism is achieved.

3.3.4 Multiobjective $\mathcal{H}_\infty$ design

This example is taken from [17] and [16]. The control objective is to minimize the $\|C_1\|_\infty + \|C_2\|_\infty$ performance for the unstable system:

$$\begin{bmatrix} z_1 \\ z_2 \\ y \end{bmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ u \end{pmatrix},$$

$$A = \begin{bmatrix} 0.5 & 1 & 1.5 & 1 \\ -1 & 3 & 2.1 & 2 \\ 1 & -1 & -0.6 & 1 \\ -2 & 2 & -1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where C_1 and C_2 represent the two performance channels from w_1 to z_1 and w_2 to z_2 respectively.

By calling GMO routines (lenqind=12, beta=10000), a lower bound and an upper bound of the $\mathcal{H}_\infty$ performance sum can be computed to be 114.4058 and 115.6487 ($\|C_1\|_\infty = 65.5327$, $\|C_2\|_\infty = 50.1160$) respectively. The compensator obtained by GMO is a 21th order unstable controller (Figure 3.8).

3.3.5 F16 longitudinal control design

Originally studied in [43], the AFTI F-16 control problem aims to synthesize an ℓ_1 robust optimal controller for the longitudinal dynamics so as to achieve certain tracking performance while satisfying constraints on control deflection, control rate, and requirements on overshoot and undershoot specifications.

Specifically, the tracking problem is to accurately command a $1 - g$ normal acceleration of the aircraft while the stabilator is limited to $\pm 25deg$ deflection angle and $\pm 60deg/s$ deflection rate ([43]). The aircraft model used in the paper consists of an actuator servo G_a and the linearized longitudinal equation of motion G_p and the continuous system is a concatenation of these two components. Since the discrete-time nature of the ℓ_1 controller calls for the sampled-data system implementation, the continuous system $G_p G_a$ is discretized at $30Hz$ using a zero-order holder (ZOH). All the simulations are conducted within this hybrid system framework and a step reference

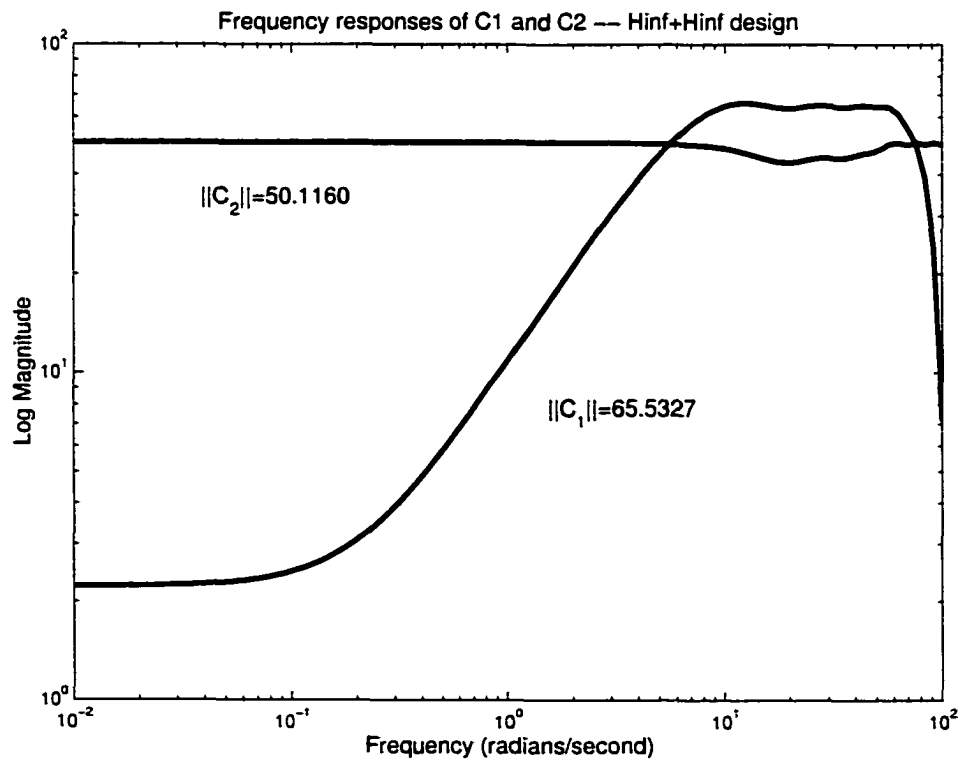
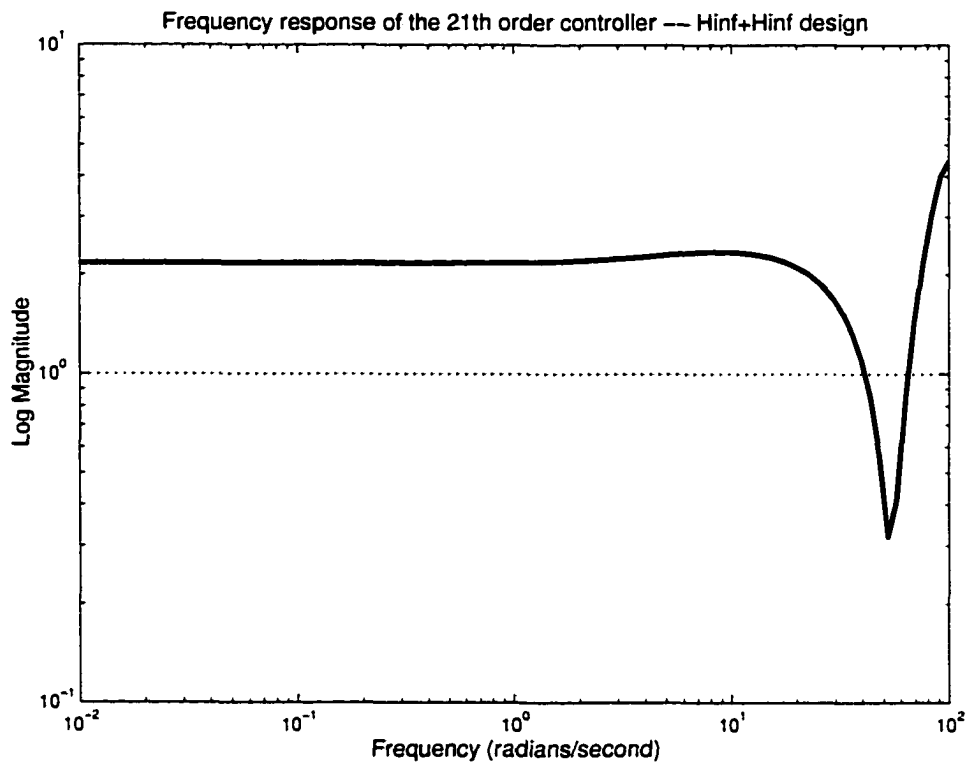


Figure 3.8 $\mathcal{H}_\infty + \mathcal{H}_\infty$ design results

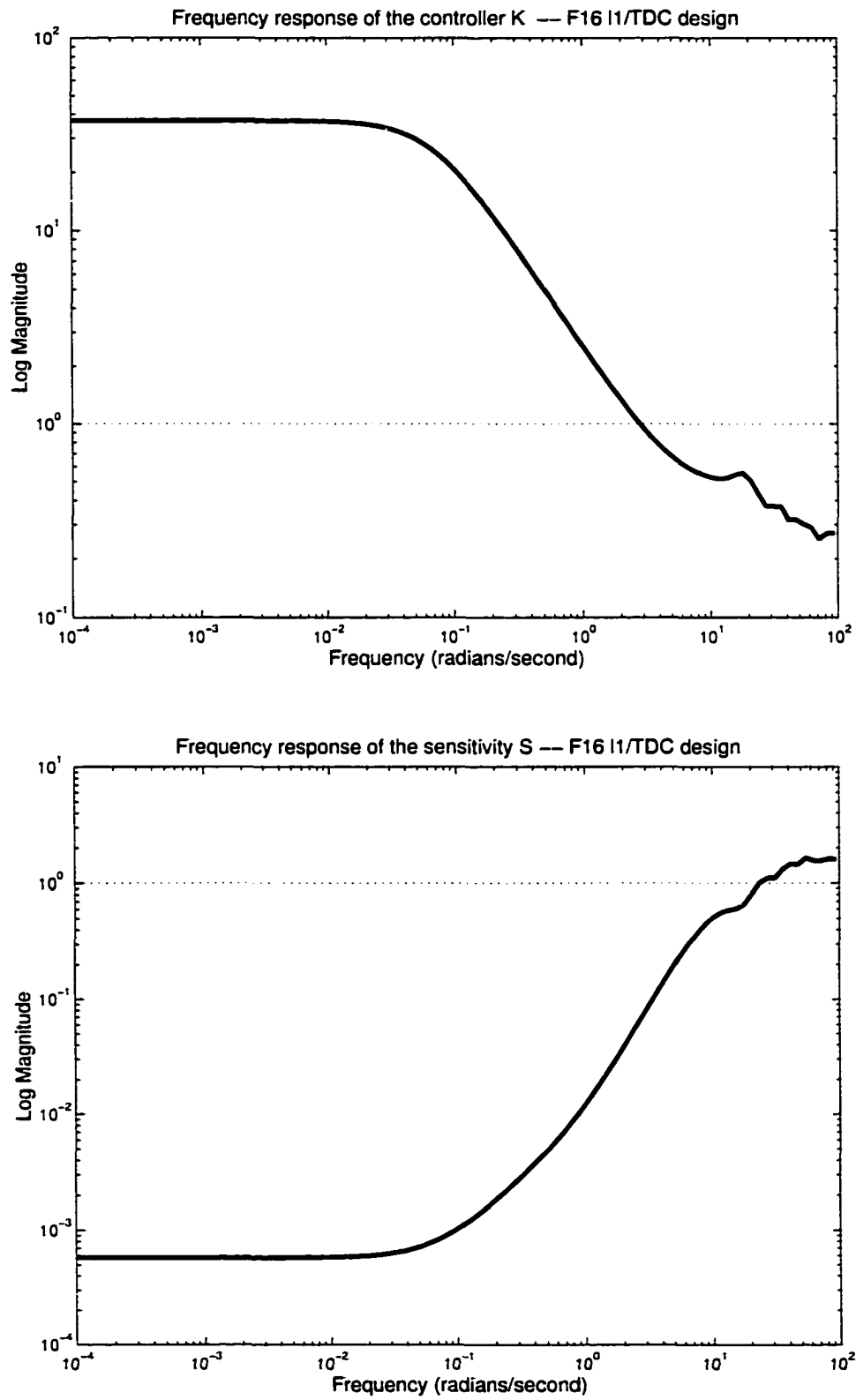


Figure 3.9 F16 longitudinal design results (a)

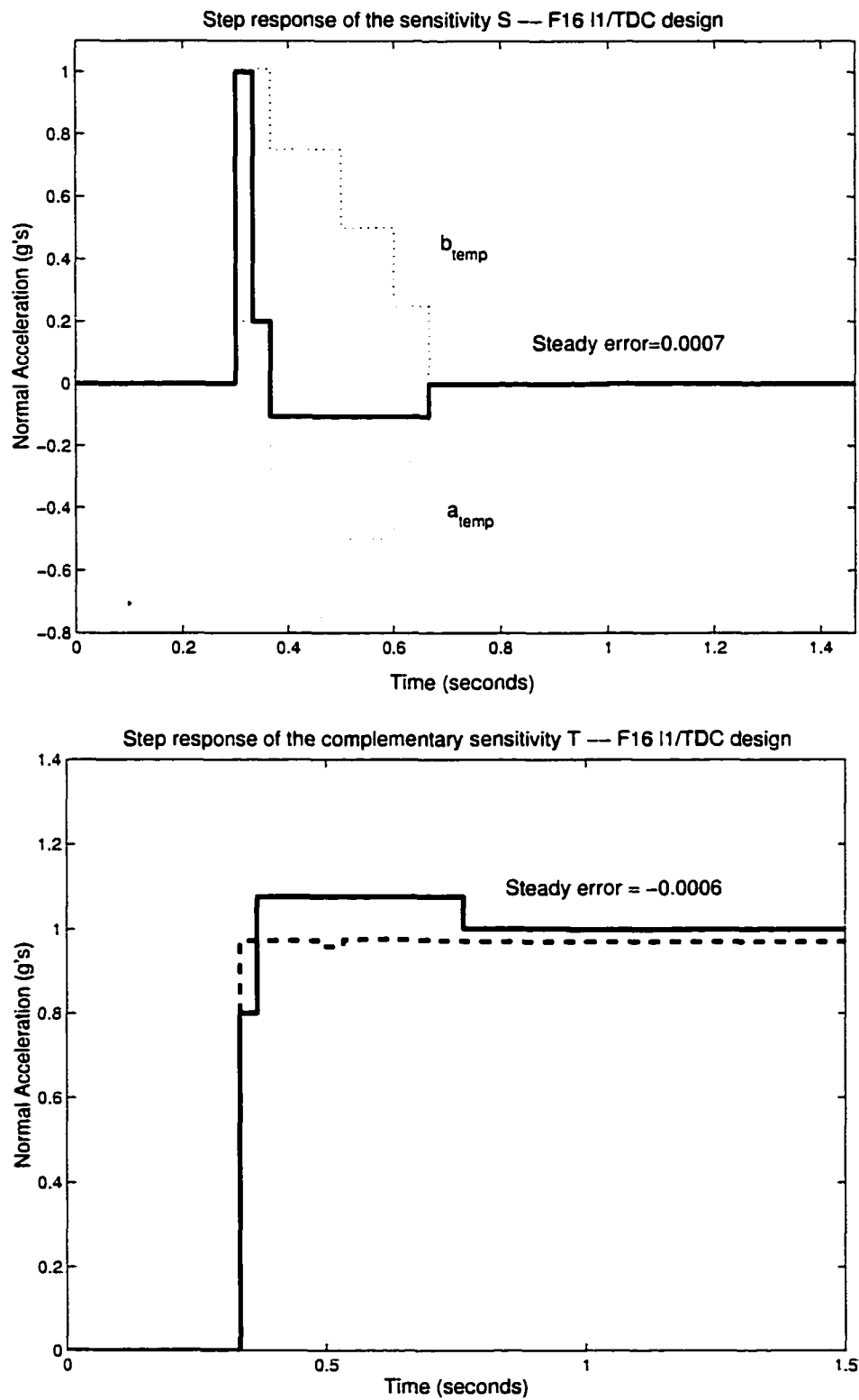


Figure 3.10 F16 longitudinal design results (b)

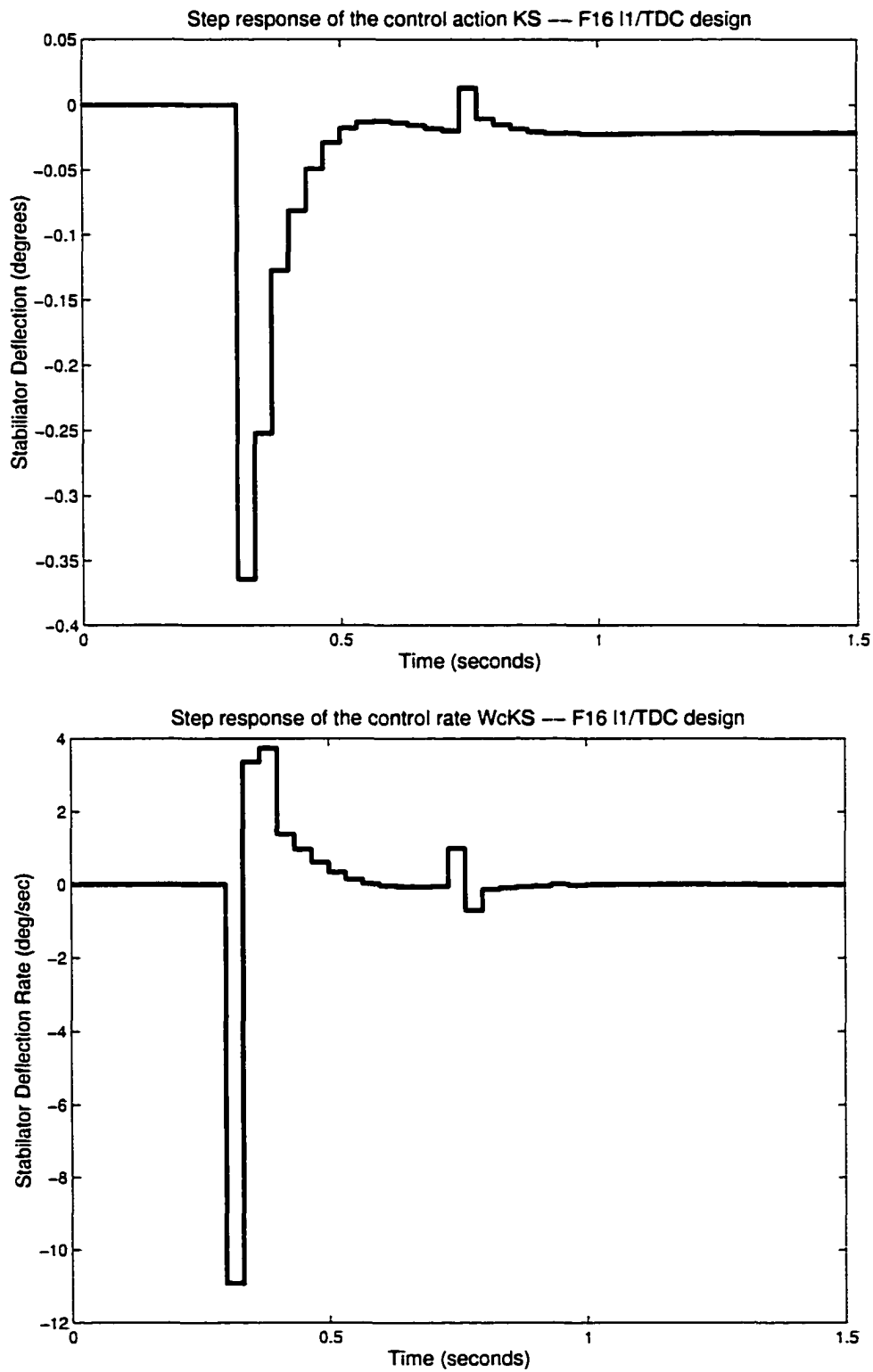


Figure 3.11 F16 longitudinal design results (c)

input of $1 - g$ normal acceleration is applied at 0.3 second(simulation time) to evaluate tracking performance.

To achieve the desired tracking performance, TDC template a_{temp} and b_{temp} are chosen in such a manner that step response of sensitivity function S is forced to converge to zero as the system proceeds into the steady state. With $lenqind = 25$ and $beta = 100$, GMO routines yield an ℓ_1 performance of 2.2127 achieved by a 15th order (sub-)optimal controller. It is interesting to note that there is an integrator (a pole at 0.9966) in this controller, which verifies the result (steady error of -0.0006) shown in Figure 3.10 from a different point of view.

Note that to effectively take out the derivative of the control signal, a discrete-time transfer function (the 'backward Euler transformation') $W_c(z) = (z - 1)/Tz$ ($T = 1/30sec$) was applied on the stabilator deflection to generate time-response output in the simulink diagram. The frequency-domain and time-domain responses of the ℓ_1/TDC design are plotted in Figure 3.9, Figure 3.10, and Figure 3.11. Note that to reduce the control action and control rate magnitude, we choose $a_{temp}(1) = a_{temp}(2) = 0.2$ in the ℓ_1/TDC design to prevent the control action becoming too large during the first two sample periods. It is clear from the step response curves of sensitivity S , control action KS , and control rate W_cKS that this objective has been effectively achieved. As a conclusion, the control design has yielded satisfactory tracking performance while satisfying all the prescribed constraints (compared to those obtained in [43]).

3.3.6 X29 pitch axis control design

To achieve certain desirable aerodynamic characteristics, the wings of the X29 aircraft are designed to be in the forward-swept shape. This renders better maneuverability to the aircraft when compared with classical wing design while leaving the aircraft statically unstable ([3]). The control objective for this plant is to design a stabilizing

discrete-time controller to minimize ℓ_1 norm of the transfer function from the disturbance w injected at the plant output to the weighted control signal z_1 and the weighted output z_2 :

$$\inf_{\kappa \text{ stabilizing}} \left\| \begin{array}{c} W_1 K S \\ W_2 S \end{array} \right\|_1$$

In this example, for the illustrative purpose, we choose $W_1 = 0.01$ and W_2 as a digital Butterworth 2nd order low-pass filter with cut-off frequency $0.1(\text{rad/sec})$. Under this setup, GMO design yields a 32th order (sub-)optimal controller with ℓ_1 performance 1.1140 (Dashed curves in Figure 3.12, 3.13, 3.14). Noticing that the step response of the sensitivity function bears an ℓ_∞ norm of 2.3647 and a steady error of 0.3901, we intend to improve the tracking performance by solving the following problem:

$$\inf_{\kappa \text{ stabilizing}} \left\{ \left\| \begin{array}{c} W_1 K S \\ W_2 S \end{array} \right\|_1 : a_{temp}(k) \leq S * \text{stepIn}(k) \leq b_{temp}(k), \forall k \right\}$$

where stepIn denotes a step and a_{temp} and b_{temp} are two prescribed time-domain template constraint (TDC). In this example, they are chosen such that the maximum absolute magnitude and the steady error of step response of S are constraint within 1.5 and 0.002.

The GMO design yields an ℓ_1 performance of 1.6227 and the step response of the sensitivity function S yields a steady error of -0.0009 with an maximum absolute magnitude of 1.5000 (Solid curves in Figure 3.12, 3.13, 3.14), which implies the desired tracking performance has been achieved. It is interesting to note that there is also an integrator (a pole at 1.0000) in the resulting suboptimal controller, which substantiate the results shown in 3.14 from a different viewpoint.

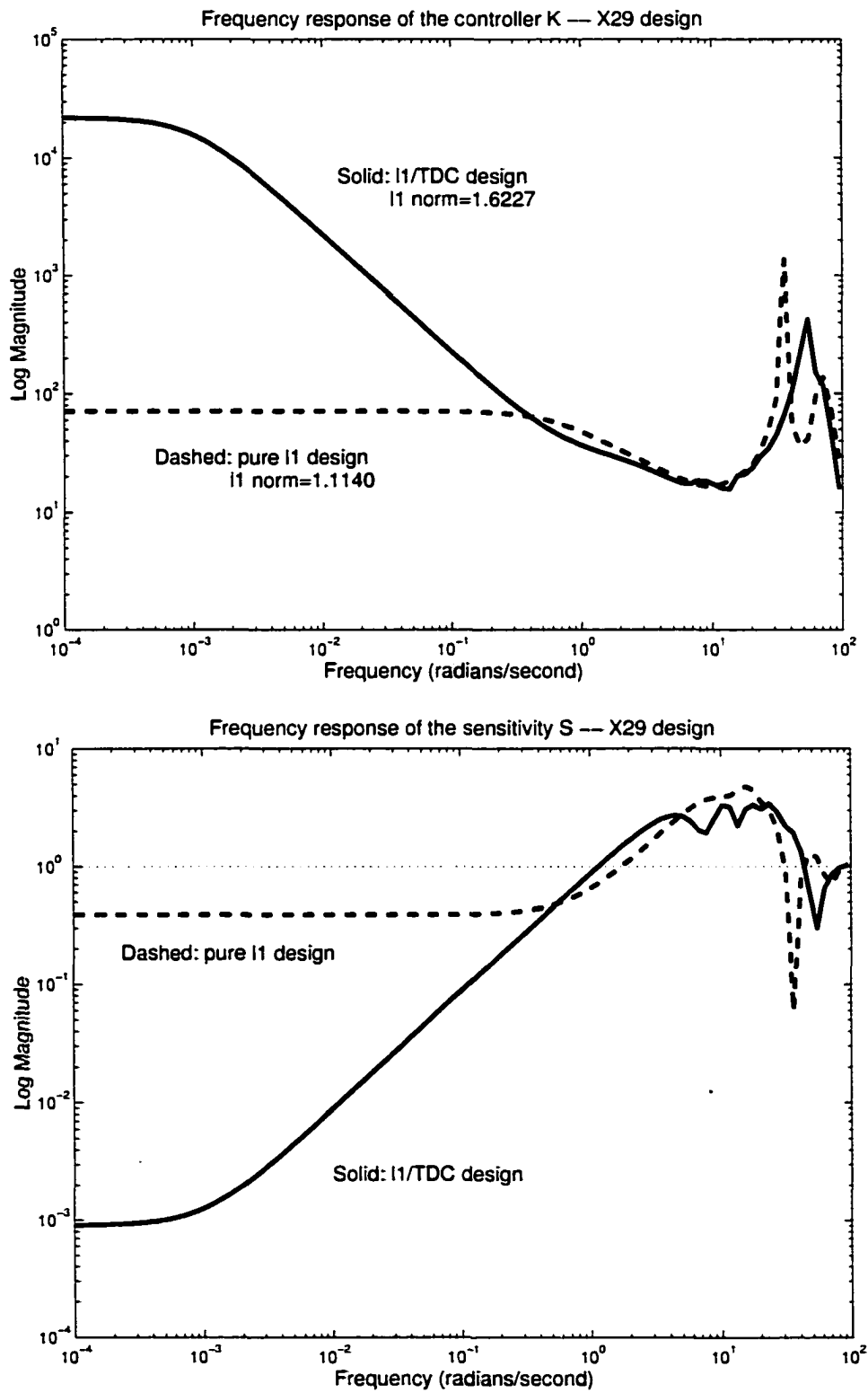


Figure 3.12 X29 pitch axis design results (a)

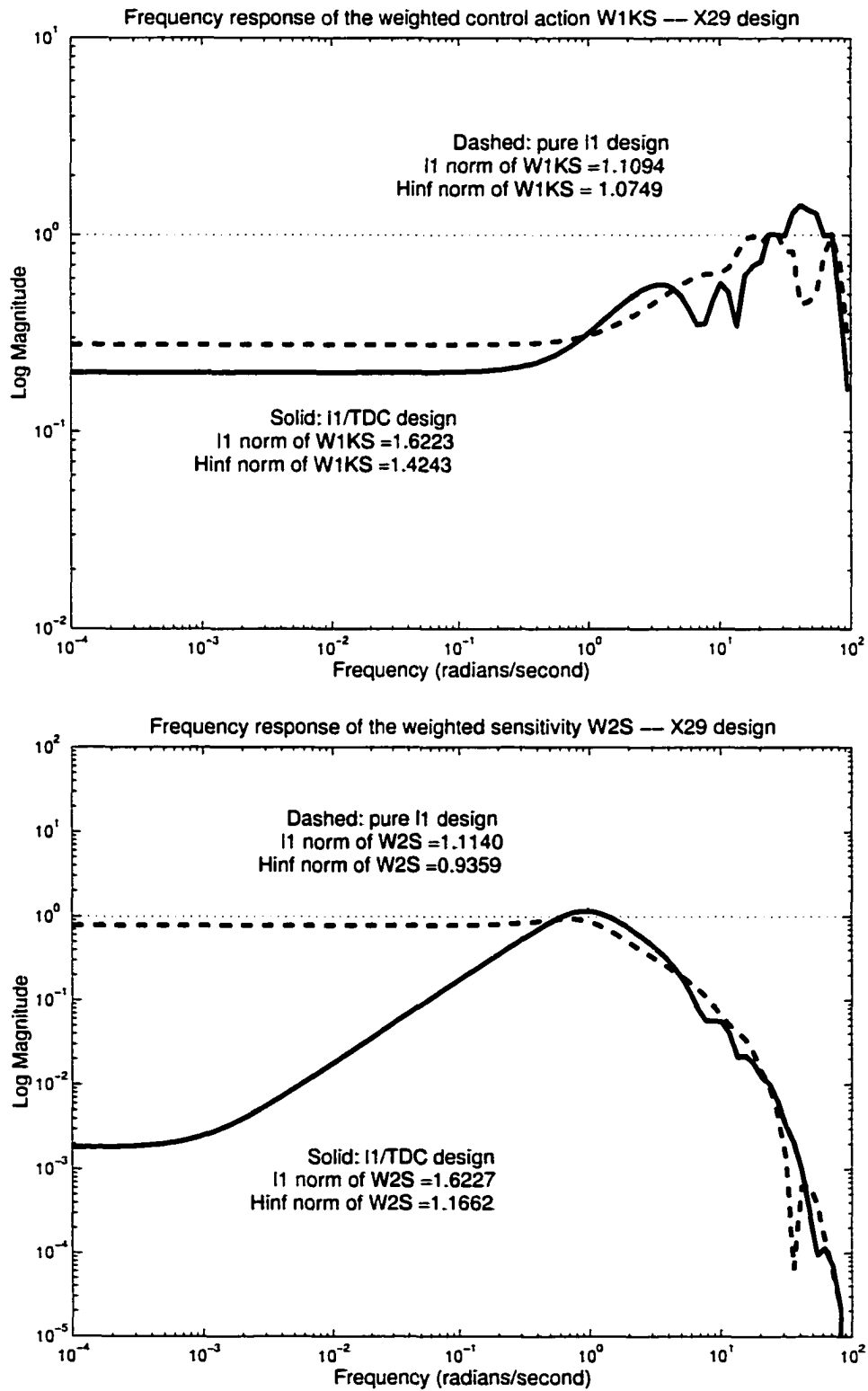


Figure 3.13 X29 pitch axis design results (b)

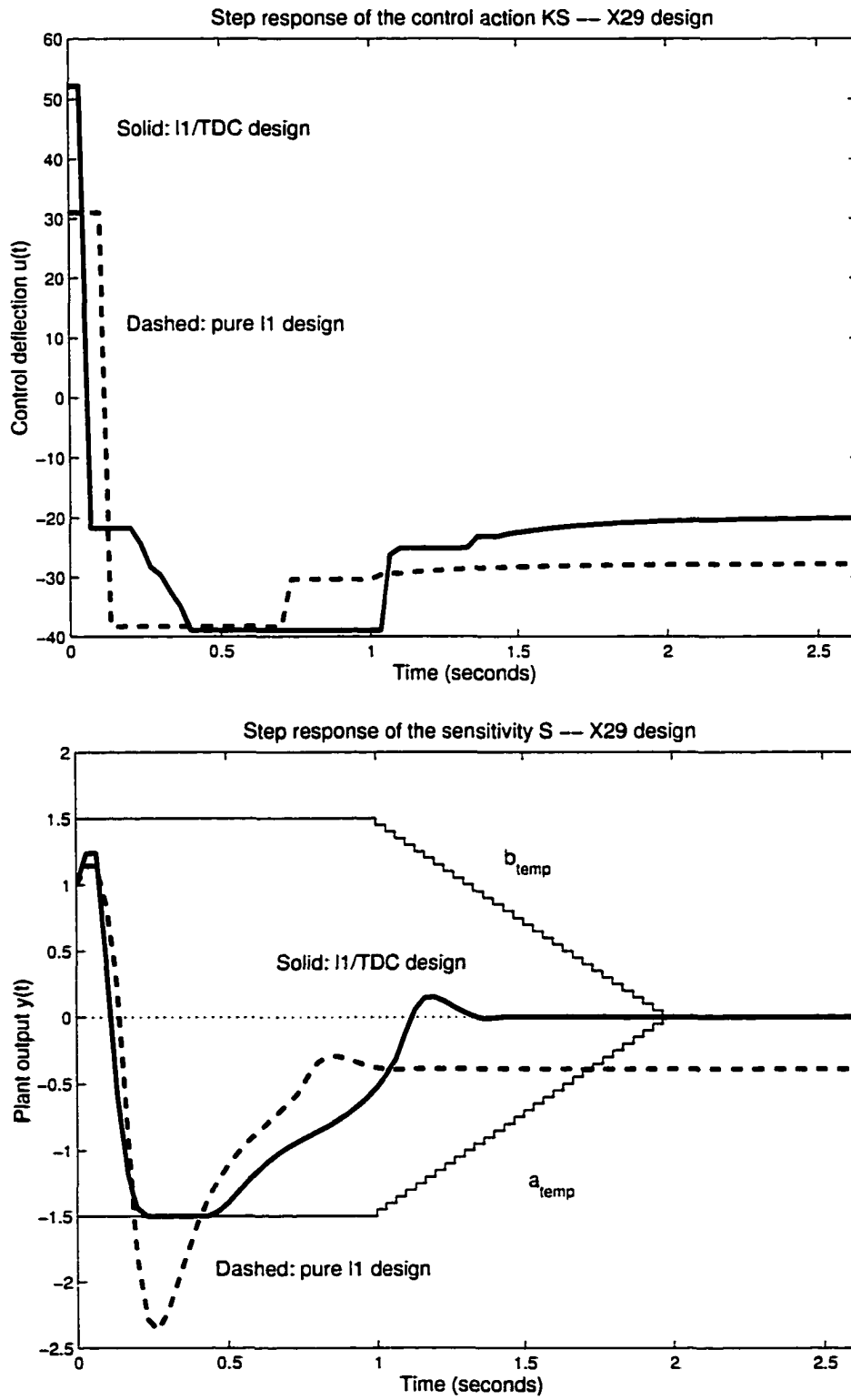


Figure 3.14 X29 pitch axis design results (c)

3.4 Conclusion Remarks

In this chapter, we have discussed the numerical implementation of the proposed GMO controller synthesis algorithm, and have illustrated the effectiveness of the proposed algorithm via several numerical examples. It is clearly seen that, by employing the proposed control synthesis tool, designers can now obtain robust optimal controllers that satisfy multiple criteria simultaneously in a straightforward manner. For example, to achieve certain desired system time response properties (such as rise time, overshoot, steady-error, etc.), the designers only need to shape the two time-domain constraint templates in the control synthesis optimization. In this way, any controller the designers obtained is guaranteed to be stabilizing and satisfies the desired time response performance. This avoids the ad-hoc effort inherent in the conventional methods and makes straightforward the synthesis of the desired multiobjective optimal controllers.

PART II

INTEGRATED PARAMETER AND CONTROL
(IPC) DESIGN

CHAPTER 4 NOMINAL PERFORMANCE WITH POLYNOMIAL DEPENDENT PARAMETERS

In this chapter, the integrated parameter and control (IPC) problem is considered where the system parameters are assumed to enter into the state-space realization in a polynomial manner. Converging finite-dimensional sub-optimal problems are constructed and solved via a linear relaxation technique, whereby a global optimal solution to the IPC problem can be computed to any prescribed tolerance.

4.1 Motivation

Conventionally the design of a controlled system is a separate two-step procedure: First, the plant is designed to satisfy certain desired static and dynamic properties. Secondly, controller is designed to satisfy closed-loop performance specifications. However, in this procedure, there is no guarantee of the optimal closed-loop system performance with respect to the possible choice of plants and controllers. It has been well recognized that system structure design and feedback control synthesis are not isolated processes ([44]). The plant design and the controller synthesis procedures are naturally iterative in a sense that good modelling should take into consideration the knowledge of the controller, and a good control design should (ideally) yield directions on how to modify the model to achieve the best possible performance. Due to the increasingly demanding performance requirements imposed on designing today's engineering systems, it is well-motivated to develop a systematic framework to conduct system structure design

and control synthesis simultaneously.

Research efforts towards this direction have yielded many algorithms to synthesize a stabilizing controller achieving certain optimal performance and to select the corresponding system structure parameters that affinely enter into the system dynamics. In particular, several numerical optimization based procedures have been proposed in [45], [46], and [47] to tackle the IPC design problem. The common practice in these approaches is to carry out the design in an iterative way. That is, the control design and plant design are repeated one after another until a certain tolerance is achieved. These approaches often yield better closed-loop performance than the traditional two-step methods. However, due to the non-convexity of the problem, these iterative algorithms usually yield a sequence of non-increasing upper bounds and do not guarantee the convergence of the bounds to the global optimal solution.

Recently a new methodology was proposed in [48] to solve the IPC problem where ℓ_1 norm or $\mathcal{H}_2$ norm were taken as the performance objectives and the system parameters were assumed to enter into the system dynamics in a polynomial manner. Evolving from the solution to the IPC design problem as in [48], in this paper, the simultaneous system and control design problem is considered for the case where plant parameters enter into the system in a rational manner. We show that globally convergent sequences of upper and lower bound problems can be formulated and solved efficiently for the IPC design problem, whereas the limitations inherited in the iterative design methods can be eliminated and a global optimal solution can be obtained within any prescribed performance tolerance.

The outline of this chapter is as follows. In Section 4.2, we formulate the problem setup and the converging sub-problems. In Section 4.3, we transform the nonlinear sub-problems into a more manageable expression. In Section 4.4, we show that the solutions to these sub-problems can be effectively computed by solving a relaxed linear

programming problem combined with a branch and bound algorithm. In Section 4.5, we summarize this chapter.

4.2 Problem Formulation

Consider the setup in Figure 4.1, where $G : [w; u] \rightarrow [z; v]$ is the generalized discrete-time linear time-invariant plant, K is the controller. w , z , u , and v are the exogenous input, regulated output, control input, and measured output of dimensions n_w , n_z , n_u , and n_v , respectively.

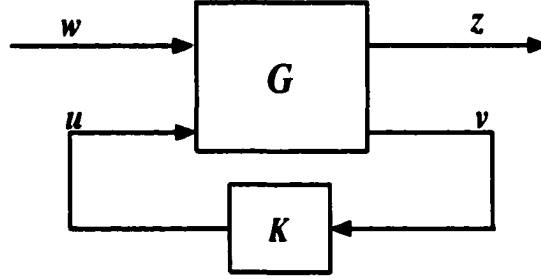


Figure 4.1 Closed-loop system

Suppose G has the following realization:

$$G(\rho) := \left[\begin{array}{c|cc} A(\rho) & B_1(\rho) & B_2(\rho) \\ \hline C_1(\rho) & D_{11}(\rho) & D_{12}(\rho) \\ C_2(\rho) & D_{21}(\rho) & D_{22}(\rho) \end{array} \right].$$

where $\rho = [\rho_1 \ \cdots \ \rho_m]^T \in R^m$, and each entry $g(\rho)$ of $G(\rho)$ is a p -degree polynomial of the form:

$$g(\rho) = \sum f_\theta \rho^\theta, \quad \rho^\theta = \prod_{j=1}^m \rho_j^{\theta_j}, \quad 0 \leq \theta_j \leq p, \quad \sum_{j=1}^m \theta_j = \phi \in \{0, 1, 2, \dots, p\}.$$

where f_θ is the coefficient of the ϕ -degree monomial ρ^θ .

Given two m -dimensional real vectors $\underline{\rho} = [\underline{\rho}_1 \ \cdots \ \underline{\rho}_m]^T$ and $\bar{\rho} = [\bar{\rho}_1 \ \cdots \ \bar{\rho}_m]^T$. In the sequel, we use the notation $\underline{\rho} \leq \rho \leq \bar{\rho}$ or $\rho \in [\underline{\rho}, \bar{\rho}]$ to denote the set of inequality

relations $\{\underline{\rho}_i \leq \rho_i \leq \bar{\rho}_i, i = 1, \dots, m\}$. Assume that $(A(\rho), B_2(\rho))$ is stabilizable and $(A(\rho), C_2(\rho))$ is detectable for any $\rho \in [\underline{\rho}, \bar{\rho}]$. The problem to be solved is formulated as:

$$\mu := \inf \{ \|\hat{R}(\hat{K}, \rho)\|_1 : \hat{K} \text{ stabilizing, } \rho \in [\underline{\rho}, \bar{\rho}] \} \quad (4.1)$$

where $\hat{R}$ denotes the closed-loop transfer matrix from w to z . From now on, we assume that the feasible set of problem (4.1) is non-empty, which includes the requirement that the optimal cost μ be finite.

Via Youla parametrization([3]), problem (4.1) is equivalently transformed into the following form:

$$\begin{aligned} \mu &:= \inf_{Q, \rho} \|R(Q, \rho)\|_1 \\ &\text{subject to} \\ R(Q, \rho) &= H(\rho) - U(\rho) * Q * V(\rho) \\ \underline{\rho} &\leq \rho \leq \bar{\rho} \end{aligned}$$

where $H \in \ell_1^{n_z \times n_w}$, $U \in \ell_1^{n_z \times n_u}$, $V \in \ell_1^{n_v \times n_w}$, Q is a free parameter in $\ell_1^{n_u \times n_v}$, and $'*'$ denotes the convolution operation. In the sequel, without loss of generality, we shall assume that H , U , and V are finitely supported.

Introducing an extra ℓ_1 norm bound on Q ([24]), we obtain the following auxiliary problem of μ :

$$\begin{aligned} \nu &:= \inf_{Q, \rho} \|R(Q, \rho)\|_1 \\ &\text{subject to} \\ \|Q\|_1 &\leq \alpha \\ R(Q, \rho) &= H(\rho) - U(\rho) * Q * V(\rho) \\ \underline{\rho} &\leq \rho \leq \bar{\rho}. \end{aligned}$$

It is clear that μ and ν are closely related. If problem μ has an optimal solution, say, Q_o , then $\mu = \nu$ for any $\alpha \geq \|Q_o\|_1$. If μ doesn't bear an optimal solution, then the

constraint $\|Q\|_1 \leq \alpha$ plays the role of a regularizing condition such that ν always has an optimal solution with a reasonable bounded gain. Thus in what follows, we shall solely focus on problem ν . Two sequences of lower and upper bounds of ν are then given by:

$$\begin{aligned}
\nu_n &:= \inf_{Q, \rho} \|P_n R(Q, \rho)\|_1 \\
&\text{subject to} \\
&\|Q\|_1 \leq \alpha \\
&R(Q, \rho) = H(\rho) - U(\rho) * Q * V(\rho) \\
&\underline{\rho} \leq \rho \leq \bar{\rho}.
\end{aligned}$$

$$\begin{aligned}
\nu^n &:= \inf_{Q, \rho} \|R(Q, \rho)\|_1 \\
&\text{subject to} \\
&\|Q\|_1 \leq \alpha \\
&R(Q, \rho) = H(\rho) - U(\rho) * Q * V(\rho) \\
&\underline{\rho} \leq \rho \leq \bar{\rho}. \\
&Q_k = 0 \text{ if } k > n.
\end{aligned}$$

Following the same argument as in [50], it can be shown that ν_n and ν^n monotonically converge to ν from below and above as n goes to infinity. In what follows, we shall demonstrate how to solve these finite-dimensional non-convex problems. The development will be based solely on ν_n , but the same technique also applies to the solution of ν^n .

4.3 Reformulation

In this section, we shall demonstrate that, by introducing two sets of auxiliary variables, the non-convex problem to be solved can be reformulated as an optimizing problem with linear and non-linear constraints, where the non-linear constraints are of the type $x = yz$ for variables x , y , and z .

Let the following be the corresponding state-space representation([51]) of H , U , and V :

$$\begin{aligned}
 H_{ss}(\rho) &= \begin{bmatrix} A_H(\rho) & B_H(\rho) \\ C_H(\rho) & D_H(\rho) \end{bmatrix} \\
 &= \begin{bmatrix} A(\rho) + B_2(\rho)F & -B_2(\rho)F & B_1(\rho) \\ 0 & A(\rho) + LC_2(\rho) & B_1(\rho) + LD_{21}(\rho) \\ C_1(\rho) + D_{12}(\rho)F & -D_{12}(\rho)F & D_{11}(\rho) \end{bmatrix} \\
 U_{ss}(\rho) &= \begin{bmatrix} A_U(\rho) & B_U(\rho) \\ C_U(\rho) & D_U(\rho) \end{bmatrix} = \begin{bmatrix} A(\rho) + B_2(\rho)F & -B_2(\rho) \\ C_1(\rho) + D_{12}(\rho)F & -D_{12}(\rho) \end{bmatrix} \\
 V_{ss}(\rho) &= \begin{bmatrix} A_V(\rho) & B_V(\rho) \\ C_V(\rho) & D_V(\rho) \end{bmatrix} = \begin{bmatrix} A(\rho) + LC_2(\rho) & B_1(\rho) + LD_{21}(\rho) \\ C_2(\rho) & D_{21}(\rho) \end{bmatrix}
 \end{aligned}$$

where we assume the existence of a pair of feedback gain F and observer gain L that stabilize the system for any $\rho \in [\underline{\rho}, \bar{\rho}]$. Note that if $A(\rho)$ is assumed to be stable for any feasible parameter vector ρ , then the zero controller ($F = 0$, $L = 0$) are to be chosen in the above realizations. By the definition of the impulse response for discrete-time systems, we infer from the above state-space representations that any entry $H_{ij}(k)$ of H is a polynomial of ρ , and so are $U_{ij}(k)$ and $V_{ij}(k)$. In what follows, for ease of notation, we shall use S_k^{ij} to denote $S_{ij}(k)$ for any variable S in $\ell_1^{n \times n}$ or $c_0^{n \times n}$.

It is easy to see from the definition of v_n that only the parameters of $R_0^{bc}, \dots, R_n^{bc}$ involves in the optimization of v_n and so, in what follows, we shall develop a new formulation for these variables. By Lemma 1 of [24], the b^{th} -row c^{th} -column entry R^{bc} of the closed-loop map R can be characterized as follow:

$$R_k^{bc}(Q, \rho) = H_k^{bc}(\rho) - \langle W^{bck}(\rho), Q \rangle$$

where

$$W^{bck}(\rho) := \{Z_k^{bc}(\rho), \dots, Z_0^{bc}(\rho), 0, \dots\} \in c_0^{n_u \times n_v}$$

$$Z^{bc}(\rho) := U_{b,\cdot}^T(\rho) * V_{\cdot,c}^T(\rho) = \{Z_0^{bc}(\rho), \dots, Z_k^{bc}(\rho), Z_{k+1}^{bc}(\rho), \dots\} \in \ell_1^{n_u \times n_v},$$

$U_{b,\cdot}$ is the b^{th} -row of U , and $V_{\cdot,c}$ is the c^{th} -column of V . Then it is clear from above that R_k^{bc} ($k = 0, 1, \dots, n$) is a polynomial of $\rho_1, \dots, \rho_m$ (up to the degree of a constant, say, o_n) and $Q_0^{st}, \dots, Q_k^{st}$.

Let

$$1, \rho_1, \dots, \rho_m, \rho_1^2, \rho_1\rho_2, \dots, \rho_1^{o_n}, \dots, \rho_m^{o_n} \quad (4.2)$$

be a basis for the o_n -degree polynomials and let d be its dimension. Define

$$\Gamma = [1 \ \rho_1 \ \dots \ \rho_m \ \rho_1^2 \ \rho_1\rho_2 \ \dots \ \rho_1^{o_n} \ \dots \ \rho_m^{o_n}]^T = [\tau_1 \ \tau_2 \ \dots \ \tau_d]^T. \quad (4.3)$$

Then each element τ_i of Γ is a d_i -degree monomial of the form

$$\tau_i = \prod_{j=1}^m \rho_j^{\theta_{ij}}, \quad 0 \leq \theta_{ij} \leq d_i, \quad \sum_{j=1}^m \theta_{ij} = d_i \leq o_n. \quad (4.4)$$

Moreover, there exist indices $i_l \in \{1, 2, \dots, d\}$ and $j_l \in \{1, 2, \dots, m\}$ ($l = 0, \dots, d_i$) such that (4.4) is equivalently characterized by the following set of equations:

$$\begin{aligned} \tau_i &= \tau_{i_0} &= \tau_{i_1} \rho_{j_1} \\ & & \vdots \\ \tau_{i_l} &= \tau_{i_{l+1}} \rho_{j_{l+1}} \\ & & \vdots \\ \tau_{i_{d_i-1}} &= \tau_{i_{d_i}} \rho_{j_{d_i}} \\ \tau_{i_{d_i}} &= 1. \end{aligned} \quad (4.5)$$

It follows that there exist constant coefficients f_i and g_{istl} such that R_k^{bc} can be expressed as

$$\begin{aligned} R_k^{bc}(Q, \rho) &= \sum_{i=1}^d f_i \tau_i + \sum_{i=1}^d \sum_{s=1}^{n_u} \sum_{t=1}^{n_v} \sum_{l=0}^k g_{istl} \tau_i Q_l^{st} \\ &= \sum_i f_i \left\{ \prod_{j=1}^m \rho_j^{\theta_{ij}} \right\} + \sum_{i,s,t,l} g_{istl} \left\{ \prod_{j=1}^m \rho_j^{\theta_{ij}} \right\} Q_l^{st} \end{aligned} \quad (4.6)$$

Note that in (4.6), f_i and g_{istl} are functions of the indices b, c , and k as well. But for the sake of notational simplicity, these three indices are omitted in the symbolic expressions of f_i and g_{istl} .

So the problem of interest becomes

$$\begin{aligned}
\nu_n &= \inf \gamma \\
&\text{subject to} \\
&\sum_{t=1}^{n_v} \sum_{k=0}^n [Q_l^{st,+} + Q_l^{st,-}] \leq \alpha, \quad \sum_{c=1}^{n_w} \sum_{k=0}^n [R_k^{bc,+} + R_k^{bc,-}] \leq \gamma \\
&Q_l^{st} = Q_l^{st,+} - Q_l^{st,-}, \quad R_k^{bc} = R_k^{bc,+} - R_k^{bc,-} \\
&(4.6) \\
&R_k^{bc,+} \geq 0, \quad R_k^{bc,-} \geq 0, \quad Q_l^{st,+} \geq 0, \quad Q_l^{st,-} \geq 0, \quad \underline{\rho} \leq \rho \leq \bar{\rho}.
\end{aligned} \tag{4.7}$$

where we have used a standard change of variables from linear programming (see for instance [3]) to reformulate the variables and constraints of ν_n . Specifically, the variable x is replaced by nonnegative variables x^+ and x^- such that $x = x^+ - x^-$. Then the ℓ_1 norm constraint $\|Q\|_1 \leq \alpha$ is replaced by the constraint $\sum_{t=1}^{n_v} \sum_{k=0}^n [Q_l^{st,+} + Q_l^{st,-}] \leq \alpha$, and $\|P_n R(Q, \rho)\|_1$, the objective function to be minimized, is replaced by introducing an auxiliary variable γ such that $\sum_{c=1}^{n_w} \sum_{k=0}^n [R_k^{bc,+} + R_k^{bc,-}] \leq \gamma$. It is also useful to mention that the optimal solution of problem (4.7) always satisfies that either $R_k^{bc,+}$ or $R_k^{bc,-}$ is zero.

To set the stage for the branch and bounding algorithm, we suppose that the rectangle-type set $[\underline{\rho}, \bar{\rho}]$ is partitioned into M subsets $[\underline{\rho}^r, \bar{\rho}^r] (r = 1, \dots, M)$ such that $[\underline{\rho}, \bar{\rho}] = \bigcup_{r=1}^M [\underline{\rho}^r, \bar{\rho}^r]$, where $\underline{\rho}^r = [\underline{\rho}_1^r \cdots \underline{\rho}_m^r]^T \in R^m$ and $\bar{\rho}^r = [\bar{\rho}_1^r \cdots \bar{\rho}_m^r]^T \in R^m$. Then a finer grid version of problem (4.7) is defined as:

$$\begin{aligned}
\nu_{n,r} &:= \inf \gamma \\
&\text{subject to} \\
&\sum_{t=1}^{n_v} \sum_{k=0}^n [Q_l^{st,+} + Q_l^{st,-}] \leq \alpha, \quad \sum_{c=1}^{n_w} \sum_{k=0}^n [R_k^{bc,+} + R_k^{bc,-}] \leq \gamma \\
&Q_l^{st} = Q_l^{st,+} - Q_l^{st,-}, \quad R_k^{bc} = R_k^{bc,+} - R_k^{bc,-} \\
&(4.6) \\
&R_k^{bc,+} \geq 0, \quad R_k^{bc,-} \geq 0, \quad Q_l^{st,+} \geq 0, \quad Q_l^{st,-} \geq 0, \quad \underline{\rho}^r \leq \rho \leq \bar{\rho}^r.
\end{aligned} \tag{4.8}$$

For notational convenience, we shall use the symbol Ψ to denote the set of $(\gamma, \rho, R_k^{bc}, R_k^{bc,+}, R_k^{bc,-}, Q_l^{st}, Q_l^{st,+}, Q_l^{st,-}) \in R^N$ ($N = 1 + m + 3n_z n_w (n+1) + 3n_u n_v (n+1)$) such that all the other constraints except the non-linear constraint (4.6) in problem (4.8) are satisfied. Thus problem $\nu_{n,r}$ is equivalently expressed as:

$$\begin{aligned} \nu_{n,r} &= \inf \gamma \\ &\text{subject to} \\ (4.6), & (\gamma, \rho, R_k^{bc}, R_k^{bc,+}, R_k^{bc,-}, Q_l^{st}, Q_l^{st,+}, Q_l^{st,-}) \in \Psi. \end{aligned} \quad (4.9)$$

To prepare for the linear relaxation scheme introduced in the next section, let us further introduce the following set of variables:

$$\lambda_{istl} := \tau_i Q_l^{st} \quad (4.10)$$

and it follows from (4.6) that

$$R_k^{bc}(Q, \rho) = \sum_i f_i \tau_i + \sum_{i,s,t,l} g_{istl} \lambda_{istl}. \quad (4.11)$$

So problem (4.9) becomes

$$\begin{aligned} \nu_n &:= \inf \gamma \\ &\text{subject to} \\ (4.5), & (4.10), (4.11), (\gamma, \rho, R_k^{bc}, R_k^{bc,+}, R_k^{bc,-}, Q_l^{st}, Q_l^{st,+}, Q_l^{st,-}) \in \Psi. \end{aligned} \quad (4.12)$$

Clearly that problem (4.12) is a non-linear optimization problem and hard to solve in general.

4.4 Problem Solution

From the formulation of problem (4.12), we can infer that the crux of solving this problem is how to deal with those non-convex product terms present in (4.5) and (4.10). For this purpose, we introduce the following result from [50]:

Lemma 4.4.1 *If the variables $x_j \in R$ satisfy the conditions $l_j \leq x_j \leq u_j$ and $t_{ij} := x_i x_j$, then*

$$\begin{aligned} t_{ij} &\geq u_j x_i + u_i x_j - u_i u_j \\ t_{ij} &\leq l_j x_i + u_i x_j - u_i l_j \\ t_{ij} &\leq u_j x_i + l_i x_j - l_i u_j \\ t_{ij} &\geq l_j x_i + l_i x_j - l_i l_j. \end{aligned} \tag{4.13}$$

Furthermore, if variables $t_{ij} \in R$ satisfy (4.13) and x_k satisfy $l_k \leq x_k \leq u_k$, then

$$|t_{ij} - x_i x_j| \leq \frac{1}{4}(u_i - l_i)(u_j - l_j).$$

Following (4.5) and (4.10), define

$$\begin{aligned} \Omega_{ijl} &:= \{(\tau_{i_l}, \tau_{i_{l+1}}, \rho_{j_{l+1}}) \in R^3 \mid \text{Inequalities in (4.13) are satisfied with} \\ &\quad (t_{ij}, x_i, x_j, u_i, l_i, u_j, l_j) \text{ replaced by } (\tau_{i_l}, \tau_{i_{l+1}}, \rho_{j_{l+1}}, \bar{\tau}_{i_{l+1}}, \underline{\tau}_{i_{l+1}}, \bar{\rho}_{j_{l+1}}, \underline{\rho}_{j_{l+1}}^r)\}. \\ \Lambda_{istl} &:= \{(\lambda_{istl}, \tau_i, Q_l^{st}) \in R^3 \mid \text{Inequalities in (4.13) are satisfied with} \\ &\quad (t_{ij}, x_i, x_j, u_i, l_i, u_j, l_j) \text{ replaced by } (\lambda_{istl}, \tau_i, Q_l^{st}, \bar{\tau}_i, \underline{\tau}_i, \alpha, -\alpha)\}. \end{aligned}$$

where $\bar{\tau}_i$ and $\underline{\tau}_i$ are upper and lower bounds for τ_i and they can be a priori computed.

Hence from (4.12) and Lemma 4.4.1, we have

$$\begin{aligned} \nu_{n,r} &= \inf \gamma \\ &\text{subject to} \\ &(\tau_{i_l}, \tau_{i_{l+1}}, \rho_{j_{l+1}}) \in \Omega_{ijl}, (\lambda_{istl}, \tau_i, Q_l^{st}) \in \Lambda_{istl} \\ &(4.5), (4.10), (4.11), (\gamma, \rho, R_k^{bc}, R_k^{bc,+}, R_k^{bc,-}, Q_l^{st}, Q_l^{st,+}, Q_l^{st,-}) \in \Psi. \end{aligned}$$

Removing the the nonlinear constraints (4.5) and (4.10), we have the following relaxed linear programming problem:

$$\begin{aligned} \nu_{n,r}^R &= \inf \gamma \\ &\text{subject to} \\ &(\tau_{i_l}, \tau_{i_{l+1}}, \rho_{j_{l+1}}) \in \Omega_{ijl}, (\lambda_{istl}, \tau_i, Q_l^{st}) \in \Lambda_{istl} \\ &(4.11), (\gamma, \rho, R_k^{bc}, R_k^{bc,+}, R_k^{bc,-}, Q_l^{st}, Q_l^{st,+}, Q_l^{st,-}) \in \Psi. \end{aligned}$$

It is clear that if the relaxed problem $\nu_{n,r}^R$ is infeasible, then so is the problem $\nu_{n,r}$. If $\nu_{n,r}^R$ is a finite real number, then $\nu_{n,r}^R \leq \nu_{n,r}$. Now we are ready to prove the main result of the paper.

Theorem 4.4.1 *Suppose an optimal solution of the relaxed problem $\nu_{n,r}^R$ is given by:*

$$(\widehat{\gamma}, \widehat{\rho}, \widehat{\tau}_i, \widehat{\lambda}_{istl}, \widehat{R}_k^{bc}, \widehat{R}_k^{bc,+}, \widehat{R}_k^{bc,-}, \widehat{Q}_l^{st}, \widehat{Q}_l^{st,+}, \widehat{Q}_l^{st,-}).$$

Then there exists a feasible solution

$$(\gamma_{feas}, \widehat{\rho}, R_k^{bc}, R_k^{bc,+}, R_k^{bc,-}, \widehat{Q}_l^{st}, \widehat{Q}_l^{st,+}, \widehat{Q}_l^{st,-}).$$

for problem $\nu_{n,r}$ (as defined in (4.9)) such that

$$\nu_{n,r}^R = \widehat{\gamma} \leq \nu_{n,r} \leq \gamma_{feas} \quad (4.14)$$

$$\gamma_{feas} - \widehat{\gamma} \leq C|\overline{\rho}^r - \underline{\rho}^r|_\infty \quad (4.15)$$

where C is a finite positive constant and $|\overline{\rho}^r - \underline{\rho}^r|_\infty = \max\{|\overline{\rho}_i^r - \underline{\rho}_i^r| : i = 1, \dots, m\}$.

Proof: Following the definition of $\nu_{n,r}$ in (4.9), we construct R_k^{bc} as defined in (4.6)

$$R_k^{bc} = \sum_i f_i \left\{ \prod_{j=1}^m \widehat{\rho}_j^{\theta_{ij}} \right\} + \sum_{i,s,t,l} g_{istl} \left\{ \prod_{j=1}^m \widehat{\rho}_j^{\theta_{ij}} \right\} \widehat{Q}_l^{st}.$$

Define

$$\gamma_{feas} := \max_b \left\{ \sum_{c=1}^{n_w} \sum_{k=0}^n [R_k^{bc,+} + R_k^{bc,-}] \right\},$$

where $R_k^{bc,+} := \max\{R_k^{bc}, 0\}$ and $R_k^{bc,-} := -\min\{R_k^{bc}, 0\}$. Then it is clear that

$$(\gamma_{feas}, \widehat{\rho}, R_k^{bc}, R_k^{bc,+}, R_k^{bc,-}, \widehat{Q}_l^{st}, \widehat{Q}_l^{st,+}, \widehat{Q}_l^{st,-})$$

is feasible for problem (4.9) and so (4.14) is established.

To show (4.15), it is useful to observe that from the definition of $\nu_{n,r}^R$ the following linear constraints hold:

$$\begin{aligned} (\widehat{\tau}_{i_l}, \widehat{\tau}_{i_{l+1}}, \widehat{\rho}_{j_{l+1}}) &\in \Omega_{ijl}, \quad (\widehat{\lambda}_{istl}, \widehat{\tau}_i, \widehat{Q}_l^{st}) \in \Lambda_{istl} \\ \widehat{R}_k^{bc} &= \sum_i f_i \widehat{\tau}_i + \sum_{i,s,t,l} g_{istl} \widehat{\lambda}_{istl}. \end{aligned} \quad (4.16)$$

Furthermore, from (4.4) and (4.5), we have $\prod_{j=1}^m \widehat{\rho_j^{\theta_{ij}}} = \prod_{l=1}^{d_i} \widehat{\rho_{j_l}}$. This, together with (4.16) and Lemma 4.4.1, implies that there exist real constant C_i and C'_i such that

$$\begin{aligned}
|\widehat{\tau_i} - \prod_{j=1}^m \widehat{\rho_j^{\theta_{ij}}}| &= |\widehat{\tau_{i_0}} - \prod_{l=1}^{d_i} \widehat{\rho_{j_l}}| \\
&\leq |\widehat{\tau_{i_0}} - \widehat{\tau_{i_1}} \widehat{\rho_{j_1}}| + |\widehat{\rho_{j_1}}| |\widehat{\tau_{i_1}} - \widehat{\tau_{i_2}} \widehat{\rho_{j_2}}| + \cdots + |\prod_{l=1}^{d_i-2} \widehat{\rho_{j_l}}| |\widehat{\tau_{i_{d_i-2}}} - \widehat{\rho_{j_{d_i}}} \widehat{\rho_{j_{d_i-1}}}| \\
&\leq (\frac{1}{4}) |\widehat{\tau_{i_1}} - \underline{\tau_{i_1}}| |\overline{\rho_{j_1}^r} - \underline{\rho_{j_1}^r}| + (\frac{1}{4}) |\widehat{\rho_{j_1}}| |\widehat{\tau_{i_2}} - \underline{\tau_{i_2}}| |\overline{\rho_{j_2}^r} - \underline{\rho_{j_2}^r}| \\
&\quad + \cdots + (\frac{1}{4}) |\prod_{l=1}^{d_i-2} \widehat{\rho_{j_l}}| |\overline{\rho_{j_{d_i}}^r} - \underline{\rho_{j_{d_i}}^r}| |\overline{\rho_{j_{d_i-1}}^r} - \underline{\rho_{j_{d_i-1}}^r}| \\
&\leq C_i |\overline{\rho^r} - \underline{\rho^r}|_\infty,
\end{aligned}$$

and

$$|\widehat{\tau_i} - \underline{\tau_i}| \leq |\overline{\rho_{j_{d_i}}^r} - \underline{\rho_{j_{d_i}}^r}| |\Phi(\overline{\rho_{j_l}^r}, \underline{\rho_{j_l}^r})| \leq C'_i |\overline{\rho^r} - \underline{\rho^r}|_\infty,$$

where $\Phi(\cdot)$ is a $(d_i - 1)$ -degree polynomial of $\overline{\rho_{j_l}^r}$ and $\underline{\rho_{j_l}^r}$. Moreover, from (4.10), (4.16), and Lemma 4.4.1, we have

$$\begin{aligned}
|\widehat{\lambda_{istl}} - \{\prod_{j=1}^m \widehat{\rho_j^{\theta_{ij}}}\} \widehat{Q_l^{st}}| &\leq |\widehat{\lambda_{istl}} - \widehat{\tau_i} \widehat{Q_l^{st}}| + |\widehat{Q_l^{st}}| |\widehat{\tau_i} - \prod_{j=1}^m \widehat{\rho_j^{\theta_{ij}}}| \\
&\leq (\frac{1}{4}) 2\alpha C'_i |\overline{\rho^r} - \underline{\rho^r}|_\infty + \alpha C_i |\overline{\rho^r} - \underline{\rho^r}|_\infty \\
&=: C_{\lambda_{istl}} |\overline{\rho^r} - \underline{\rho^r}|_\infty
\end{aligned}$$

Thus it follows that

$$\begin{aligned}
&|[\widehat{R_k^{bc,+}} + \widehat{R_k^{bc,-}}] - [R_k^{bc,+} + R_k^{bc,-}]| \\
&= ||\widehat{R_k^{bc}}| - |R_k^{bc}|| \\
&\leq |\widehat{R_k^{bc}} - R_k^{bc}| \\
&\leq \sum_i |f_i| |\widehat{\tau_i} - \{\prod_{j=1}^m \widehat{\rho_j^{\theta_{ij}}}\}| + \sum_{i,s,t,l} |g_{istl}| |\widehat{\lambda_{istl}} - \{\prod_{j=1}^m \widehat{\rho_j^{\theta_{ij}}}\} \widehat{Q_l^{st}}| \\
&\leq \sum_i |f_i| C_i |\overline{\rho^r} - \underline{\rho^r}|_\infty + \sum_{i,s,t,l} |g_{istl}| C_{\lambda_{istl}} |\overline{\rho^r} - \underline{\rho^r}|_\infty \\
&=: C_{R_k^{bc}} |\overline{\rho^r} - \underline{\rho^r}|_\infty.
\end{aligned}$$

Define

$$C := \max_b \{ \sum_{c=1}^{n_w} \sum_{k=0}^n C_{R_k^{bc}} \}$$

Then we have

$$\begin{aligned}
\gamma_{feas} - \hat{\gamma} &= \max_b \left\{ \sum_{c=1}^{n_w} \sum_{k=0}^n [R_k^{bc,+} + R_k^{bc,-}] \right\} - \max_b \left\{ \sum_{c=1}^{n_w} \sum_{k=0}^n [\widehat{R_k^{bc,+}} + \widehat{R_k^{bc,-}}] \right\} \\
&\leq \max_b \left\{ \sum_{c=1}^{n_w} \sum_{k=0}^n |[R_k^{bc,+} + R_k^{bc,-}] - [\widehat{R_k^{bc,+}} + \widehat{R_k^{bc,-}}]| \right\} \\
&\leq C |\bar{\rho}^r - \underline{\rho}^r|_\infty,
\end{aligned}$$

which completes the proof. ■

Having established the relationship between $\hat{\gamma}$ and γ_{feas} , we can compute the optimal solution of ν_n within any prescribed tolerance $\epsilon > 0$ as shown in what follows. First, it is clear that to compute a cost with an ϵ tolerance, the number of problems to be solved is no larger than the order of $1/\epsilon^m$. Moreover, if the lower bound $\nu_{n,r}$ for any given sub-grid $[\underline{\rho}_r, \bar{\rho}_r]$ is greater than any upper bound on any other region, then we can infer that the global optimality must be achieved outside of $[\underline{\rho}_r, \bar{\rho}_r]$. This can then be combined with certain branch and bound algorithm to compute a global optimal solution. More explicitly, so long as the lower bound obtained on a sub-region of the parameter space is smaller than the best available global upper bound, we can further prune this region. This algorithm is guaranteed to converge and yield a global optimal value up to the given tolerance ϵ (see [52] for more details). Hence, for a fixed tapping length n of Q , if $\nu^n - \nu_n$ is less than the prescribed tolerance, we can stop the iteration and recover a globally optimal controller for problem ν from the corresponding optimizing variables. Otherwise we can increase n until the desired performance is achieved.

If the performance measure used in problem (4.1) is $\mathcal{H}_2$ norm instead of ℓ_1 norm, then the exactly same procedure as above would enable us to arrive at the same conclusion of Theorem 4.4.1 by additionally observing the fact that

$$|[R_k^{bc,+} + R_k^{bc,-}]^2 - [\widehat{R_k^{bc,+}} + \widehat{R_k^{bc,-}}]^2| \leq C_{\mathcal{H}_2} |[R_k^{bc,+} + R_k^{bc,-}] - [\widehat{R_k^{bc,+}} + \widehat{R_k^{bc,-}}]|$$

where $C_{\mathcal{H}_2}$ is a finite constant that can be computed a priori from the known parameters.

4.5 Summary

In this chapter, we have presented a global optimal solution to the IPC problem. The solutions are obtained by solving linear/quadratic programming problems.

CHAPTER 5 IPC DESIGN WITH RATIONAL DEPENDENT PARAMETERS

In this chapter, we consider the integrated parameter and control (IPC) optimization problem where the system structure parameters enter the state-space representation of the system in a rational manner. Converging finite-dimensional sub-optimal problems are constructed and solved via a linear relaxation technique, whereby a global optimal solution to the IPC problem is computed within any given performance tolerance. A numerical example is presented to illustrate effectiveness of the proposed methodology.

Throughout the chapter, unless mentioned explicitly, the superscript of a variable denotes the power of that variable, the time instant index of a variable is put inside braces, and all the other indices appear as subscripts.

Suppose $\rho = [\rho_1 \ \cdots \ \rho_m]^T \in \mathcal{R}^m$ denotes an m -dimensional parameter vector. We use the notation $\underline{\rho} \leq \rho \leq \bar{\rho}$ or $\rho \in [\underline{\rho}, \bar{\rho}]$ to denote the set of inequality relations $\{\underline{\rho}_i \leq \rho_i \leq \bar{\rho}_i, i = 1, \dots, m\}$, where $\underline{\rho} = [\underline{\rho}_1 \ \cdots \ \underline{\rho}_m]^T$ and $\bar{\rho} = [\bar{\rho}_1 \ \cdots \ \bar{\rho}_m]^T$ are any two given m -dimensional real vectors.

5.1 Problem Setup

Consider the setup shown in Figure 5.1, where $G : [w; u] \rightarrow [z; v]$ is the generalized linear time-invariant plant, K is the controller to be designed. w , z , u , and v denote the exogenous input, regulated output, control input, and measured output of dimensions n_w , n_z , n_u , and n_v , respectively.

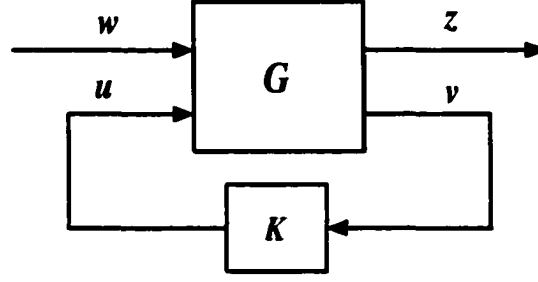


Figure 5.1 Closed-loop system

Suppose the generalized plant G admits the following state-space realization:

$$G := \left[\begin{array}{c|cc} A(\rho) & B_1(\rho) & B_2(\rho) \\ \hline C_1(\rho) & D_{11}(\rho) & D_{12}(\rho) \\ C_2(\rho) & D_{21}(\rho) & D_{22}(\rho) \end{array} \right]$$

where $\rho = [\rho_1 \ \cdots \ \rho_m]^T \in \mathcal{R}^m$ is a bounded parameter vector of interest such that $\underline{\rho} \leq \rho \leq \bar{\rho}$ for prescribed lower and upper bounds $\underline{\rho}$ and $\bar{\rho}$. Possible candidates of ρ include the mass, stiffness, damping coefficients of a mechanical system, the coefficients of the weighting functions for the robust control problem, and the sampling time of a sampled-data digital control system.

To ease the notation, we stack all the entries of the generalized plant $G(\rho)$ into a single vector variable g such that

$$g := [g_1 \ g_2 \ \cdots \ g_{\bar{m}}]^T \in \mathcal{R}^{\bar{m}}$$

where $\bar{m} = (n_w + n_u) \cdot (n_z + n_v)$. Each entry g_i is assumed to be a rational function of ρ of the form:

$$g_i = \frac{\sum_k e_{ik} \Theta_{ik}}{\sum_l f_{il} \Upsilon_{il}} \quad (5.1)$$

where

$$\begin{aligned} \Theta_{ik} &:= \prod_{j=1}^m \rho_j^{\pi_{ikj}}, \quad \sum_{j=1}^m \pi_{ikj} = \phi_{ik} \in \{0, 1, \dots, \varphi\}, \\ \Upsilon_{il} &:= \prod_{j=1}^m \rho_j^{\sigma_{ilj}}, \quad \sum_{j=1}^m \sigma_{ilj} = \vartheta_{il} \in \{0, 1, \dots, \varphi\}, \end{aligned}$$

e_{ik} is the coefficient of the ϕ_{ik} -degree monomial Θ_{ik} in the numerator of g_i , and f_{il} is the coefficient of the ϑ_{il} -degree monomial Υ_{il} in the denominator of g_i . Without loss of generality, we assume that there exists a positive constant β such that all the denominators of g_i 's are bounded as follows:

$$|\sum_l f_{il} \Upsilon_{il}| \geq \beta > 0, \forall \rho \in [\underline{\rho}, \bar{\rho}], \forall i \in \{1, 2, \dots, \bar{m}\}.$$

Moreover, as a necessary condition for the existence of stabilizing controllers, we assume that $(A(\rho), B_2(\rho))$ is stabilizable and $(A(\rho), C_2(\rho))$ is detectable for any ρ in the m -dimensional hyperrectangle $[\underline{\rho}, \bar{\rho}]$. The procedure of how to verify these two assumptions will be clarified in the next section.

Problem Statement

The integrated parameter and control synthesis problem considered in this paper is to compute a global optimal solution to optimization problems of the following form:

$$\begin{aligned} \nu := & \inf_{\Phi, Q, \rho} f_{obj}(\Phi, Q, \rho) \\ \text{subject to } & \|Q\|_1 \leq \gamma \\ & \underline{\rho} \leq \rho \leq \bar{\rho} \\ & \Phi = f_{con}(Q, \rho) \end{aligned} \tag{5.2}$$

where f_{obj} is a rational function of the vector (Φ, Q, ρ) , f_{con} is a rational function of the vector (Q, ρ) , γ , $\underline{\rho}$ and $\bar{\rho}$ are constants of appropriate dimensions. Here the vectors Φ , Q , and ρ , are assumed to be finite dimensional.

The formulation defined in (5.2) incorporates the finite dimensional approximations of several important IPC synthesis problems as special cases. Explicitly, we shall show that the finite dimensional approximations of ℓ_1 and $\mathcal{H}_2$ IPC design problems can be formulated into the form of problem (5.2). Moreover, the finite dimensional approximations of the robust ℓ_1 IPC synthesis problem can be also posed into optimization problems of the form of (5.2).

ℓ_1 IPC design problem

The ℓ_1 IPC design problem is formulated as follows:

$$\begin{aligned}
\nu &:= \inf_{\Phi, Q, \rho} \|\Phi\|_1 \\
\text{s. t. } &\|Q\|_1 \leq \gamma \\
&\underline{\rho} \leq \rho \leq \bar{\rho} \\
&\Phi = H(\rho) - U(\rho) * Q * V(\rho)
\end{aligned} \tag{5.3}$$

where $H \in \ell_1^{n_z \times n_w}$, $U \in \ell_1^{n_z \times n_u}$, $V \in \ell_1^{n_v \times n_w}$, Q is a free parameter in $\ell_1^{n_u \times n_v}$, and $'*'$ denotes the convolution operation. The stable operators H , U , and V are obtained using the well-known Youla parametrization ([3]). Problem (5.3) is an infinite dimensional non-convex optimization problem and for each fixed parameter vector ρ , problem (5.3) becomes an ℓ_1 control design problem.

The polynomial version of problem (5.3) is solved in [48], where the parameter vector ρ is assumed to enter into the system state-space in a polynomial manner. Here we consider the more general rational case. That is, each entry g_i of the generalized plant is a rational function of ρ as defined in Equation (5.1). In the sequel, without loss of generality, we assume that H , U , and V are finitely supported. If $\hat{U}$ and $\hat{V}$ were rational matrices in λ , doubly-coprime factorizations can be performed on $\hat{U}$ and $\hat{V}$ and the resulting right and left coprime factors of $\hat{U}$ and $\hat{V}$ can be readily incorporated into $\hat{Q}$ ([3]). This assumption on the finite supportedness of H is justified by the fact that H is an operator in the $\ell_1^{n_z \times n_w}$ space.

In this case, finite dimensional lower and upper bound problems of ν are given by:

$$\begin{aligned}
\nu_n &:= \inf_{\Phi, Q, \rho} \|P_n \Phi\|_1 \\
\text{s. t. } &\|Q\|_1 \leq \gamma \\
&\underline{\rho} \leq \rho \leq \bar{\rho} \\
&\Phi = H(\rho) - U(\rho) * Q * V(\rho)
\end{aligned} \tag{5.4}$$

$$\begin{aligned}
\nu^n &:= \inf_{\Phi, Q, \rho} \|\Phi\|_1 \\
\text{s. t. } &\|Q\|_1 \leq \gamma \\
&\underline{\rho} \leq \rho \leq \bar{\rho} \\
&\Phi = H(\rho) - U(\rho) * Q * V(\rho) \\
&Q(k) = 0 \text{ if } k > n.
\end{aligned}$$

Following the same argument as in [52], it can be shown that the non-convex optimization problems ν_n and ν^n monotonically converge to ν from below and above as n goes to infinity. Clearly, they admit the form of problem (5.2).

$\mathcal{H}_2$ IPC design problem

The $\mathcal{H}_2$ IPC design problem is formulated as follows:

$$\begin{aligned}
\nu &:= \inf_{\Phi, Q, \rho} \|\Phi\|_2^2 \\
\text{s. t. } &\|Q\|_1 \leq \gamma \\
&\underline{\rho} \leq \rho \leq \bar{\rho} \\
&\Phi = H(\rho) - U(\rho) * Q * V(\rho)
\end{aligned}$$

where we follow the exactly same notations and assumption as made in the definition of the ℓ_1 IPC design problem.

Two convergent sequences of lower and upper bounds of ν are given by:

$$\begin{aligned}
\nu_n &:= \inf_{\Phi, Q, \rho} \|P_n \Phi\|_2^2 \\
\text{s. t. } &\|Q\|_1 \leq \gamma \\
&\underline{\rho} \leq \rho \leq \bar{\rho} \\
&\Phi = H(\rho) - U(\rho) * Q * V(\rho)
\end{aligned}$$

$$\begin{aligned}
\nu^n &:= \inf_{\Phi, Q, \rho} \|\Phi\|_2^2 \\
\text{s. t. } &\|Q\|_1 \leq \gamma \\
&\underline{\rho} \leq \rho \leq \bar{\rho} \\
&\Phi = H(\rho) - U(\rho) * Q * V(\rho) \\
&Q(k) = 0 \text{ if } k > n.
\end{aligned}$$

These non-convex optimization problems are finite dimensional and they admit the form of problem (5.2).

Robust ℓ_1 IPC design problem

The Robust ℓ_1 IPC design problem is formulated as:

$$\begin{aligned}
\nu &:= \inf_{\Phi, Q, L, \rho} \|L^{-1}\Phi L\|_1 \\
\text{s. t. } &\|Q\|_1 \leq \gamma \\
&\underline{\rho} \leq \rho \leq \bar{\rho} \\
&L \in \mathcal{L} \\
&\Phi = H(\rho) - U(\rho) * Q * V(\rho)
\end{aligned} \tag{5.5}$$

where $\mathcal{L} := \{\text{diag}(\ell_1, \dots, \ell_{n_x}) \mid \ell_i > 0\}$ and n_x is a positive integer. Note that for each fixed scaling matrix $L = \text{diag}(\ell_1, \dots, \ell_{n_x})$, problem (5.5) is a standard ℓ_1 IPC design problem. And for each fixed parameter vector ρ , (5.5) becomes the ℓ_1 robust performance problem([52]).

Following the similar argument as in [52], it can be shown that problem (5.5) is equivalent to an infinite dimensional optimization problem of the following form:

$$\begin{aligned}
\nu &= \inf_{\Phi, Q, L, \rho} \|L^{-1}\Phi L\|_1 \\
\text{subject to } &\|Q\| \leq \alpha \\
&\Phi = H(\rho) - U(\rho) * Q * V(\rho) \\
&\underline{\rho} \leq \rho \leq \bar{\rho}, \underline{\ell} \leq \ell \leq \bar{\ell}
\end{aligned}$$

where $\underline{\ell}$ and $\bar{\ell}$ are vectors in $\mathcal{R}^{n_x}$ such that $0 < \underline{\ell} \leq \bar{\ell}$. Two sequences of lower and upper bounds of ν are given by:

$$\begin{aligned}
\nu_n &:= \inf_{\Phi, Q, L, \rho} \|L^{-1}(P_n \Phi)L\|_1 \\
&\text{s. t. } \|Q\|_1 \leq \alpha \\
&\quad \underline{\rho} \leq \rho \leq \bar{\rho}, \quad \underline{\ell} \leq \ell \leq \bar{\ell} \\
&\quad \Phi = H(\rho) - U(\rho) * Q * V(\rho)
\end{aligned}$$

$$\begin{aligned}
\nu^n &:= \inf_{\Phi, Q, L, \rho} \|L^{-1}R(Q, \rho)L\|_1 \\
&\text{s. t. } \|Q\|_1 \leq \alpha \\
&\quad \underline{\rho} \leq \rho \leq \bar{\rho}, \quad \underline{\ell} \leq \ell \leq \bar{\ell} \\
&\quad \Phi = H(\rho) - U(\rho) * Q * V(\rho) \\
&\quad Q(k) = 0 \text{ if } k > n.
\end{aligned}$$

Using the same argument as in [52], it can be shown that finite dimensional optimization problems ν_n and ν^n monotonically converge to ν from below and above as n goes to infinity, and that they are also in the form of problem (5.2).

In what follows, we shall demonstrate how to solve the finite dimensional non-convex optimization problem of the form (5.2). For the ease of exposition, we shall carry out the development based solely on the formulation given in (5.4) while the exact same technique applies the other cases that fall into the general setup defined in (5.2).

As a concluding remark for this section, it should be mentioned that following the same framework developed here, the $\ell_1/\mathcal{H}_2$ multiobjective IPC design problem can also be defined, where the objective function is composed as the nonnegative linear combination of ℓ_1 and $\mathcal{H}_2$ norms of the closed-loop system. And the corresponding convergent finite dimensional approximation problems can be formulated in a straightforward manner and shown to admit the same form as problem (5.2).

5.2 Reformulation

In this section, we show that, by introducing several sets of auxiliary variables, the non-convex optimization problem to be solved can be reformulated as an optimizing problem with linear and non-linear constraints, where the non-linear constraints are of the type $x = yz$ for variables x , y , and z .

Verification of the stabilization and detectability assumptions

Let the following be the state-space representations([51]) of stable operators H , U , and V in (5.4):

$$\begin{aligned} H_{ss} &= \left[\begin{array}{c|c} A_H & B_H \\ \hline C_H & D_H \end{array} \right] = \left[\begin{array}{cc|c} A + B_2F & -B_2F & B_1 \\ 0 & A + LC_2 & B_1 + LD_{21} \\ \hline C_1 + D_{12}F & -D_{12}F & D_{11} \end{array} \right] \\ U_{ss} &= \left[\begin{array}{c|c} A_U & B_U \\ \hline C_U & D_U \end{array} \right] = \left[\begin{array}{c|c} A + B_2F & -B_2 \\ \hline C_1 + D_{12}F & -D_{12} \end{array} \right] \\ V_{ss} &= \left[\begin{array}{c|c} A_V & B_V \\ \hline C_V & D_V \end{array} \right] = \left[\begin{array}{c|c} A + LC_2 & B_1 + LD_{21} \\ \hline C_2 & D_{21} \end{array} \right] \end{aligned}$$

where F and L denote the feedback and observer gains that stabilize the system for any given $\rho \in [\underline{\rho}, \bar{\rho}]$. Note that if $A(\rho)$ is assumed to be stable for any feasible parameter vector ρ , then the zero controller ($F = 0$, $L = 0$) are to be chosen in the above realizations.

As discussed above, the gain matrices F and L vary as the parameter vector ρ changes. In what follows, we show that, given the stabilization and detectability assumptions on $(A(\rho), B_2(\rho), C_2(\rho))$, there necessarily exist a finite number (say, M) of subsets $[\underline{\rho}_r, \bar{\rho}_r]$ of $\mathcal{R}^m$, and corresponding gain matrices F_r and L_r such that $[\underline{\rho}, \bar{\rho}] \subset \bigcup_{r=1}^M [\underline{\rho}_r, \bar{\rho}_r]$ and $A(\rho) + B_2(\rho)F_r$ and $A(\rho) + L_rC_2(\rho)$ are stable for all $\rho \in [\underline{\rho}_r, \bar{\rho}_r]$.

In the sequel, denote $\underline{\rho}_r = [\underline{\rho}_{r1} \cdots \underline{\rho}_{rm}]$ and $\bar{\rho}_r = [\bar{\rho}_{r1} \cdots \bar{\rho}_{rm}]$. For the given parameter vector $\underline{\rho}$, a pair of feedback gain F_1 and observer gain L_1 can be computed such that $A(\underline{\rho}) + B_2(\underline{\rho})F_1$ and $A(\underline{\rho}) + L_1C_2(\underline{\rho})$ are stable. By solving a robust analysis problem with respect to the parameter ρ at the nominal point $\underline{\rho}$ (see [56]), we can obtain a positive constant c_1 such that $A(\rho) + B_2(\rho)F_1$ and $A(\rho) + L_1C_2(\rho)$ are guaranteed to be stable for any $\rho \in [\underline{\rho} - c_1, \underline{\rho} + c_1] := \{\rho \in \mathcal{R}^m \mid \underline{\rho}_i - c_1 \leq \rho_i \leq \underline{\rho}_i + c_1, i = 1, \dots, m\}$. Let $\underline{\rho}_1 = \underline{\rho}$ and $\bar{\rho}_1 = \underline{\rho} + c_1$. If $c_1 \geq \|\bar{\rho} - \underline{\rho}\|_\infty$, we are done. Otherwise, define $\underline{\rho}_2 := \{\underline{\rho}_{21} = \bar{\rho}_{11} \text{ and } \underline{\rho}_{2i} = \underline{\rho}_{1i}, i = 2, \dots, m\}$. Following the same argument as above, we can find a pair of gain matrices F_2 and L_2 and a positive radius c_2 such that $A(\rho) + B_2(\rho)F_2$ and $A(\rho) + L_2C_2(\rho)$ are stable for any $\rho \in [\underline{\rho}_2, \underline{\rho}_2 + c_2]$. Let $\bar{\rho}_2 = \underline{\rho}_2 + c_2$. Continue the above iteration and the compactness of the set $[\underline{\rho}, \bar{\rho}]$ implies that, after a finite number (say, M) of steps, the set $[\underline{\rho}, \bar{\rho}]$ will be covered by the union of all the sets $[\underline{\rho}_r, \bar{\rho}_r]$.

Therefore, the problem ν_n defined in (5.4) can be restated as:

$$\begin{aligned} \nu_n := & \min_{1 \leq r \leq M} \inf_{\Phi, Q, \rho} \|P_n \Phi\|_1 \\ \text{s. t. } & \|Q\|_1 \leq \gamma \\ & \underline{\rho}_r \leq \rho \leq \bar{\rho}_r \\ & \Phi = H(\rho) - U(\rho) * Q * V(\rho). \end{aligned}$$

Without loss of generality, we can assume $M = 1$ and so the problem ν_n would still admit the same formulation as defined in (5.4).

Let

$$1, \rho_1, \dots, \rho_m, \rho_1^2, \rho_1\rho_2, \dots, \rho_1^\varphi, \dots, \rho_m^\varphi$$

be a basis for all the polynomials of elements of ρ up to φ -degree and let d be its dimension. Define

$$\Pi = [1 \ \rho_1 \ \dots \ \rho_m \ \rho_1^2 \ \rho_1\rho_2 \ \dots \ \rho_1^\varphi \ \dots \ \rho_m^\varphi]^T = [\eta_1 \ \eta_2 \ \dots \ \eta_d]^T.$$

where η_i 's is a set of new variables to be used in the reformulation of the problem ν_n .

Each element η_i of Π is a d_i -degree monomial of the form

$$\eta_i = \prod_{j=1}^m \rho_j^{\theta_{ij}}, \quad 0 \leq \theta_{ij} \leq d_i, \quad \sum_{j=1}^m \theta_{ij} = d_i \leq \varphi. \quad (5.6)$$

Moreover, there exist indices $i_l \in \{1, 2, \dots, d\}$ ($l = 0, \dots, d_i$) and $j_l \in \{1, 2, \dots, m\}$ ($l = 1, \dots, d_i$) such that η_i is equivalently characterized by the following set of equations:

$$\begin{aligned} \eta_i &= \eta_{i_0} &= \eta_{i_1} \rho_{j_1} \\ & & \vdots \\ \eta_{i_{d_i-1}} &= \eta_{i_{d_i}} \rho_{j_{d_i}} \\ \eta_{i_{d_i}} &= 1. \end{aligned} \quad (5.7)$$

For example, suppose $m = 2$ and $\varphi = 2$. Then

$$\Pi = [1 \ \rho_1 \ \rho_2 \ \rho_1^2 \ \rho_1 \rho_2 \ \rho_2^2]^T = [\eta_1 \ \eta_2 \ \eta_3 \ \eta_4 \ \eta_5 \ \eta_6]^T.$$

Hence, η_2 can be characterized by

$$\begin{aligned} \eta_2 &= \eta_1 \cdot \rho_1 & [i_0 = 2, j_1 = 1] \\ \eta_1 &= 1 & [i_1 = 1] \end{aligned}$$

and η_4 can be expressed as

$$\begin{aligned} \eta_4 &= \eta_2 \cdot \rho_1 & [i_0 = 4, j_1 = 1] \\ \eta_2 &= \eta_1 \cdot \rho_1 & [i_1 = 2, j_2 = 1] \\ \eta_1 &= 1 & [i_2 = 1]. \end{aligned}$$

Other entries of Π can be characterized in a similar way.

Denote the denominator of g_i by

$$\omega_i := \frac{1}{\sum_l f_{il} \prod_{j=1}^m \rho_j^{\sigma_{ilj}}}$$

and so

$$\begin{aligned} 1 &= \omega_i \sum_l f_{il} \left\{ \prod_{j=1}^m \rho_j^{\sigma_{ilj}} \right\} \\ g_i &= \omega_i \sum_k e_{ik} \left\{ \prod_{j=1}^m \rho_j^{\pi_{ikj}} \right\} \end{aligned}$$

It follows that there exist constant coefficients u_{ik} and v_{ik} such that

$$1 = \omega_i \sum_k u_{ik} \eta_k = \omega_i \sum_{k=1}^d u_{ik} \left\{ \prod_{j=1}^m \rho_j^{\theta_{kj}} \right\} \quad (5.8)$$

$$g_i = \omega_i \sum_k v_{ik} \eta_k = \omega_i \sum_{k=1}^d v_{ik} \left\{ \prod_{j=1}^m \rho_j^{\theta_{kj}} \right\}. \quad (5.9)$$

New characterization of Φ

By the definition of the impulse response for discrete-time systems, we infer from the state-space representations of H , U , and V that any entry $H_{ij}(k)$ of the impulse response matrix sequence H is a polynomial function of the vector variable g , and so are $U_{ij}(k)$ and $V_{ij}(k)$. It is easy to see from the definition of ν_n that only the parameters of $\Phi_{bc}(k), \dots, \Phi_{bc}(k), \dots, \Phi_{bc}(k)$ are involved in the optimization of ν_n . Moreover, based on the definition of the convolution operation, it is clear that $\Phi_{bc}(k)$ ($k = 0, 1, \dots, n$) is a polynomial function of $g_1, \dots, g_{\bar{m}}$ (up to the degree of a constant, say, o_n) and $Q_{st}(0), \dots, Q_{st}(k)$.

Similar to the case of the parameter vector variable ρ , let

$$1, g_1, \dots, g_{\bar{m}}, g_1^2, g_1 g_2, \dots, g_1^{o_n}, \dots, g_{\bar{m}}^{o_n}$$

be a basis for the o_n -degree polynomials of g and let $\bar{d}$ be its dimension. Define

$$\Gamma = [1 \ g_1 \ \dots \ g_{\bar{m}} \ g_1^2 \ g_1 g_2 \ \dots \ g_1^{o_n} \ \dots \ g_{\bar{m}}^{o_n}]^T = [\tau_1 \ \tau_2 \ \dots \ \tau_{\bar{d}}]^T.$$

where τ_i 's is a set of new variables to be used in the reformulation of the problem ν_n .

Then each element τ_i of Γ is a $\bar{d}_i$ -degree monomial of the form

$$\tau_i = \prod_{j=1}^{\bar{m}} g_j^{\theta_{ij}}, \quad 0 \leq \theta_{ij} \leq \bar{d}_i, \quad \sum_{j=1}^{\bar{m}} \theta_{ij} = \bar{d}_i \leq o_n. \quad (5.10)$$

Moreover, there exist indices $i_l \in \{1, 2, \dots, \bar{d}\}$ ($l = 0, \dots, \bar{d}_i$) and $j_l \in \{1, 2, \dots, \bar{m}\}$ ($l = 1, \dots, \bar{d}_i$) such that τ_i is equivalently characterized by the following set of equations:

$$\begin{aligned} \tau_i &= \tau_{i_0} = \tau_{i_1} g_{j_1} \\ &\quad \vdots \\ \tau_{i_{\bar{d}_i-1}} &= \tau_{i_{\bar{d}_i}} g_{j_{\bar{d}_i}} \\ \tau_{i_{\bar{d}_i}} &= 1. \end{aligned} \tag{5.11}$$

It follows that there exist constant coefficients f_{bcki} and $h_{bckistl}$ such that $\Phi_{bc}(k)$ can be expressed as

$$\begin{aligned} \Phi_{bc}(k) &= \sum_{i=1}^{\bar{d}} f_{bcki} \left\{ \prod_{j=1}^{\bar{m}} g_j^{\theta_{ij}} \right\} + \sum_{i=1}^{\bar{d}} \sum_{s=1}^{n_u} \sum_{t=1}^{n_v} \sum_{l=0}^k h_{bckistl} \left\{ \prod_{j=1}^{\bar{m}} g_j^{\theta_{ij}} \right\} Q_{st}(l) \\ &= \sum_i f_{bcki} \tau_i + \sum_{i,s,t,l} h_{bckistl} \tau_i Q_{st}(l). \end{aligned} \tag{5.12}$$

Reformulation of problem ν_n

The problem of interest becomes

$$\begin{aligned} \nu_n &= \inf \quad \gamma \\ \text{s. t.} \quad &\sum_{k=0}^n [\Phi_{bc}^+(k) + \Phi_{bc}^-(k)] \leq \gamma, \quad \sum_{t=1}^{n_v} \sum_{l=0}^n [Q_{st}^+(l) + Q_{st}^-(l)] \leq \alpha \\ &\Phi_{bc}(k) = \Phi_{bc}^+(k) - \Phi_{bc}^-(k), \quad Q_{st}(l) = Q_{st}^+(l) - Q_{st}^-(l) \\ &(5.8), (5.9), (5.12) \\ &\Phi_{bc}^+(k) \geq 0, \quad \Phi_{bc}^-(k) \geq 0, \quad Q_{st}^+(l) \geq 0, \quad Q_{st}^-(l) \geq 0, \quad \underline{\rho} \leq \rho \leq \bar{\rho} \end{aligned}$$

where the optimization variable set is taken as $(\gamma_{feas}, \rho, g, \omega_i, \Phi_{bc}(k), \Phi_{bc}^+(k), \Phi_{bc}^-(k), Q_{st}(l), Q_{st}^+(l), Q_{st}^-(l))$, and we have used a standard change of variables from linear programming (see for instance [3]) to reformulate the variables and constraints of ν_n . Specifically, the variable x is replaced by nonnegative variables x^+ and x^- such that $x = x^+ - x^-$. Then the ℓ_1 norm constraint $\|Q\|_1 \leq \alpha$ is replaced by the constraint $\sum_{t=1}^{n_v} \sum_{k=0}^n [Q_{st}^+(l) + Q_{st}^-(l)] \leq \alpha$, and $\|P_n \Phi\|_1$, the objective function to be minimized, is

replaced by introducing an auxiliary variable γ such that $\sum_{k=0}^n [\Phi_{bc}^+(k) + \Phi_{bc}^-(k)] \leq \gamma$. It is also useful to mention that the optimal solution of the above programming problem always satisfies that either $\Phi_{bc}^+(k)$ or $\Phi_{bc}^-(k)$ is zero.

To set the stage for the branch and bound algorithm, we suppose that the rectangle-type set $[\underline{\rho}, \bar{\rho}] \in \mathcal{R}^m$ is partitioned into $\bar{M}$ subsets $[\underline{\rho}_r, \bar{\rho}_r]$ ($r = 1, \dots, \bar{M}$) such that $[\underline{\rho}, \bar{\rho}] = \bigcup_{r=1}^{\bar{M}} [\underline{\rho}_r, \bar{\rho}_r]$, where $\underline{\rho}_r = [\underline{\rho}_{r1} \cdots \underline{\rho}_{rm}]^T$ and $\bar{\rho}_r = [\bar{\rho}_{r1} \cdots \bar{\rho}_{rm}]^T \in \mathcal{R}^m$. Then a finer grid version of problem ν_n is defined as:

$$\begin{aligned}
\nu_{n,r} = \quad & \inf \quad \gamma \\
\text{s. t.} \quad & \sum_{k=0}^n [\Phi_{bc}^+(k) + \Phi_{bc}^-(k)] \leq \gamma, \quad \sum_{t=1}^{n_u} \sum_{l=0}^n [Q_{st}^+(l) + Q_{st}^-(l)] \leq \alpha \\
& \Phi_{bc}(k) = \Phi_{bc}^+(k) - \Phi_{bc}^-(k), \quad Q_{st}(l) = Q_{st}^+(l) - Q_{st}^-(l) \\
& (5.8), (5.9), (5.12) \\
& \Phi_{bc}^+(k) \geq 0, \quad \Phi_{bc}^-(k) \geq 0, \quad Q_{st}^+(l) \geq 0, \quad Q_{st}^-(l) \geq 0, \quad \underline{\rho}^r \leq \rho \leq \bar{\rho}^r.
\end{aligned} \tag{5.13}$$

where the variable set is $(\gamma_{feas}, \rho, g, \omega_i, \Phi_{bc}(k), \Phi_{bc}^+(k), \Phi_{bc}^-(k), Q_{st}(l), Q_{st}^+(l), Q_{st}^-(l))$.

For notational convenience, we shall use the symbol Ψ to denote the set of $(\gamma, \rho, \Phi_{bc}(k), \Phi_{bc}^+(k), \Phi_{bc}^-(k), Q_{st}(l), Q_{st}^+(l), Q_{st}^-(l)) \in \mathcal{R}^N$ ($N = 1 + m + 3n_z n_w (n+1) + 3n_u n_v (n+1)$) such that all the linear constraints in problem (5.13) are satisfied. Thus problem $\nu_{n,r}$ is expressed as:

$$\begin{aligned}
\nu_{n,r} = \quad & \inf \quad \gamma \\
\text{s. t.} \quad & (5.8), (5.9), (5.12) \\
& (\gamma, \rho, \Phi_{bc}(k), \Phi_{bc}^+(k), \Phi_{bc}^-(k), Q_{st}(l), Q_{st}^+(l), Q_{st}^-(l)) \in \Psi.
\end{aligned} \tag{5.14}$$

To prepare for the linear relaxation scheme introduced in the next section, let us further introduce the following variables:

$$\begin{aligned}
\lambda_{istl} &:= \tau_i Q_{st}(l) \\
\delta_{ik} &:= \omega_i \eta_k
\end{aligned} \tag{5.15}$$

and it follows from (5.8), (5.9), and (5.12) that

$$1 = \sum_{k=1}^d u_{ik} \delta_{ik} \quad (5.16)$$

$$g_i = \sum_{k=1}^d v_{ik} \delta_{ik}. \quad (5.17)$$

$$\Phi_{bc}(k) = \sum_i f_{bcki} \tau_i + \sum_{i,s,l} h_{bckistl} \lambda_{istl} \quad (5.18)$$

So problem (5.14) becomes

$$\begin{aligned} \nu_{n,r} := \quad & \inf \quad \gamma \\ \text{s. t.} \quad & (5.7), (5.11), (5.15 - 5.18) \\ & (\gamma, \rho, \Phi_{bc}(k), \Phi_{bc}^+(k), \Phi_{bc}^-(k), Q_{st}(l), Q_{st}^+(l), Q_{st}^-(l)) \in \Psi. \end{aligned} \quad (5.19)$$

where the variable set is taken as $(\gamma, \rho, \tau_i, g, \eta_i, \lambda_{istl}, \delta_{ik}, \omega_i, \Phi_{bc}(k), \Phi_{bc}^+(k), \Phi_{bc}^-(k), Q_{st}(l), Q_{st}^+(l), Q_{st}^-(l))$. Clearly problem (5.19) is a non-linear optimization problem and hard to solve in general.

5.3 Problem Solution

Following (5.7), (5.11), and (5.15), define

$$\begin{aligned} \Lambda_{ijl}^{\tau} &:= \{(\tau_{i_l}, \tau_{i_{l+1}}, g_{j_{l+1}}) \in \mathcal{R}^3 \mid \text{Inequalities in (4.13) are satisfied with} \\ &\quad (t_{ij}, x_i, x_j, u_i, l_i, u_j, l_j) \text{ replaced by } (\tau_{i_l}, \tau_{i_{l+1}}, g_{j_{l+1}}, \bar{\tau}_{i_{l+1}}, \underline{\tau}_{i_{l+1}}, \bar{g}_{j_{l+1}}, \underline{g}_{j_{l+1}})\} \\ \Lambda_{ijl}^{\eta} &:= \{(\eta_{i_l}, \eta_{i_{l+1}}, \rho_{j_{l+1}}) \in \mathcal{R}^3 \mid (4.13) \text{ are satisfied with} \\ &\quad (\eta_{i_l}, \eta_{i_{l+1}}, \rho_{j_{l+1}}, \bar{\eta}_{i_{l+1}}, \underline{\eta}_{i_{l+1}}, \bar{\rho}_{j_{l+1}}, \underline{\rho}_{j_{l+1}})\} \\ \Lambda_{istl}^{\lambda} &:= \{(\lambda_{istl}, \tau_i, Q_{st}(l)) \in \mathcal{R}^3 \mid (4.13) \text{ are satisfied with } (\lambda_{istl}, \tau_i, Q_{st}(l), \bar{\tau}_i, \underline{\tau}_i, \alpha, -\alpha)\} \\ \Lambda_{ik}^{\delta} &:= \{(\delta_{ik}, \omega_i, \eta_k) \in \mathcal{R}^3 \mid (4.13) \text{ are satisfied with } (\delta_{ik}, \omega_i, \eta_k, \bar{\omega}_i, \underline{\omega}_i, \bar{\eta}_k, \underline{\eta}_k)\} \end{aligned}$$

where $\bar{x}$ and $\underline{x}$ denote upper and lower bounds for the variable x and they can be a priori computed.

Hence from (5.19) and Lemma 4.4.1, we have

$$\begin{aligned}
\nu_{n,r} &:= \inf \quad \gamma \\
\text{s. t.} \quad &(\tau_{i_l}, \tau_{i_{l+1}}, g_{j_{l+1}}) \in \Lambda_{ijl}^\tau, \quad (\eta_{i_l}, \eta_{i_{l+1}}, \rho_{j_{l+1}}) \in \Lambda_{ijl}^\eta \\
&(\lambda_{istl}, \tau_i, Q_{st}(l)) \in \Lambda_{istl}^\lambda, \quad (\delta_{ik}, \omega_i, \eta_k) \in \Lambda_{ik}^\delta \\
&(5.7), \quad (5.11), \quad (5.15 - 5.18) \\
&(\gamma, \rho, \Phi_{bc}(k), \Phi_{bc}^+(k), \Phi_{bc}^-(k), Q_{st}(l), Q_{st}^+(l), Q_{st}^-(l)) \in \Psi.
\end{aligned}$$

Removing the nonlinear constraints (5.7), (5.11), and (5.15), we have the following relaxed linear programming problem:

$$\begin{aligned}
\nu_{n,r}^R &:= \inf \quad \gamma \\
\text{s. t.} \quad &(\tau_{i_l}, \tau_{i_{l+1}}, g_{j_{l+1}}) \in \Lambda_{ijl}^\tau, \quad (\eta_{i_l}, \eta_{i_{l+1}}, \rho_{j_{l+1}}) \in \Lambda_{ijl}^\eta \\
&(\lambda_{istl}, \tau_i, Q_{st}(l)) \in \Lambda_{istl}^\lambda, \quad (\delta_{ik}, \omega_i, \eta_k) \in \Lambda_{ik}^\delta \\
&(5.16 - 5.18), \quad (\gamma, \rho, \Phi_{bc}(k), \Phi_{bc}^+(k), \Phi_{bc}^-(k), Q_{st}(l), Q_{st}^+(l), Q_{st}^-(l)) \in \Psi.
\end{aligned}$$

It is clear that if the relaxed problem $\nu_{n,r}^R$ is infeasible, then so is the problem $\nu_{n,r}$. If $\nu_{n,r}^R$ is a finite real number, then $\nu_{n,r}^R \leq \nu_{n,r}$. Now we are ready to prove the main result of the paper.

Theorem 5.3.1 *Suppose an optimal solution of the relaxed problem $\nu_{n,r}^R$ is given by:*

$$(\widehat{\gamma}, \widehat{\rho}, \widehat{\tau}_i, \widehat{g}, \widehat{\eta}_i, \widehat{\lambda}_{istl}, \widehat{\delta}_{ik}, \widehat{\omega}_i, \widehat{\Phi}_{bc}(k), \widehat{\Phi}_{bc}^+(k), \widehat{\Phi}_{bc}^-(k), \widehat{Q}_{st}(l), \widehat{Q}_{st}^+(l), \widehat{Q}_{st}^-(l)).$$

Then there exists a feasible solution

$$(\gamma_{feas}, \widehat{\rho}, g, \Phi_{bc}(k), \Phi_{bc}^+(k), \Phi_{bc}^-(k), \widehat{Q}_{st}(l), \widehat{Q}_{st}^+(l), \widehat{Q}_{st}^-(l))$$

for problem $\nu_{n,r}$ (as defined in (5.13)) such that

$$\nu_{n,r}^R = \widehat{\gamma} \leq \nu_{n,r} \leq \gamma_{feas} \tag{5.20}$$

$$\gamma_{feas} - \widehat{\gamma} \leq C d_{r,\infty} \tag{5.21}$$

where C is a finite positive constant and $d_{r,\infty} = \max\{|\bar{\rho}_{ri} - \underline{\rho}_{ri}| : i = 1, \dots, m\}$.

Proof: Following the definition of $\nu_{n,r}$ in (5.13), we construct a candidate feasible solution of $\nu_{n,r}$ as follows:

$$\begin{aligned}
g_i &:= \frac{\sum_k v_{ik} \{\prod_{j=1}^m \widehat{\rho}_j^{\theta_{kj}}\}}{\sum_k u_{ik} \{\prod_{j=1}^m \widehat{\rho}_j^{\theta_{kj}}\}} && [\text{ from (5.8) and (5.9) }] \\
\Phi_{bc}(k) &:= \sum_i f_{bcki} \{ \prod_{j=1}^{\bar{m}} g_j^{\theta_{ij}} \} + \sum_{i,s,t,l} h_{bckistl} \{ \prod_{j=1}^{\bar{m}} g_j^{\theta_{ij}} \} \widehat{Q_{st}}(l) && [\text{ from (5.12) }] \\
\Phi_{bc}^+(k) &:= \max\{\Phi_{bc}(k), 0\}, \quad \Phi_{bc}^-(k) := -\min\{\Phi_{bc}(k), 0\} \\
\gamma_{feas} &:= \max_b \{ \sum_{c=1}^{n_w} \sum_{k=0}^n [\Phi_{bc}^+(k) + \Phi_{bc}^-(k)] \}.
\end{aligned}$$

Then it is clear that

$$(\gamma_{feas}, \widehat{\rho}, g, \Phi_{bc}(k), \Phi_{bc}^+(k), \Phi_{bc}^-(k), \widehat{Q_{st}}(l), \widehat{Q_{st}^+}(l), \widehat{Q_{st}^-}(l))$$

is feasible for problem (5.13) and so (5.20) is established.

To show (5.21), it is useful to observe that from the definition of $\nu_{n,r}^R$ the following linear constraints hold:

$$\begin{aligned}
(\widehat{\tau}_{i_l}, \widehat{\tau}_{i_{l+1}}, \widehat{g}_{j_{l+1}}) &\in \Lambda_{ijl}^\tau, \quad (\widehat{\eta}_{i_l}, \widehat{\eta}_{i_{l+1}}, \widehat{\rho}_{j_{l+1}}) \in \Lambda_{ijl}^\eta \\
(\widehat{\lambda}_{istl}, \widehat{\tau}_i, \widehat{Q_{st}}(l)) &\in \Lambda_{istl}^\lambda, \quad (\widehat{\delta}_{ik}, \widehat{\omega}_i, \widehat{\eta}_k) \in \Lambda_{ik}^\delta \\
1 &= \sum_{k=1}^d u_{ik} \widehat{\delta}_{ik}, \quad \widehat{g}_i = \sum_{k=1}^d v_{ik} \widehat{\delta}_{ik} \\
\widehat{\Phi_{bc}}(k) &= \sum_i f_{bcki} \widehat{\tau}_i + \sum_{i,s,t,l} h_{bckistl} \widehat{\lambda}_{istl}.
\end{aligned} \tag{5.22}$$

Furthermore, from (5.6) and (5.7), we have $\prod_{j=1}^m \widehat{\rho}_j^{\theta_{ij}} = \prod_{l=1}^{d_i} \widehat{\rho}_{j_l}$. This, together with

(5.22) and Lemma 4.4.1, implies that there exist real constant C_{η_i} and C'_{η_i} such that

$$\begin{aligned}
|\widehat{\eta}_i - \prod_{j=1}^m \widehat{\rho}_j^{\theta_{ij}}| &= |\widehat{\eta}_{i_0} - \prod_{l=1}^{d_i} \widehat{\rho}_{j_l}| \\
&\leq |\widehat{\eta}_{i_0} - \widehat{\eta}_{i_1} \widehat{\rho}_{j_1}| + |\widehat{\rho}_{j_1}| |\widehat{\eta}_{i_1} - \widehat{\eta}_{i_2} \widehat{\rho}_{j_2}| + \cdots + |\prod_{l=1}^{d_i-2} \widehat{\rho}_{j_l}| |\widehat{\eta}_{i_{d_i-2}} - \widehat{\rho}_{j_{d_i}} \widehat{\rho}_{j_{d_i-1}}| \\
&\leq (\frac{1}{4}) |\widehat{\eta}_{i_1} - \underline{\eta}_{i_1}| |\bar{\rho}_{r_{j_1}} - \underline{\rho}_{r_{j_1}}| + (\frac{1}{4}) |\widehat{\rho}_{j_1}| |\widehat{\eta}_{i_2} - \underline{\eta}_{i_2}| |\bar{\rho}_{r_{j_2}} - \underline{\rho}_{r_{j_2}}| \\
&\quad + \cdots + (\frac{1}{4}) |\prod_{l=1}^{d_i-2} \widehat{\rho}_{j_l}| |\bar{\rho}_{r_{j_{d_i}}} - \underline{\rho}_{r_{j_{d_i}}}| |\bar{\rho}_{r_{j_{d_i-1}}} - \underline{\rho}_{r_{j_{d_i-1}}}| \\
&\leq C_{\eta_i} d_{r,\infty} \\
|\bar{\eta}_i - \underline{\eta}_i| &\leq |\bar{\rho}_{r_{j_{d_i}}} - \underline{\rho}_{r_{j_{d_i}}}| |\Phi(\bar{\rho}_{r_{j_l}}, \underline{\rho}_{r_{j_l}})| \leq C'_{\eta_i} d_{r,\infty}
\end{aligned}$$

where $\Phi(\cdot)$ is a $(d_i - 1)$ -degree polynomial of $\bar{\rho}_{r_{j_l}}$ and $\underline{\rho}_{r_{j_l}}$. Similarly, from (5.8), (5.9), (5.22), and Lemma 4.4.1, we infer that there exist real constant C_{ω_i} , C_{g_i} , and C'_{g_i} such that

$$\begin{aligned}
|1 - \widehat{\omega}_i \sum_k u_{ik} \{ \prod_{j=1}^m \widehat{\rho}_j^{\theta_{kj}} \}| &= | \sum_k u_{ik} \widehat{\delta}_{ik} - \widehat{\omega}_i \sum_k u_{ik} \{ \prod_{j=1}^m \widehat{\rho}_j^{\theta_{kj}} \}| \\
&\leq \sum_k |u_{ik}| \{ |\widehat{\delta}_{ik} - \widehat{\omega}_i \widehat{\eta}_k| + |\widehat{\omega}_i| |\widehat{\eta}_k - \{ \prod_{j=1}^m \widehat{\rho}_j^{\theta_{kj}} \}| \} \\
&\leq \sum_k |u_{ik}| \{ (\frac{1}{4}) |\bar{\omega}_i - \underline{\omega}_i| |\bar{\eta}_k - \underline{\eta}_k| + |\widehat{\omega}_i| C_{\eta_k} d_{r,\infty} \} \\
&\leq C_{\omega_i} d_{r,\infty}
\end{aligned}$$

$$\begin{aligned}
|\widehat{g}_i - g_i| &\leq \sum_k |v_{ik}| \left| \widehat{\delta}_{ik} - \frac{\{ \prod_{j=1}^m \widehat{\rho}_j^{\theta_{kj}} \}}{\sum_k u_{ik} \{ \prod_{j=1}^m \widehat{\rho}_j^{\theta_{kj}} \}} \right| \\
&\leq \sum_k |v_{ik}| \{ |\widehat{\delta}_{ik} - \widehat{\omega}_i \widehat{\eta}_i| + |\widehat{\eta}_i| |\widehat{\omega}_i - \frac{1}{\sum_k u_{ik} \{ \prod_{j=1}^m \widehat{\rho}_j^{\theta_{kj}} \}}| \\
&\quad + |\frac{1}{\sum_k u_{ik} \{ \prod_{j=1}^m \widehat{\rho}_j^{\theta_{kj}} \}}| |\widehat{\eta}_i - \prod_{j=1}^m \widehat{\rho}_j^{\theta_{ij}}| \} \\
&\leq \sum_k |v_{ik}| \{ (\frac{1}{4}) |\bar{\omega}_i - \underline{\omega}_i| |\bar{\eta}_k - \underline{\eta}_k| + |\widehat{\eta}_i| (\frac{1}{\beta}) |\widehat{\omega}_i \sum_k u_{ik} \{ \prod_{j=1}^m \widehat{\rho}_j^{\theta_{kj}} \} - 1| + (\frac{1}{\beta}) C_{\eta_k} d_{r,\infty} \} \\
&\leq C_{g_i} d_{r,\infty}
\end{aligned}$$

$$| \prod_{j=1}^{\bar{m}} \widehat{g}_j^{\theta_{ij}} - \prod_{j=1}^{\bar{m}} g_j^{\theta_{ij}} | \leq C'_{g_i} d_{r,\infty}.$$

Similar arguments as above show that for some constants C_{τ_i} , C'_{τ_i} , and $C_{\lambda_{istl}}$

$$\begin{aligned}
|\widehat{\tau_i} - \prod_{j=1}^{\bar{m}} g_j^{\theta_{ij}}| &\leq |\widehat{\tau_i} - \prod_{j=1}^{\bar{m}} \widehat{g_j^{\theta_{ij}}}| + |\prod_{j=1}^{\bar{m}} \widehat{g_j^{\theta_{ij}}} - \prod_{j=1}^{\bar{m}} g_j^{\theta_{ij}}| \leq C_{\tau_i} d_{r,\infty} \\
|\widehat{\tau_i} - \tau_i| &\leq C'_{\tau_i} d_{r,\infty} \\
|\widehat{\lambda_{istl}} - \{\prod_{j=1}^{\bar{m}} g_j^{\theta_{ij}}\} \widehat{Q_{st}(l)}| &\leq |\widehat{\lambda_{istl}} - \widehat{\tau_i} \widehat{Q_{st}(l)}| + |\widehat{Q_{st}(l)}| |\widehat{\tau_i} - \prod_{j=1}^{\bar{m}} g_j^{\theta_{ij}}| \\
&\leq (\frac{1}{4}) 2\alpha C'_{\tau_i} d_{r,\infty} + \alpha C_{\tau_i} d_{r,\infty} \\
&\leq C_{\lambda_{istl}} d_{r,\infty}.
\end{aligned}$$

Thus it follows that for some constant $C_{\Phi_{bc}(k)}$,

$$\begin{aligned}
&|[\widehat{\Phi_{bc}^+}(k) + \widehat{\Phi_{bc}^-}(k)] - [\Phi_{bc}^+(k) + \Phi_{bc}^-(k)]| \\
&= ||\widehat{\Phi_{bc}(k)}| - |\Phi_{bc}(k)|| \\
&\leq |\widehat{\Phi_{bc}(k)} - \Phi_{bc}(k)| \\
&\leq \sum_i |f_{bcki}| |\widehat{\tau_i} - \{\prod_{j=1}^{\bar{m}} g_j^{\theta_{ij}}\}| + \sum_{i,s,t,l} |h_{bckistl}| |\widehat{\lambda_{istl}} - \{\prod_{j=1}^{\bar{m}} g_j^{\theta_{ij}}\} \widehat{Q_{st}(l)}| \\
&\leq \sum_i |f_{bcki}| C_{\tau_i} d_{r,\infty} + \sum_{i,s,t,l} |h_{bckistl}| C_{\lambda_{istl}} d_{r,\infty} \\
&=: C_{\Phi_{bc}(k)} d_{r,\infty}.
\end{aligned}$$

Define

$$C := \max_b \{ \sum_{c=1}^{n_w} \sum_{k=0}^n C_{R_{bck}} \}.$$

Then we have

$$\begin{aligned}
\gamma_{feas} - \widehat{\gamma} &= \max_b \{ \sum_{c=1}^{n_w} \sum_{k=0}^n [\Phi_{bc}^+(k) + \Phi_{bc}^-(k)] \} - \max_b \{ \sum_{c=1}^{n_w} \sum_{k=0}^n [\widehat{\Phi_{bc}^+}(k) + \widehat{\Phi_{bc}^-}(k)] \} \\
&\leq \max_b \{ \sum_{c=1}^{n_w} \sum_{k=0}^n |[\Phi_{bc}^+(k) + \Phi_{bc}^-(k)] - [\widehat{\Phi_{bc}^+}(k) + \widehat{\Phi_{bc}^-}(k)]| \} \\
&\leq C d_{r,\infty},
\end{aligned}$$

which completes the proof. ■

Having established the relationship between $\widehat{\gamma}$ and γ_{feas} , we can follow the exactly same procedure as discussed in Section 4.4 to compute a global optimal solution for problem ν .

5.4 An Example

Suppose that we are interested in designing a digital controller for a simple model of a mechanical system. The mathematical model of the plant is assumed to be a second order transfer function:

$$P(s) = \frac{1}{ms^2 + cs + k} \quad (5.23)$$

where m , c , and k denote the mass, damping, and stiffness coefficients, respectively. We fix the parameters c and k such that $c = 1$ and $k = 1$, and we are supposed to have the freedom to ascertain the value for the mass of the system, which lies in a given interval $[\underline{m}, \overline{m}] = [0.25, 4.00]$.

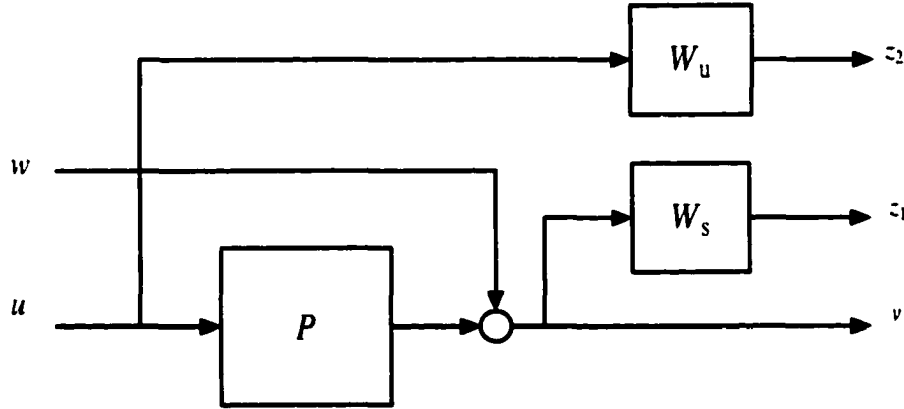


Figure 5.2 Framework for Control Synthesis

The closed-loop feedback control system design is expected to satisfy two objectives. The first is the ℓ_1 performance goal: For the setup shown in Figure 5.2, the discrete-time controller shown stabilize the system so that the ℓ_1 norm of the transfer function from the disturbance w , to the weighted output z_1 and the weighted control action z_2 , is as small as possible. The second goal incorporates the cost efficiency requirement. That is, since the cost of building the system increases as the mass decreases, smaller mass is penalized.

Given the design objectives, the optimization problem of interest can be formulated as follows:

$$\begin{aligned} \inf \quad & \left\| \begin{array}{c} W_s S \\ W_u K S \end{array} \right\|_{\ell_1} + d \cdot f\left(\frac{1}{m}\right) \\ \text{s.t.} \quad & K \text{ stabilizing} \\ & m \in [\underline{m}, \overline{m}] \end{aligned}$$

where d is a weighting coefficient, $f(\cdot)$ represents the cost function of the system with respect to the mass element, W_s is a low-pass filter:

$$W_s = \frac{2.4524z + 2.4524}{z - 0.5095}$$

and

$$W_u = 1.$$

Under the proposed framework, $f(\cdot)$ can be an arbitrary rational function of its variable. Here we choose $d = 1$ and $f(\cdot) = \frac{1}{m}$. Hence the optimization problem of interest becomes:

$$\begin{aligned} \inf \quad & \left\| \begin{array}{c} W_s S \\ W_u K S \end{array} \right\|_{\ell_1} + \frac{1}{m} \\ \text{s.t.} \quad & K \text{ stabilizing} \\ & m \in [\underline{m}, \overline{m}]. \end{aligned} \tag{5.24}$$

The λ -domain model of the plant, $P(\lambda)$, is obtained by using the standard bilinear transformation from Equation (5.23) at a sampling frequency of $f_s = 5$ Hz, which is 5 times larger than all the possible system frequencies as the mass varies in the given interval $[0.25, 4.00]$. The state-space realization of the discrete-time generalized plant

$G(m)$ is given by:

$$G(m) := \left[\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ -\frac{100m-9}{100m+11} & \frac{200m-2}{100m+11} & 0 & 0 & 1 \\ \hline \frac{1}{100m+11} - \frac{100m-9}{(100m+11)^2} & \frac{2}{100m+11} + \frac{200m-2}{(100m+11)^2} & 0.5095 & 1 & \frac{1}{100m+11} \\ \frac{2.4524}{100m+11} - \frac{2.4524(100m-9)}{(100m+11)^2} & \frac{4.9048}{100m+11} + \frac{2.4524(200m-2)}{(100m+11)^2} & 3.7019 & 2.4524 & \frac{2.4524}{100m+11} \\ 0 & 0 & 0 & 0 & 1 \\ \hline \frac{1}{100m+11} - \frac{100m-9}{(100m+11)^2} & \frac{2}{100m+11} + \frac{200m-2}{(100m+11)^2} & 0 & 1 & \frac{1}{100m+11} \end{array} \right] \quad (5.25)$$

from which it can be seen that the entries of the generalized plant G depend on the system structure parameter m in a rational manner, and the denominators are uniformly bounded away from zero. Moreover, it can be concluded from Equation (5.23) that the plant is stable for all the possible mass values in the interval $[0.25, 4.00]$ and the generalized plant (5.25) is stabilizable by zero controller, whereby zero observer and feedback gains ($F = 0$, $L = 0$) are chosen in the simulation. It was determined that the finite dimensional approximation of tapping length $n = 15$ would yield a sufficiently good suboptimum to the infinite dimensional optimal cost and thus in the sequel, we shall present the results for solving problem (5.24) with $n = 15$.

The optimization problem (5.24) was solved by combining the linear relaxation technique and the branch and bound algorithm (see details in [52]) on a P4 1.4G PC under Windows XP system and Matlab 6.1/Cplex 6.5 environment. With a tolerance of $\epsilon = 0.01$, the algorithm took 391 steps to reach the optimum and another 631 steps for verification. The total execution time was 4590 seconds. The optimum is achieved at $m_o = 1.22$ and the corresponding ℓ_1 performance and manufacturing cost are 7.65 and 0.82, respectively. The associated stabilizing digital controller is of 6th order and is given by:

$$K = \frac{-2.2879(1 - 1.819\lambda + 0.8493\lambda^2)(1 - 1.819\lambda + 0.8495\lambda^2)(1 + 1.85\lambda + 0.8828\lambda^2)}{(1 + 0.3045\lambda)(1 - 0.5105\lambda)(1 - 1.819\lambda + 0.8492\lambda^2)(1 + 0.1708\lambda + 0.09512\lambda^2)}.$$

The plot of the cost function of (5.24) versus the variable m is given in Figure 5.3, from which we can see that the optimum is achieved in the region of $[1, 1.5]$ and this coincides with the result obtained from the simulation.

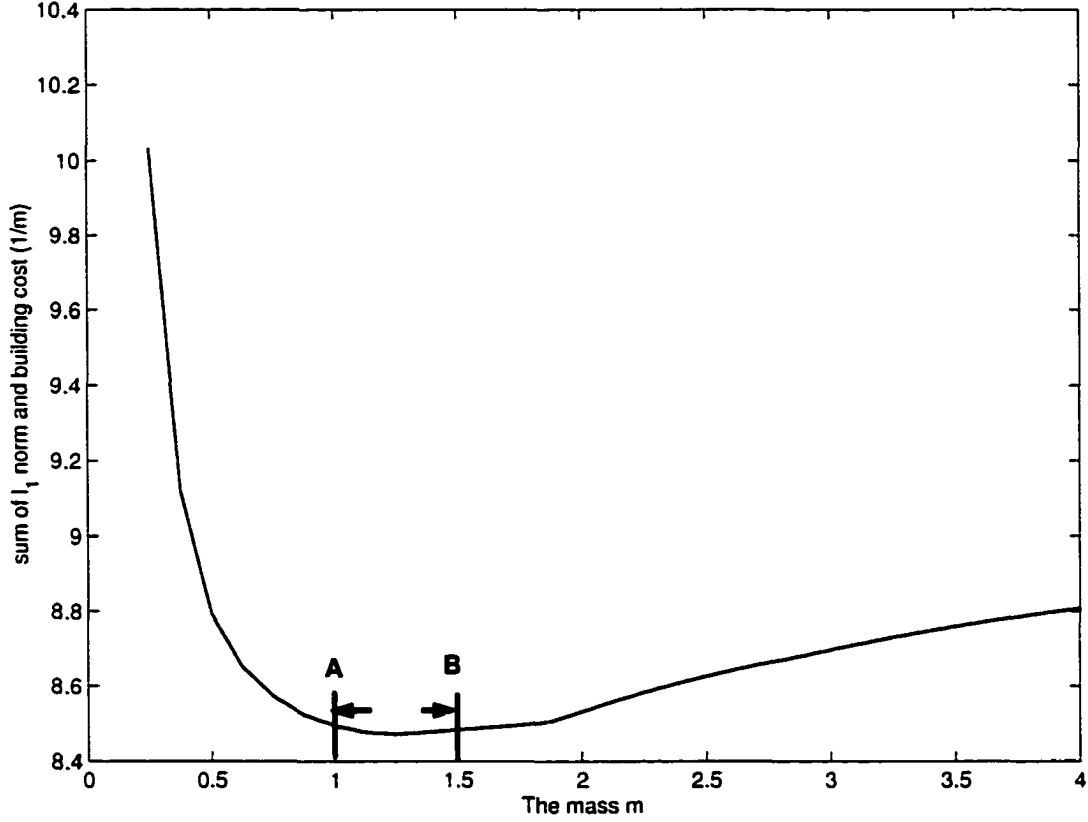


Figure 5.3 Cost function of (5.24) versus the variable m

Now we suppose we are also given the freedom to choose the spring constant k , whose range of choice is given to be $[\underline{k}, \bar{k}] = [1, 20]$ and an additional control design objective is to penalize larger k . To accommodate the requirements of the Shannon sampling theorem, the sampling frequency of $f_s = 10\text{Hz}$ is chosen and the possible range for the mass is restricted to be $[\underline{m}, \bar{m}] = [1, 1.4]$. Here we consider the following

optimization problem of interest:

$$\begin{aligned}
& \inf \left\| \begin{bmatrix} W_s S \\ W_u K S \end{bmatrix} \right\|_{\ell_1} + \frac{1}{m} + 0.004k \\
& \text{s.t. } K \text{ stabilizing} \\
& m \in [\underline{m}, \overline{m}] \\
& k \in [\underline{k}, \overline{k}].
\end{aligned} \tag{5.26}$$

The state-space realization of the discrete-time generalized plant $G(m, k)$ is given by:

$$G(m, k) := \left[\begin{array}{c|c} A_G(m, k) & B_G(m, k) \\ \hline C_G(m, k) & D_G(m, k) \end{array} \right] \tag{5.27}$$

with

$$\begin{aligned}
A_G(m, k) &= \begin{bmatrix} 0 & 1 & 0 \\ -\frac{400m+k-20}{400m+k+20} & \frac{800m-2k}{400m+k+20} & 0 \\ \frac{1}{400m+k+20} - \frac{400m+k-20}{(400m+k+20)^2} & \frac{2}{400m+k+20} + \frac{800m-2k}{(400m+k+20)^2} & 0.5095 \end{bmatrix} \\
B_G(m, k) &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & \frac{1}{400m+k+20} \end{bmatrix} \\
C_G(m, k) &= \begin{bmatrix} \frac{2.4524}{400m+k+20} - \frac{2.4524(400m+k-20)}{(400m+k+20)^2} & \frac{4.9048}{400m+k+20} + \frac{2.4524(800m-2k)}{(400m+k+20)^2} & 3.7019 \\ 0 & 0 & 0 \\ \frac{1}{400m+k+20} - \frac{400m+k-20}{(400m+k+20)^2} & \frac{2}{400m+k+20} + \frac{800m-2k}{(400m+k+20)^2} & 0 \end{bmatrix} \\
D_G(m, k) &= \begin{bmatrix} 2.4524 & \frac{2.4524}{400m+k+20} \\ 0 & 1 \\ 1 & \frac{1}{400m+k+20} \end{bmatrix}
\end{aligned}$$

from which it can be seen that the entries of the generalized plant G depend on the system structure parameters m and k in a rational manner, and the denominators

are uniformly bounded away from zero. Moreover, it can be seen that the plant is stable for all the possible parameter pairs in the rectangle $[1, 1.4] \times [1, 20]$ and the generalized plant (5.27) is stabilizable by zero controller, whereby zero observer and feedback gains ($F = 0$, $L = 0$) are chosen in the simulation. It was determined that the finite dimensional approximation of tapping length $n = 15$ is good enough and in the sequel, we shall present the results for solving problem (5.26) with $n = 15$.

With a tolerance of $\epsilon = 0.01$, the algorithm took 14335 steps to reach the optimum. The total execution time was eight hours and three minutes. The optimum is achieved at $m_o = 1.325$ and $k_o = 15.992$. The corresponding ℓ_1 performance, manufacturing cost and spring cost are 8.669, 0.755, and 0.064, respectively. The associated stabilizing digital controller is of 10th order and is given by:

$$K = \frac{-3.6237(1 - 1.816\lambda + 0.9293\lambda^2)(1 + 1.816\lambda + 0.9294\lambda^2)}{(1 - 0.5066\lambda)(1 + 0.2068\lambda)(1 + 0.6217\lambda + 0.1964\lambda^2)} \cdot \frac{(1 - 1.816\lambda + 0.9293\lambda^2)(1 - (7.743 \times 10^{-5})\lambda + 0.577\lambda^2)}{(1 - 1.816\lambda + 0.9293\lambda^2)(1 - 0.3463\lambda + 0.1669\lambda^2)}.$$

5.5 Summary

We have presented a global optimal solution to the simultaneous parameter and robust control synthesis problem in the paper. The structure parameters are assumed to enter into the system dynamics in a general rational manner and the structured uncertainty under consideration admits a bounded ℓ_∞ to ℓ_∞ induced norm. Global suboptimal solutions are obtained by solving linear programming problems for which powerful numerical softwares exist.

PART III

CONCLUSIONS AND DIRECTIONS

In this part, we briefly summarize our main contributions, and outline some possible directions of future research.

Summary

In Chapter 1 and 2, a general multiobjective (GMO) control design framework involving several important performance measures was formulated. Based on results from functional analysis and linear algebra, we showed that the problem resulting after imposing a regularizing condition always admits an optimal solution. Suboptimal solutions with performance arbitrarily close to the optimal cost can be obtained by constructing two sequences of finite dimensional convex optimization problems whose objective values converge to the optimum from below and above.

In Chapter 3, we showed that the finite dimensional upper and lower bound optimization problems formulated in Chapter 2 can be formulated as LMI optimization problems and solved using semidefinite programming techniques. We introduced a multiobjective control design matlab package, GMO 1.0, which was developed by the authors to implement the GMO algorithms. Using this package, we successfully computed solutions to several control design problems that have diverse performance requirements, which illustrated the effectiveness of the proposed theory and the software.

In Chapter 4, based on a linear relaxation technique, we developed a global optimal solution to the integrated parameter and control (ISC) design problem, where the system structural parameters are assumed to enter into the system dynamics in a general polynomial manner. Before this work, no known result is available on how to compute a global optimal solution for the ISC problem, even for the simplified case that system dynamics depend on structural parameters in an affine manner. Another advantage of the proposed algorithm is that it only requires the solution of linear/quadratic pro-

gramming optimizations for which powerful and efficient numerical tools are available.

In Chapter 5, following the similar idea as in the polynomial case, we presented a global optimal solution to ISC design problem where the structural parameters are assumed to enter into the system dynamics in a rational manner. The ISC problem setup considered is rather general, which, for example, includes as a special case the robust ℓ_1 ISC problem. The effectiveness of the proposed algorithms in Chapter 4 and 5 was illustrated via two numerical examples whose solutions were obtained by solving linear programming optimization problems.

As a summary, the main contributions of the thesis are highlighted (compared with the current approaches to GMO and ISC problems as surveyed in Section 1.3 and 4.1):

- (GMO PART) For the first time, a methodology is developed to solve the general multiobjective control synthesis problem involving ℓ_1 norm, $\mathcal{H}_2$ norm, $\mathcal{H}_\infty$ norm, time response constraints, and controller structural constraints, which furnishes the designers with a design framework, while all other current available approaches can only address a subset (two or three) of the objectives listed above. Moreover:
 - for the ℓ_1 optimization with infinite horizontal TDC case, the GMO approach developed here is less conservative than the solution proposed in [5] in the sense that it does not assume the existence of an FIR feasible solution while the latter one does.
 - the GMO approach not only provides a much simpler solution to the $\ell_1/\mathcal{H}_\infty$ problem, compared with the solution in [19], but also presents a solution for the $\mathcal{H}_\infty/\ell_1$ problem, for which no other known solution exists.
 - global optimal solutions are furnished for the well-known active suspension multiobjective $\ell_1/\mathcal{H}_2/\mathcal{H}_\infty$ control design for transportation vehicles, while other known methods (e.g. [41]) can only yield local optimal solutions.

- a Matlab-based software package, GMO v1.0, has been developed by the author to implement the proposed GMO methodology and this provides a tool for system designers.
- (ISC PART) For the first time, a global optimal solution is proposed for the integrated structure and control design problem, while all other known methods can only yield (at most) local optimal solution. Moreover,
 - the parameter dependence considered here is the general rational case, while all other known methods can only deal with the case of linear (affine) cases.
 - the proposed ISC design framework enables the designers to achieve the best possible system performance with respect to all the stabilizing controllers, all the possible system parameters, and any given ℓ_∞ induced norm bounded structured uncertainty block. Currently no other approaches can compute a global optimal solution for such types of problems, even for the simpler case when the system depends on the structural parameters affinely.

Future Research

As a future research direction, it would be interesting to explore the possibility of incorporating into the GMO setup some controller order constraints, since in many engineering applications designers are interested in achieving the best optimal controller that admits an order of less or equal to a fixed number given a priori. As a closely related open problem, it is also intriguing to examine how to reduce the order of a given system with ℓ_∞ induced norm as the reduction criterion.

To design a (sub)optimal closed-loop system that admits performance within a given tolerance ϵ to the global optimum, the current practice is to compute both upper and

lower bounds for an increasing sequence of tapping length of the optimization variable, the Youla parameter Q , until the difference between the upper and lower bound is less than ϵ . Depending on the nature of the given system, it might be computational expensive to achieve the desired performance. Hence it would be very attractive if the correspondence between the tapping length and the desired tolerance ϵ can be established. Moreover, to reduce the computation cost of the LMI optimization associated with the GMO setup when $\mathcal{H}_\infty$ objective/constraint is involved, it would be interesting to examine the effect of using alternative LMI formulation for $\mathcal{H}_\infty$ specification. In this direction, the result presented in [16] might prove useful.

APPENDIX GMO 1.0 USER MANUAL

Please refer to Chapter 1 – 3 for the theoretical background on which the GMO 1.0 package is based on. Here we present a simplified version of the user manual for the multiobjective control design package, GMO 1.0.

A calling synopsis of the main function (GMO.m) of GMO 1.0 package is summarized in Table A.1. In what follows, via a simple example plant, we show how to set up the three parameters *nwvec*, *nzvec*, and *coeff*, for a given generalized plant to solve various robust control design problems of interest. For the setup of other parameters, please refer to Table A.1 at the end of this Appendix and the template file GMOexample.m in the root directory of the GMO package. A good way to read through and understand this section is to follow the illustration of this section with the template file GMOexample.m given in the root directory of the GMO package.

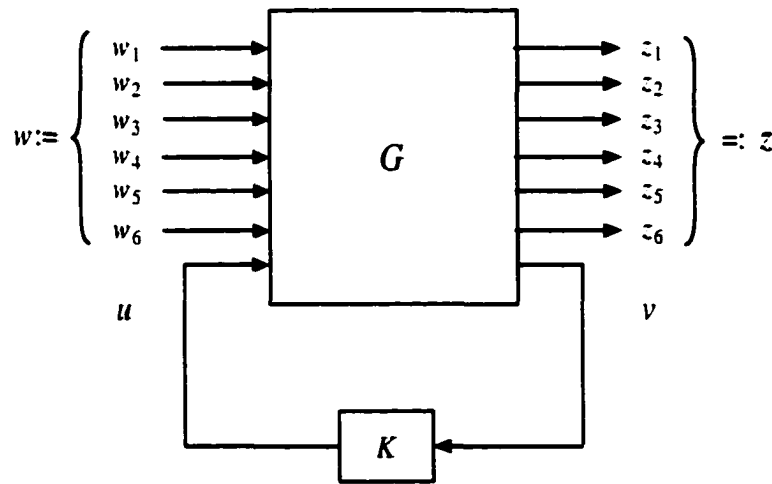


Figure A.1 Closed-loop system

Define two transfer matrices $\hat{R}^1$ and $\hat{R}^2$ (Figure A.1) as follows:

$$\hat{R}^1 : \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} \rightarrow \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

$$\hat{R}^2 : \begin{bmatrix} w_5 \\ w_6 \end{bmatrix} \rightarrow \begin{bmatrix} z_4 \\ z_5 \\ z_6 \end{bmatrix}$$

We now show how to setup the parameters *nwvec*, *nzvec*, and *coeff* in Matlab to solve robust control design problems involving the optimization of $\mathcal{H}_2$ and ℓ_1 performance for $\hat{R}^1$ and $\hat{R}^2$. We will also show how to incorporate time-domain constraint (TDC) into the control synthesis setup.

$\mathcal{H}_2$ optimization

Suppose one is interested in minimizing the $\mathcal{H}_2$ norm performance for the transfer matrix $\hat{R}^1$, that is, to minimize

$$\|\hat{R}^1\|_{\mathcal{H}_2}^2 = \left\| \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} \rightarrow \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \right\|_{\mathcal{H}_2}^2.$$

As shown in `GMOexample.m`, the following setup in matlab would correspond to the above minimization objective function:

$$\begin{aligned} nwvec &= \begin{bmatrix} 1 & 4; \end{bmatrix}; \\ nzvec &= \begin{bmatrix} 1 & 3; \end{bmatrix}; \\ coeff &= \begin{bmatrix} 1 & 1 & 2; \end{bmatrix}; \end{aligned}$$

and the configurations for the three parameters are given in the Figure A.2.

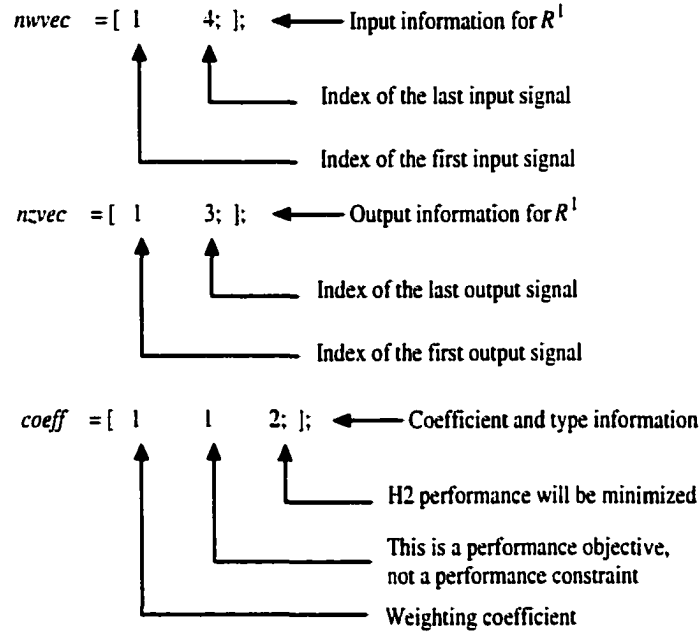


Figure A.2 Parameter setup for $\hat{R}^1$

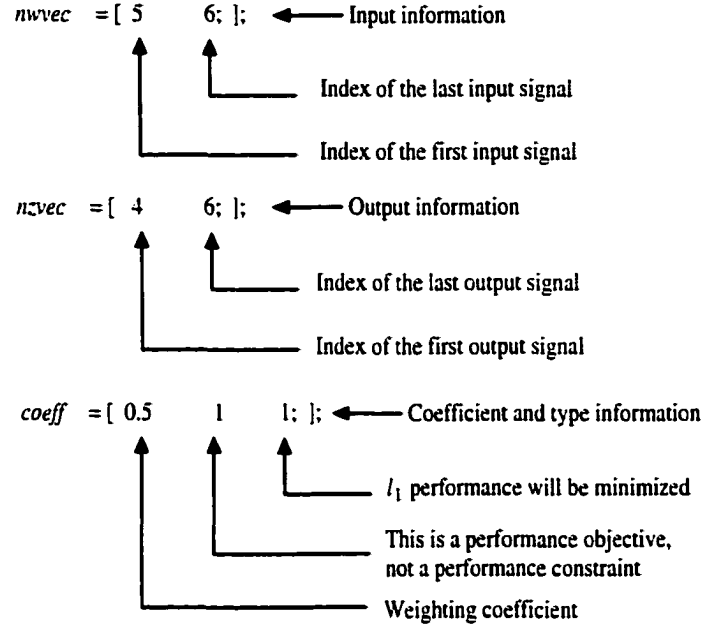
ℓ_1 optimization

Suppose one is interested to minimize the following linear combination of $\mathcal{H}_2$ and ℓ_1 objectives:

$$\|\hat{R}^2\|_{\ell_1} = 0.5 * \left\| \begin{bmatrix} w_5 \\ w_6 \end{bmatrix} \rightarrow \begin{bmatrix} z_4 \\ z_5 \\ z_6 \end{bmatrix} \right\|_{\ell_1}.$$

Then the following setup in matlab would correspond to the above objective function:

$$\begin{aligned} nwvec &= \begin{bmatrix} 5 & 6; \end{bmatrix}; \\ nzvec &= \begin{bmatrix} 4 & 6; \end{bmatrix}; \\ coeff &= \begin{bmatrix} 0.5 & 1 & 1; \end{bmatrix}; \end{aligned}$$

Figure A.3 Parameter setup for $\hat{R}^2$

and the configurations for the three parameters are given in the Figure A.3:

$\mathcal{H}_2 + \ell_1$ multiobjective optimization

Suppose one is interested to minimize the following objective function:

$$\|\hat{R}^1\|_{\mathcal{H}_2}^2 + 0.5 * \|\hat{R}^2\|_{\ell_1} = \left\| \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} \rightarrow \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \right\|_{\mathcal{H}_2}^2 + 0.5 * \left\| \begin{bmatrix} w_5 \\ w_6 \end{bmatrix} \rightarrow \begin{bmatrix} z_4 \\ z_5 \\ z_6 \end{bmatrix} \right\|_{\ell_1}.$$

Then the following setup in matlab would correspond to the above objective func-

tion:

$$\begin{aligned} nwvec &= \begin{bmatrix} 1 & 4; \\ 5 & 6; \end{bmatrix}; \\ nzvec &= \begin{bmatrix} 1 & 3; \\ 4 & 6; \end{bmatrix}; \\ coeff &= \begin{bmatrix} 1 & 1 & 2; \\ 0.5 & 1 & 1; \end{bmatrix}; \end{aligned}$$

and the configurations for the three parameters are given in the Figure A.4:

ℓ_1/TDC multiobjective optimization

Suppose one is interested to minimize the following objective function:

$$\|\hat{R}^2\|_{\ell_1} = \left\| \begin{bmatrix} w_5 \\ w_6 \end{bmatrix} \rightarrow \begin{bmatrix} z_4 \\ z_5 \\ z_6 \end{bmatrix} \right\|_{\ell_1}$$

such that the map $\hat{R}^3 : w_1 \rightarrow z_1$ satisfies the step response constraint templates *atemp* and *btemp*:

$$atemp(k) \leq R^3(k) \leq btemp(k), \forall k > 0.$$

Then the following setup in matlab would correspond to the above objective function (Figure A.5):

$$\begin{aligned} nwvec &= \begin{bmatrix} 5 & 6; \\ 1 & 1; \\ 4 & 6; \end{bmatrix}; \\ nzvec &= \begin{bmatrix} 1 & 1; \\ 0.5 & 0 & 1; \\ 1 & 1 & 5; \end{bmatrix}; \\ coeff &= \begin{bmatrix} 0.5 & 0 & 1; \\ 1 & 1 & 5; \end{bmatrix}; \end{aligned}$$

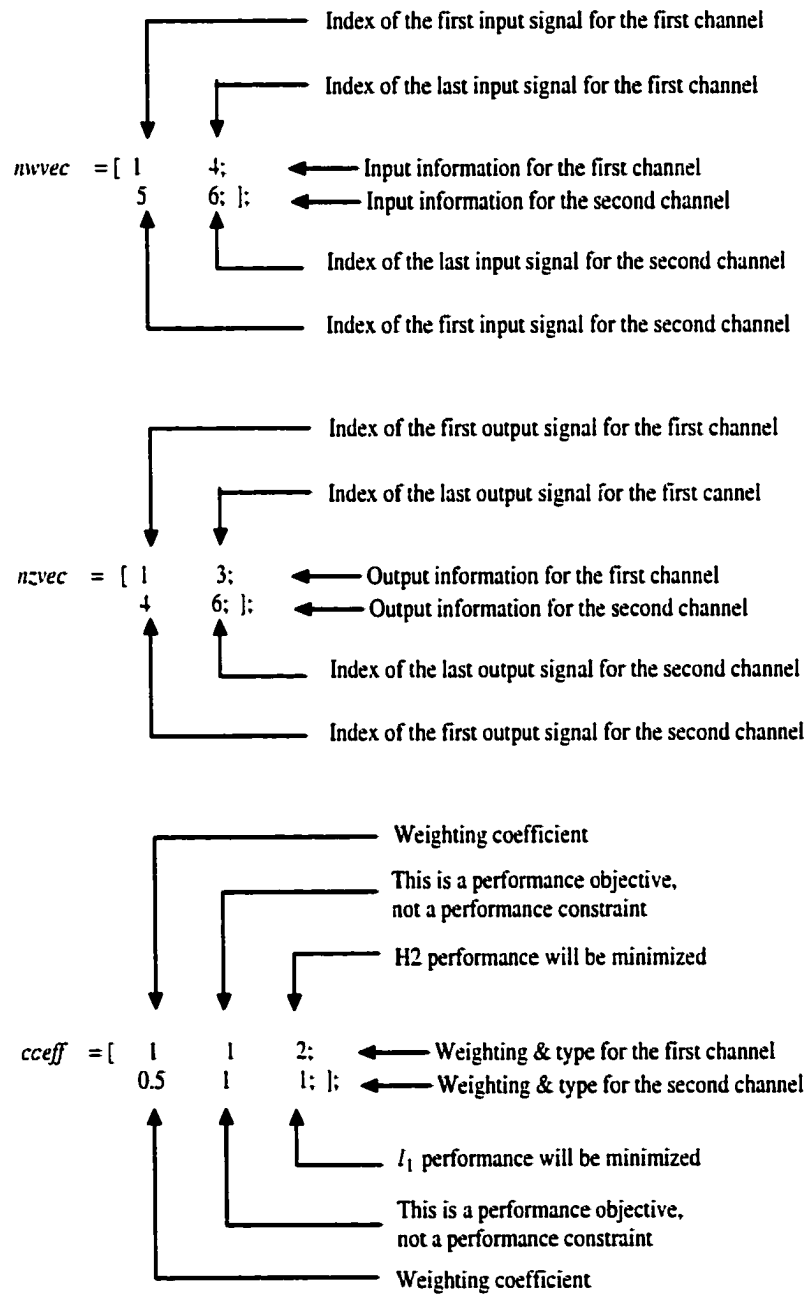
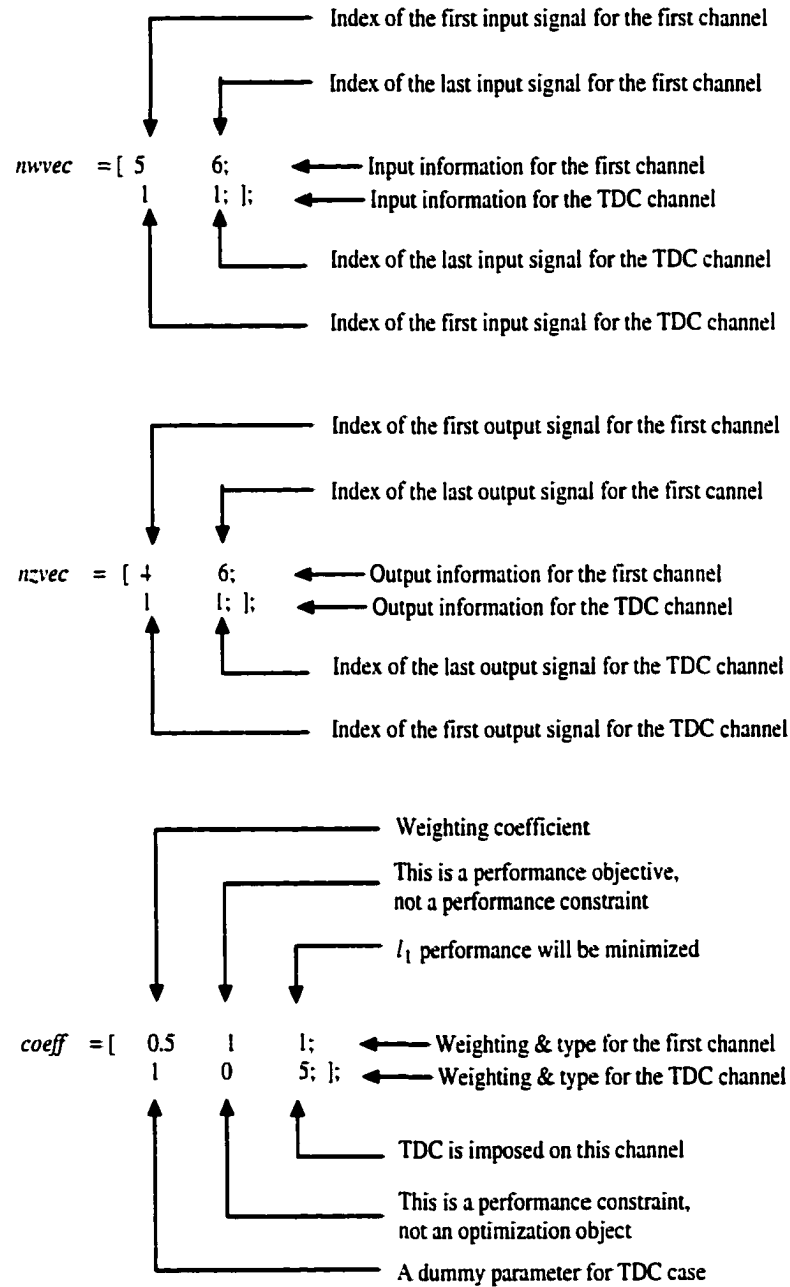


Figure A.4 $\|\hat{R}^1\|_{\mathcal{H}_2}^2 + 0.5 * \|\hat{R}^2\|_{\ell_1}$ optimization

Figure A.5 $0.5 * \|\hat{R}^2\|_{\ell_1}$ optimization

function [Ksys,Rsys,Qsys,obj,Rnorm,Q,bounds] = GMO(Gsys,nz,nw,nzvec,nwvec, nu,ny,coeff, Qcoeff,tol,tol2,Ts,lenqind,lbflag1,lbflag2, LMIsolver,PPmethod,beta,atemp,btemp,ctemp,dtemp)	
Output variables	
Ksys	discrete-time controller
Rsys	discrete-time closed-loop system
Qsys	Youla parameter
obj	Objective function value
Rnorm	closed-loop system norm
Q	Impulse response of the Youla parameter Q
bounds	Lower and upper bounds with respect to lenq
Input variables	
Gsys	discrete-time generalized plant
nz	column vector of the dimension of input channels
nw	column vector of the dimension of output channels
nzvec	l-by-2 matrix containing the dimension information of the l output channels, where l=dim(nzvec(:,i))=dim(nwvec(:,i)), nzvec(i,1:2)=[(starting output channel index) (starting input channel index)];
nwvec	l-by-2 matrix containing the dimension information of the l input channels, where l=dim(nzvec(:,i))=dim(nwvec(:,i)), nwvec(i,1:2)=[(starting output channel index) (starting input channel index)];
nu	number of controller outputs
ny	number of controller inputs
coeff	coefficient matrix with the structure: coeff=[coeff(1)' ... coeff(l)']', where coeff(i)=[ci obj/con type] ci: ith channel performance weighting ci obj/con: 1--Objective 0--Constraint type: 1--l1 2--H2^2 3--H2 4--Hinf 5--SRC 6--IRC
Qcoeff	nu-by-ny coefficient matrix, whose nonzero elements indicate the zero elements of the Youla parameter matrix Q
tol	relative difference tolerance between final objective values
tol2	FIR approximation tolerance
Ts	sampling period
lenqind	length of Q variable
lbflag1	1--lenq=lenu+lenv-2 (faster convergence) 0--lenq=lenqind
lbflag2	1--compute the lower bound 0--omit the computation of lower bound
LMIsolver	LMI solution via: 1--sp code 2--sdp code 3--sdpha code 4--Cplex(no Hinf obj/cons;no H22/H2 cons;no H2 obj) 5--LP/QP(no Hinf obj/cons;no H22/H2 cons;no H2 obj) LP--linprog.m, QP--quadprog.m in Matlab optim toolbox
PPmethod	Pole placement via: 1--linear quadratic method 2--matlab place(.) function
beta	one norm bound on Q
atemp	lower template matrix for SRC constraint
btemp	upper template matrix for SRC constraint
ctemp	lower template matrix for IRC constraint
dtemp	upper template matrix for IRC constraint

Table A.1 Calling Syntax of GMO(.) function

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