

Resilient and Situationally Aware Multi-Microgrid: A Stochastic Control and Resource Pooling Framework

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Dedication

To ...

Ma, Babai, Didi, Umanjan, my little Advik, and Didubhai

... for their love.

Abstract

In recent times, the proliferation of distributed energy resources, propelled by a mounting interest in the generation and utilization of clean renewable energy, has fostered a paradigm change in the operation and control of power distribution networks. Further, the impetus to create carbon-free power systems has incentivized the rapid growth of interconnected multi-microgrid frameworks that are powered by renewable energy sources (RESs). However, the inherent uncertainty of RESs poses a challenge in meeting the power demand of the microgrids (MGs) unless sufficient energy storage, such as battery energy resources (BESs), is maintained. While the energy storage systems can supply the deficit in the case of underproduction, an excess provisioning of energy storage systems can result in overproduction which in turn increases the operating cost. Since both overproduction and underproduction are undesirable, an important question that needs to be answered is: how can the MGs guarantee the demand-supply balance? In traditional distribution systems, MGs depend on the main grid for demand-supply balance and assistance with ancillary services. However, excessive grid dependency adversely affects system reliability as grid outages are becoming more frequent due to climate-related contingencies. Hence, designing an ecosystem of self-sufficient MGs that are capable of supplying local loads with local generations, while being minimally dependent on a centralized grid, is of considerable importance. In this context, it is crucial to devise a strategy for controlling the power output capability of the BESs to counter the uncertainty of RESs production and ensure that there will be no under and over-production simultaneously to improve the system reliability.

Another crucial design aspect of the BESs is to ensure that they have sufficient energy capacity to minimize generation and load curtailments, arising due to over and underproduction scenarios. The BESs energy capacity should be designed in such a way that: (i) if the RESs produce more power than what is demanded by the load, then the BESs can be charged with the surplus generation, and (ii) when the RESs production is less than the required load, the BESs have sufficient energy to supply the deficit power. The above conditions are satisfied if the maximum capacity of BESs is not reached and the BESs never get depleted under the uncertainty presented by the RESs. Such a challenge in deciding the BESs capacity is further exacerbated by the uncertainty of RESs. While BESs with excess capacity can address the technical challenges of minimizing

the generation and load curtailments, it increases the installation and operation cost of the entire system thus restricting the economical viability of maintaining BESs with huge capacities. Hence, in the planning stage of the system, it is essential to decide the minimum required BESs capacity to mitigate the RESs uncertainty with provable guarantee. For multi-microgrid systems, the facility to share power has the potential to harness the benefits of spatial diversity of generation to further minimize needed BESs capacity. In this context, it is essential to explore strategies to interconnect the MGs and to decide how the MGs should share power. While it is possible to reduce the required BESs capacity by interconnecting the MGs, the economic benefits of such a reduction may be nullified by the cost associated with the transmission lines connecting them. As the transmission line installation and maintenance cost primarily depends on the line flow capacity, it is essential to analyze the required capacity of both BESs and transmission lines in unison.

In this thesis, Chapter 2 contributes towards developing a dynamic battery allocation and control strategy, under a novel electricity market mechanism, to optimize the power the battery reserve provides, while ensuring that demand is met with a provable guarantee. A power distribution system with multiple microgrids with stochastic RESs, and a distribution network operator (DNO) is considered. The objective of the DNO is to control the battery power output capability in such a way that the demands of all the microgrids, at the requested future time, are met with probability *one* without any over or underproduction. Our solution is built upon stochastic feedback control techniques. Under our proposed scheme, the DNO responds to the evolving uncertainty by dynamically balancing the RESs and battery resources, via contracts made in the electricity market, and eliminates the risk of over or underproduction.

Chapter 3 contributes towards designing a methodology to determine battery capacity, for MGs, which ensures that during the operational phase, BESs' operating range is respected with pre-specified high probability, under renewable uncertainties. We provide an explicit analytical expression of the required BESs energy capacity for a single microgrid with stochastic RESs as the main power source. Leveraging insights from the single microgrid case, the article focuses on the design and planning of BESs for an interconnected two-microgrid scenario. Utilizing stochastic feedback control theory, we propose an optimization framework to determine how the microgrids should share power, while respecting the transmission line capacity, to minimize the battery capacity requirement. Furthermore, we provide a theoretical analysis of the trade-offs between

transmission line capacity and BESs energy capacity and show that for a large operation time horizon, it is even possible to reduce the transmission line capacity while maintaining the least amount of battery.

Another important aspect of distribution system planning and control is to design a system monitoring infrastructure providing real-time situational awareness. To efficiently operate the system, the operator needs real-time measurements of bus voltages, power injection and flow through the branches, etc. Apart from the real-time measurements, another crucial information is the knowledge of the underlying power network topology. The network topology is utilized for various techno-economical purposes, such as economic dispatch, electricity price determination, etc. Traditionally the topology information is obtained from the prior knowledge of the system. However, the operational flexibility of renewable-powered MGs can alter the topology over time. Further, weather-related extreme events can result in branch faults in the system which changes the topology. Therefore, a methodology to detect and localize power network topology changes in real-time is extremely important to efficiently operate the system. To aid with such requirement, Information and Communication Technology (ICT) infrastructure is utilized. ICT infrastructure includes various modern measurement devices such as smart meters, phasor measurement units, etc. However, the wireless nature of such devices renders the system vulnerable to cyber-attacks. Many incidents of cyber-attacks in the energy sector have been reported in the last decade resulting in severe economic loss and electricity outages. The most common type of cyber-attack in this context is false data injection (FDI) attacks, where the attacker manipulates the field measurements to disrupt the system operation. The attack objectives range from technical interruptions, such as power outages, system faults, etc., to intrusion in economic operations, such as electricity markets. To this end, it is of utmost importance to devise cyber-attack detection and protection mechanisms to improve the resiliency of the system.

In this thesis, Chapter 4 contributes towards developing a machine-learning method to detect and localize changes in a power network topology. The proposed strategy utilizes only bus voltages and phase angles to detect any change and further localize the change. The proposed strategy is capable of detecting multiple topology changes (branch outages or restoration) occurring simultaneously or over a time horizon. Real-time hardware-in-the-loop experiments on a modified IEEE 33-bus, 226-bus, and 1262-bus network, emulated on OPAL-RT real-time simulator, are conducted to demonstrate the efficacy of the proposed methodology.

Chapter 5 of the thesis investigates system resiliency to cyber-attacks. In the first part of the chapter, cyber-attacks in breaker status measurements are considered. Such type of attack is capable of falsifying the topology information conveyed to the operator. Since the topology information is extensively used in deciding the electricity prices at different buses in the network, we demonstrate the impact of cyber-topology attacks in disrupting the electricity market. Utilizing reinforcement learning, we provide a system vulnerability analysis, by identifying the critical branches in the network, which must be protected to avoid such disruption. In the second part of the chapter we demonstrate the impact of FDI attacks in tertiary control of microgrids on system stability. We consider FDI attacks on measurements and control commands in a state-feedback-based controller and devise an attack detection and mitigation strategy. We conduct real-time controller hardware-in-the-loop validation on a microgrid emulated inside OPAL-RT to illustrate the utility of our proposed attack mitigation strategy.

In Chapter 6 we describe an end-to-end cyber-physical testbed infrastructure for grid control and monitoring. The testbed is completely developed in the lab. The infrastructure is equipped with databases and a data-driven algorithm execution framework for experimental analysis of cyber-physical systems. Further, the testbed has real-life or emulated power system integration and visualization framework to assist with situational awareness. Some detailed real-time demonstrations on the testbed are illustrated to showcase the flexibility and utility of the same.

Contents

Acknowledgements	i
Dedication	iii
Abstract	iv
Contents	viii
List of Tables	xiii
List of Figures	xiv
List of Symbols	xxi
List of Abbreviations	xxiii
1 Introduction	1
1.1 Motivation for Renewable-Powered Microgrid	1
1.1.1 Microgrid: The Grid of the Future	2
1.1.2 Operational Architecture of Microgrids	2
1.2 Impact of Renewable Penetration	3
1.2.1 Uncertainty of Renewable Power Generation	3
1.2.2 Requirement of Energy Storage Systems	5
1.2.3 System Design Challenges of Microgrids	6
1.3 System Monitoring and Cyber-attack Resiliency	8
1.3.1 Situational Awareness Challenges	9
1.4 Contributions and Overview of the Thesis	10
1.4.1 Uncertainty Mitigation of Renewable-Powered Microgrids	10

1.4.2	Situational Awareness and Cyber-Attack Resiliency	11
2	Optimal Multi-microgrid Feedback Policies: Guaranteed Demand Fulfillment under Renewable Uncertainty	14
2.1	Gist of the Chapter	14
2.2	State-of-the-art on Load Demand Fulfillment under Renewable Uncertainty	15
2.2.1	Prior Works	16
2.2.2	Key Contributions of the Chapter	18
2.3	System Description	19
2.4	Problem formulation	21
2.5	Solution Methodology	26
2.5.1	Individualized Multi-microgrid System	26
2.5.2	Interconnected Multi-microgrid System	27
2.6	Experimental results	30
2.6.1	Model assumption verification	30
2.6.2	Sample size determination	31
2.6.3	Simulation Results: Algorithmic Efficacy	32
2.6.4	Real-time Simulation on a Distribution Network	36
3	Guaranteeing Service in Connected Microgrids: Storage Planning and Optimal Power Sharing Policy	40
3.1	Gist of the Chapter	40
3.2	State-of-the-art: Energy Storage Planning and Power Sharing for Multi-microgrid Systems	41
3.2.1	Prior Works	42
3.2.2	Key Contributions of the Chapter	43
3.3	Battery Storage Sizing for a Single Microgrid System	43
3.3.1	System Description	43
3.3.2	Problem Formulation	44
3.3.3	Solution Methodology	46
3.4	Battery Sizing and Power Sharing Policy for Connected multi-microgrids	48
3.4.1	Problem formulation	48
3.4.2	Solution methodology	51
3.5	Trade-offs Between Battery Storage Energy Capacity and Transmission Line Capacity	52

3.5.1	Limiting Cases of Transmission Line Capacity	52
3.5.2	Necessary and Sufficient Condition on Transmission Line Capacity	54
3.6	Experimental Results	63
4	Topology Change Estimation in Power Distribution Networks: A Machine Learning Approach	66
4.1	Gist of the Chapter	66
4.2	State-of-the-art on Topology Change Estimation	67
4.2.1	Prior Works	68
4.2.2	Key Contributions of the Chapter	70
4.3	Problem Formulation	70
4.4	Solution Methodology	72
4.4.1	Expectation-Maximization	75
4.4.2	Finding the Number of Different Topologies	78
4.4.3	Change instant detection	79
4.4.4	Topology Estimation and Change Localization	82
4.5	Hardware-in-the-loop Validation	83
4.5.1	Setup description	83
4.5.2	Topology change identification performance	87
4.5.3	Topology estimation performance	92
5	Resilient System Operation: A Cyber-Security Analysis	97
5.1	Cyber-topology Attack to Disrupt Electricity Market Operation	98
5.1.1	State-of-the-art	98
5.1.2	Attack Model and Problem Description	100
5.1.3	Proposed Reinforcement Learning Solution	103
5.1.4	Simulation Results	105
5.2	False Data Injection Attack on Microgrid Secondary Control	109
5.2.1	State-of-the-art	110
5.2.2	Attack Model and Problem Description	111
5.2.3	Proposed Solution	116
5.2.4	Controller Hardware-in-the-loop based Validation	121
6	An End-to-End Cyber-Physical Infrastructure for Smart Grid Control and Monitoring	126
6.1	Gist of the Chapter	126

6.2	State-of-the-art: Cyber-Physical Laboratory Testbeds	127
6.2.1	Prior Works and Key Contributions of the Chapter	127
6.3	Overview of the Architecture	128
6.3.1	System Layer	128
6.3.2	Backend and Data Analysis Layer	130
6.3.3	Visualization Layer	131
6.4	Case-study Demonstration	132
6.4.1	Architecture Specifications	132
6.4.2	Topology Estimation of modified IEEE-33 bus system	133
6.4.3	Demonstration-1	133
6.4.4	Demonstration-2	134
6.4.5	Demonstration-3	135
7	Conclusions and Future Research	138
7.1	Summary of Completed Works	138
7.2	Future Works	140
	References	143
	Appendix A.	161
A.1	Proof of Proposition 1	161
A.2	Proof of Theorem 1	161
A.3	Derivation of stochastic differential equation of Γ_t in IMM for RESs uncertainty mitigation	164
A.4	Proof of Theorem 2	165
A.5	Proof of Theorem 3	166
	Appendix B.	169
B.1	Proof of Lemma 1	169
B.2	Proof of Theorem 4	170
B.3	Proof of Theorem 5	173
B.4	Power transfer policy determination	173
B.5	Gaussian Tail Inequality	186
B.6	Inequalities involving Wiener process with drift	186
B.7	Proof of Proposition 2	188
B.8	Proof of Theorem 6	190

B.9 Proof of Lemma 2	193
B.10 Proof of Corollary 1	195
B.11 Proof of Lemma 3	197
B.12 Proof of Lemma 4	200
B.13 Proof of Theorem 7	202
B.14 Proof of Lemma 5	204
B.15 Proof of Lemma 6	206
Appendix C.	208
C.1 Probability distribution function of Phasor Data	208
C.2 A brief overview of EM algorithm	209

List of Tables

2.1	Average reduction (%) in portfolio value in IMM for Case 1,2,3	35
2.2	Surplus/deficit comparison for 1000 RESs realizations, for case 2	36
4.1	IEEE 33-bus: Change localization performance	89
4.2	Change estimation performance	91
5.1	IEEE-39 bus system parameters and Q-learning parameters	106
5.2	IEEE 39 bus system vulnerability analysis results	108
5.3	IEEE 300 bus system vulnerability analysis results	108

List of Figures

1.1	A schematic representation of a generic microgrid.	2
1.2	Generation curtailment due to increased renewable penetration in the regions under California Independent System Operator (CAISO)	4
1.3	Top 5 U.S. states with highest number of power outages	5
1.4	Energy storage capacity is expected to grow to about 200 GW by 2050, deploying a range of durations. This translates to about 1,200 gigawatt-hours (GWh) of stored energy.	6
1.5	Utility scale 60 MW Li-ion Battery Storage Cost for 2 – 10 hours duration	7
1.6	Energy storage capacity is expected to grow to about 200 GW by 2050, deploying a range of durations. This translates to about 1,200 gigawatt-hours (GWh) of stored energy.	9
2.1	IMMG architecture with centralized BESS and microgrids with RESs. .	20
2.2	Forward and backward propagation to calculate Γ_{t_n} with 2 MGs	29
2.3	Histograms of $P_{g1}(t)$ for $t \in \{1, 2, \dots, 5\}$ with two sample sets each containing 10000 samples of $P_{g1}(t)$ with terminal condition $P_{g1}(T_f) > D_{c1}$. .	33
2.4	Excess generation from MG 1 is utilized to supply demand of MG 2 in IMMG at T_f , whereas in IDMMG, average power from portfolio is 6 kW to guarantee the demand is met. Power portfolio value is lower in IMMG than IDMMG. Zoomed inset, showing an example of 95% CI around the mean of power portfolio for IMMG at 2 hrs, indicates high confidence in our scheme.	34
2.5	Reduction in power portfolio value in IMMG as a percentage of the same in IDMMG for three cases. For all $t < T_f$, in IMMG, DNO requires less amount of power from portfolio compared to IDMMG	34

2.6	Incorrect initial battery allocation leads to risk of over or underproduction. $\bar{\Gamma}_u^{\Delta t}(t)$ denotes the power portfolio trajectory when initial battery allocation is 20% more than optimal, $\Gamma^{\Delta t}(0)$, derived in Case 3. Here, at the end of the horizon, DNO has overproduction. $\bar{\Gamma}_d^{\Delta t}(t)$ shows the scenario when initial battery allocation is 20% less than optimal. Here, DNO cannot produce required power to support the loads at requested time T_f	35
2.7	Real-time simulation on 33 bus network: The variation of power output of the GFM battery is minimal denoting negligible impact of renewable uncertainties on it. Most of the uncertainties are mitigated by the GFL battery, delivering power as per the reference set-points generated by our algorithm.	38
2.8	Simulation on 33 bus network in PYPOWER for 10000 realizations of $P_{g1}(t)$, $P_{g2}(t)$: The narrow standard deviation of power output from the GFM battery indicates high confidence in the outcomes of the algorithm.	39
2.9	Real-time simulation on 225 bus network with 10 MGs: The variation of power output of GFM battery is minimal. Most of the uncertainties are mitigated by the GFL battery controlled as per our algorithm.	39
3.1	Microgrid architecture	44
3.2	Connected two microgrid architecture	49
3.3	Illustration of epochs for a sample trajectory of $X_d^{\pi'}(t)$	58
3.4	(a) Wind, load variation, and GFM battery power for 5 hours in real-time in the IEEE-33 bus single MG system. (b) The battery does not get fully charged or get depleted throughout the time horizon.	65
3.5	Two MG system: Battery capacity of each MG is 10 kWh with a power line capacity of 15 kW. Out of 5000 realizations of the available energy in the batteries of MG-1 and MG-2, only 0.4% realizations, hits either the upper limit of 10 kWh or the lower limit of 0 kWh.	65
3.6	Required BES capacity for each MG decreases, under the optimal power transfer control policy, as power line capacity increases.	65
4.1	Change detection flowchart	74
4.2	Illustration of the proposed change instant detection technique	81
4.3	Change detection and topology estimation flowchart	84

4.4	Figure of (a) the laboratory-based controller-hardware-in-the-loop test system with OP5700 RT-simulator (RTS), (b) the end-to-end cyber-physical framework for grid monitoring.	85
4.5	IEEE 33-bus: (a) Mean silhouette score of the phasor samples obtained by running Algorithm 4 for each $K \in \{2, \dots, \overline{K} = 10\}$. The highest value of the average silhouette score is 0.72 with $K = 2$, indicating two different topologies are observed during the sample collection window, (b) Estimated density functions of the timestamps of the phasor samples classified under topology 0 and 1. The density functions intersect at the timestamp value of 3160 indicating the change instant, (c) Topology is actually changed upon collecting the phasor sample with timestamp 3160, indicating the actual time instant. The value of the timestamp at the intersection of the density functions in (b) matches with the actual instant.	88
4.6	Line outage detected by the proposed topology change identification and proposed topology estimation technique.	89
4.7	226-bus: (a) Mean silhouette score of the phasor samples obtained by running Algorithm 4 for each $K \in \{2, \dots, \overline{K} = 10\}$. The highest value of the average silhouette score is 0.84 with $K = 3$, indicating three different topologies are observed during the sample collection window, (b) Estimated density functions of the timestamps of the phasor samples classified under topology 0, 1, 2. The density functions intersect at the timestamp value of 9115, and 18232, indicating the change instants, (c) Topology is actually changed upon collecting the phasor sample with timestamp 9115 and 18232, indicating the actual time instant. The values of the timestamps at the intersections of the density functions in (b) match with the actual instants.	90

4.8	1262-bus: (a) Mean silhouette score of the phasor samples obtained by running Algorithm 4 for each $K \in \{2, \dots, \overline{K} = 10\}$. The highest value of the average silhouette score is 0.96 with $K = 2$, indicating two different topologies are observed during the sample collection window, (b) Estimated density functions of the timestamps of the phasor samples classified under topology 0 and 1. The density functions intersect at the timestamp value of 2736 indicating the change instant, (c) Topology is actually changed upon collecting the phasor sample with timestamp 2736, indicating the actual time instant. The value of the timestamp at the intersection of the density functions in (b) matches with the actual instant.	92
4.9	IEEE 33-bus: (a) Topology estimation error, (b) False positive percentage, (c) False negative percentage, (d) Number of islands mistakenly detected, for 1000 runs with various numbers of phasor samples. Dashed lines indicate the mean values of the corresponding metrics.	94
4.10	226-bus: (a) Topology estimation error, (b) False positive percentage, (c) False negative percentage, (d) Number of islands mistakenly detected, for 1000 runs with various numbers of phasor samples. Dashed lines indicate the mean values of the corresponding metrics. The extremely loopy structure of the network contributes to a lower estimation accuracy that translates to a higher number of erroneously identified islands. . . .	95
4.11	1262-bus: (a) Topology estimation error, (b) False positive percentage, (c) False negative percentage, (d) Number of islands mistakenly detected, for 1000 runs with various numbers of phasor samples. Dashed lines indicate the mean values of the corresponding metrics. A very large network with numerous leaf nodes can have a higher number of erroneously detected islands even with a lower topology estimation error when phasor measurement noise is present.	96
5.1	Line removing attack	101
5.2	Attacker performance in IEEE-39 bus system, averaged over 1000 runs, when only the breakers in $\mathcal{E}_p$, are secured.	107
5.3	IEEE-39 bus: Attacker performance, averaged over 1000 runs, when grid operator protects breakers installed at branch 7, 12 in addition to $\mathcal{E}_p$. . .	107
5.4	IEEE-39 bus: Histogram of attacked branches, upon convergence of our proposed algorithm for 1000 runs.	108

5.5	Attacker performance in IEEE-300 bus, averaged over 100 runs.	109
5.6	Schematic of a MG with renewable and diesel generation based resources supplying load of a critical infrastructures along with load frequency control and communication medium prone to possible cyber-attacks.	112
5.7	Block diagram of the proposed cyber-secured MG control architecture .	119
5.8	Schematic of the CHIL-based demonstration of the load frequency control with the proposed defence mechanism against cyber-attack.	121
5.9	Actual false data injection in active power reference of EVS-1 and output active power measurements of EVS-1 and EVS-2.	122
5.10	Performance of the proposed estimator, (a) internal state (e.g. active power output of diesel generation, P_{Diesel}) estimation, (b) Estimate of FDI in control unit, f_a) estimation.	123
5.11	Frequency of the MG as a result of the the load frequency control with proposed defense mechanism-based observer under cyber-attacks both in measurements and control signals.	124
5.12	Output active power of the EVS-1 as a result of the the load frequency control with proposed defense mechanism-based observer under cyber-attacks both in measurements and control signals.	125
5.13	Output active power of the EVS-2 as a result of the the load frequency control with proposed defense mechanism-based observer under cyber-attacks both in measurements and control signals.	125
6.1	Overview of the Proposed End-to-End Cyber-Physical Framework for Smart Grid Control and Monitoring, (a) generic architecture showing interconnected System, Backend, Data Analysis and Visualization layers, (b) deployment of the proposed framework on IEEE-33 bus distribution network	129
6.2	Lab prototype showing the software-hardware co-simulation workflow .	131
6.3	Demonstration-1: DSO reconstructs grid topology, in real time, using comprehensive set of PMU data utilizing proposed cyber-physical framework	134
6.4	Topology estimation error plot for 1000 runs with various number of samples of phasor measurements from 32 PMUs in IEEE 33 bus network used for Demonstration-1. Dashed line indicates the mean error rate. . .	135
6.5	Demonstration-2: PMUs at buses 4 – 6 and 19 – 22 are malfunctioning. DSO reconstructs the grid topology using rest of the available PMU data.	136

6.6	Demonstration-3: DSO-1 and 2 utilize PMU data from Sub-network-1 and 2 respectively and reconstruct the corresponding topologies, in real time.	136
6.7	Topology estimation error plot for 1000 runs with various number of phasor measurement samples from PMUs deployed in their respective sub-networks used for Demonstration-3. Dashed line indicates the mean error rate.	137

List of Symbols

D_{ci}	Load demand of microgrid i (kW)
N_m	Number of microgrids
P_b	Power output of a single battery unit (kW)
$P_{GFL}(t)$	Power provided by the battery, running in Grid-following mode, at time t (kW)
$P_{GFM}(t)$	Power provided by the battery, running in Grid-forming mode, at time t (kW)
$P_{gi}(t)$	Power output from renewable energy sources in microgrid i at t (kW)
T_f	Future time when demand is requested by the microgrids
W_{ti}	Weiner process capturing the uncertainty of renewable power output of microgrid i at t
$\Gamma(\mathbf{P}_g(t), t)$	Total power available in the portfolio of DNO at time t in IMMIG (kW)
Γ_t	Shorthand for $\Gamma(\mathbf{P}_g(t), t)$ (kW)
$\Gamma_t^{\Delta t}$	Total power available in the portfolio of DNO at time t in discrete time in IMMIG (kW)
$\mathbf{P}_g(t)$	Vector containing power output from renewable energy sources from all N_m microgrids
$\mathbf{P}_g^{\Delta t}(t)$	Vector containing power output from renewable energy sources from all N_m microgrids in discrete time

$\gamma_i(P_{gi}(t), t)$	Power available in the portfolio of DNO for microgrid i at time t in IDMMG (kW)
$\hat{\Gamma}(\mathbf{P}_g(t), t)$	Total available power in the portfolio of DNO at time t in IDMMG (kW)
$\hat{\Gamma}_t^{\Delta t}$	Total power available in the portfolio of DNO at time t in discrete time in IDMMG (kW)
$\hat{b}_t^{\Delta t}$	total number of battery units in the portfolio of the DNO at time t in discrete time in IDMMG
$\hat{b}_t$	Total number of battery units in the portfolio of DNO at time t in IDMMG
$\mathcal{I}$	Set of microgrids, $\mathcal{I} = \{1, 2, \dots, N_m\}$
$\mathcal{I}_d$	Set of all microgrids i for which $P_{gi}(T_f) < D_{ci}$
$\mathcal{I}_s$	Set of all microgrids i for which $P_{gi}(T_f) \geq D_{ci}$
μ_{gi}	Drift term of Wiener process W_{ti} (sec^{-1})
ρ_{ij}	Correlation coefficient between renewable power output of microgrids i and j
σ_{gi}	Volatility term of Wiener process W_{ti}
a_{it}	Number of RES assets in the portfolio of the DNO at t in IMM
$b(t)$	Number of battery units in the portfolio of the DNO at time t in IMM
$b_t^{\Delta t}$	Number of battery units in the portfolio of the DNO at time t in discrete time in IMM
b_{it}	Number of battery units in the portfolio of DNO for microgrid i at time t in IDMMG

List of Abbreviations

DER	- Distributed Energy Resources
RES	- Renewable Energy Source
MG	- Microgrid
DNO	- Distribution Network Operator
BES	- Battery Energy Storage
IDMMG	- Individualized Multi-microgrid
IMMG	- Interconnected Multi-microgrid
GFL	- Grid-Following
GFM	- Grid-Forming
CHIL	- Controller hardware-in-the-loop
CPS	- Cyber-Physical System
ICT	- Information and Communication Technology
FDI	- False Data Injection
LFC	- Load-Frequency Control
PMU	- Phasor Measurement Unit
UDP	- User Datagram Protocol
SoC	- State-of-Charge
UIO	- Unknown Input Observer

Chapter 1

Introduction

1.1 Motivation for Renewable-Powered Microgrid

In recent years, the momentum toward carbon footprint reduction has incentivized the proliferation of renewable energy sources (RESs), which in turn has led to a paradigm shift in the landscape of power distribution system operations and control. In the US, RESs are expected to contribute 60% of total generation by 2050. Apart from environmental benefits, RESs are a promising solution to enhance system reliability. In a traditional distribution system, the energy producers are connected to a main grid and supply the load demand of the downstream consumers. Such a vertically integrated system is critically dependent on the main grid where any disruption in the operation of the main grid results in a complete system outage. In the past few years, such disruption has increased in alarming frequency due to weather-related catastrophes. For instance, from 2000 to 2023, 80% of all major U.S. power outages (over 1700 outages), were due to weather-related events such as severe weather, tropical cyclone, wildfire, etc. [1]. It is reported that the US experienced two times more weather-related power outages during the last 10 years (2014 – 2023) than during the previous decade. The NOAA National Centers for Environmental Information (NCEI) reported that about 28 climate disasters in 2023 exceeded billion-dollar damage [2]. Therefore, it is of utmost importance to devise a resilient distribution system architecture to safeguard the system from grid-related contingencies. Here, a potential solution is to utilize RESs since they can be distributed across the system at the consumer level. Such clusters, with local RESs generation and loads, form microgrids (MGs).

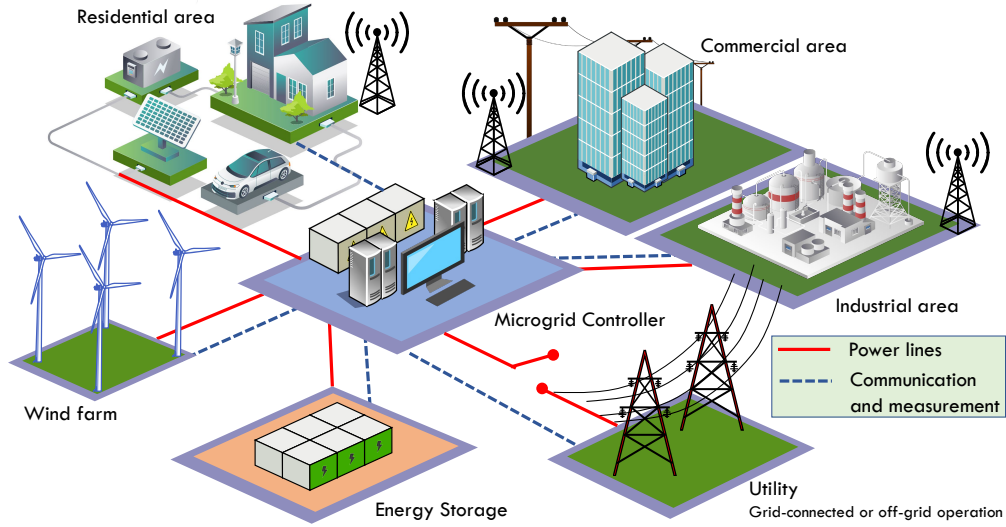


Figure 1.1: A schematic representation of a generic microgrid.

1.1.1 Microgrid: The Grid of the Future

Renewable-powered MGs are envisioned to operate self-sufficiently, utilizing an interconnected network of RESs scattered throughout the network. Fig. 1.1 shows a schematic representation of such MG consisting of RESs, such as rooftop photovoltaic (PV) panels, local wind farms, etc, and energy storage systems such as battery energy sources (BESs). The MGs provide flexibility to operate in two modes (i) grid-connected, and (ii) islanded mode. In grid-connected mode, the main power grid serves as an electrical power source, whereas in the islanded mode of operation, the MGs are disconnected from the main grid. In the islanded mode, the MGs primarily depend on RESs and energy storage systems to support the load demand and to maintain the demand-supply balance. Such flexibility to operate in an islanded mode renders the MGs a viable option for system resiliency during grid outages.

1.1.2 Operational Architecture of Microgrids

The ability to operate the MGs in both grid-connected and islanded modes, along with the advancement of communication and measurement infrastructure has transformed the operational architecture of traditional power systems. Apart from the traditional objective of maintaining voltage and frequency stability, the ability to distribute the

RESs across the network has incentivized techno-economic operational objectives such as coordination and power-sharing between multiple MGs, economic efficiency by participating in electricity markets, etc. To fulfill such objectives, three different levels of control strategies are deployed; (i) primary-level, (ii) secondary-level, and (iii) tertiary-level controls. The objectives of the aforementioned control strategies are briefly described below:

1. *Primary-level* Control: The prime objective of primary-level control is to maintain the voltage and frequency of the system within acceptable limits and to control the active and reactive power-sharing of the distributed energy resources. The synchronization of the resources, to enable plug-and-play operation, is also taken care of in this control layer. The time-scale lies usually between a few *milliseconds* to a few *seconds*.
2. *Secondary-level* Control: Secondary controllers compensate for voltage and frequency deviation, caused by the *primary-level* controllers, and regulate the steady-state deviations of the same. The time-scale lies usually between a few *seconds* to a few *minutes*.
3. *Tertiary-level* Control: The power flow between microgrids and the utility grid, power-sharing strategies between multiple microgrids to facilitate the economic operation of the system are managed by the tertiary control. The interaction with the electricity markets and power procurement decisions are made in this layer utilizing various optimization methods. The time scale lies usually between a few *minutes* to a few *hours*, or in some cases to a few days where interaction with the electricity market is prioritized.

1.2 Impact of Renewable Penetration

1.2.1 Uncertainty of Renewable Power Generation

To maintain system stability, it is essential to maintain demand-supply balance. In this context, renewable penetration poses a major challenge to the reliable operation of the MGs as RESs are inherently uncertain. Such uncertainties lead to overproduction, when generation is more than the demand, and underproduction when demand exceeds the generation. Apart from degrading the system stability, such over and underproduction scenarios result in the risk of generation and load curtailment, causing

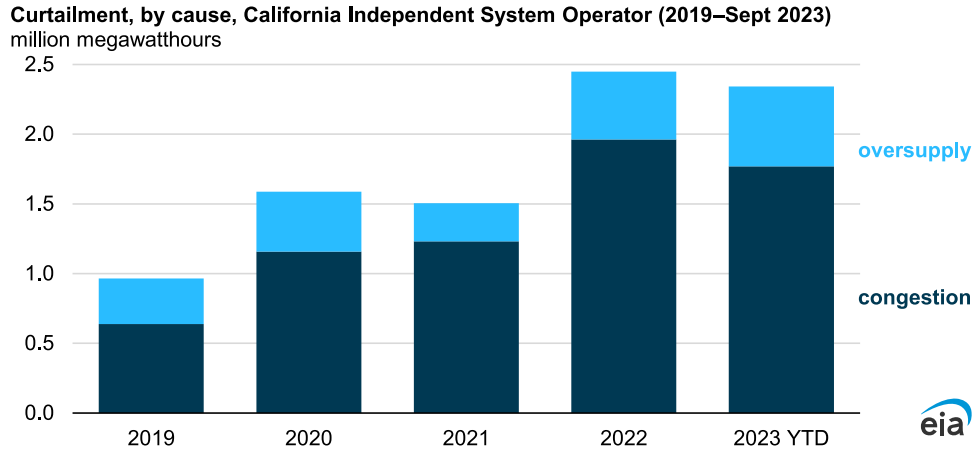


Figure 1.2: Generation curtailment due to increased renewable penetration in the regions under California Independent System Operator (CAISO)

economic loss. In this context, an interconnection between multiple MGs provide a solution to curtailment problems by diversifying different renewable resources through regional collaboration. The utility of such interconnection largely depends on the transmission network design and power line capacities. A poor design of the power network, in terms of the insufficient power line capacities, may lead to network congestion prohibiting the extra amount of generation in one MGs being transferred to other MGs to supply loads. The uncertainty of RESs further exacerbates the network congestion problems thus contributing to additional generation and load curtailment. In 2023 (till September) California Independent System Operator (CAISO) has curtailed more than 2.3 million MWh of wind and solar output [3]. As reported by CAISO and shown in Fig. 1.2, the congestion-related curtailments have increased significantly since 2019 because solar generation has been outpacing upgrades in transmission capacity.

On the other hand, the load curtailment directly affects the electricity consumers resulting into severe economic loss. US Energy Information Administration (EIA) reported a total of 155 events in Texas, spanning over 4500 hours of power outage from 2019 – 2023. In 2023, Texas experienced about 386 hours of power outage [4], as shown in Fig. 1.3. Therefore, it is pertinent to devise a robust and reliable solution to counter the uncertainty of RESs to harness the environmental benefits of the same.

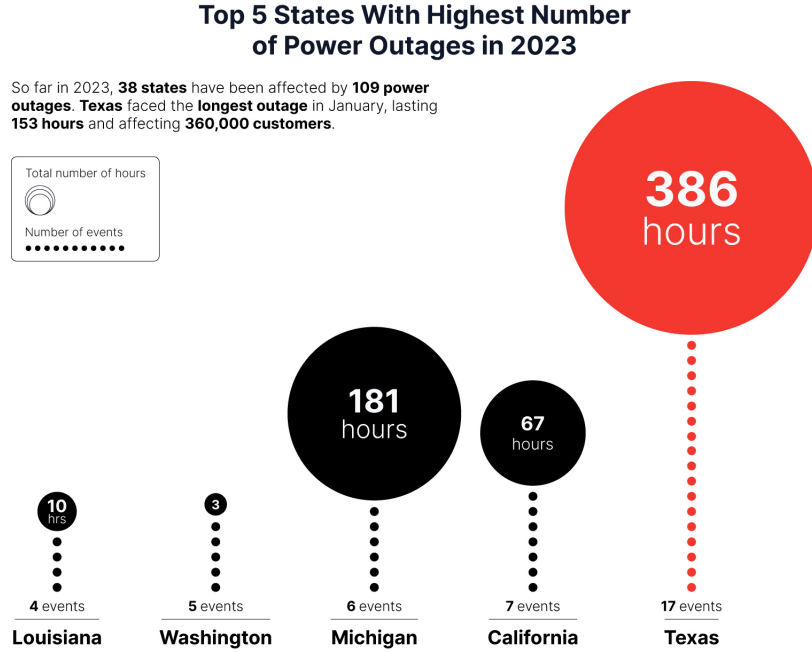


Figure 1.3: Top 5 U.S. states with highest number of power outages

1.2.2 Requirement of Energy Storage Systems

Energy Storage Systems, such as BESs, provide an attractive option to counter both generation and demand uncertainties with the ability to store excess energy when supply exceeds demand and to provide energy when demand exceeds generation. It is projected that the deployment of energy storage will increase significantly, reaching about 200 GW total capacity in 2050 as shown in Fig. 1.4 [5]. Although currently, the most widely deployed storage device is pumped-hydro storage, in the future, Lithium-ion batteries of 4 – 6 hours duration will have the highest share of the total expected capacity [5]. Naturally, BESs with high energy capacity are beneficial in terms of countering the RESs uncertainty, however, it increases the investment and operational cost of the system. For example, Fig. 1.5 shows the increase in system capital cost as the size of a 60 MW Utility-scale battery size is increased [5]. Therefore, for an efficient techno-economic operation of the system, it is essential to determine the required energy capacity without compromising the advantages of self-sufficiency of the future microgrid architecture, primarily powered by RESs and BESs.

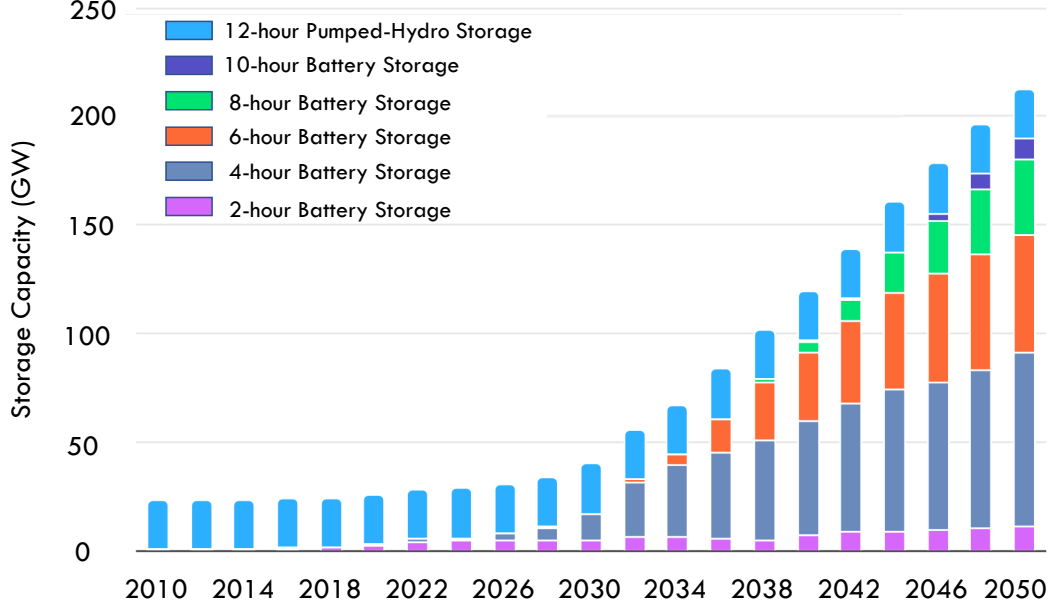


Figure 1.4: Energy storage capacity is expected to grow to about 200 GW by 2050, deploying a range of durations. This translates to about 1,200 gigawatt-hours (GWh) of stored energy.

1.2.3 System Design Challenges of Microgrids

The major challenge in the landscape of RESs-based power distribution system, with multiple MGs, is to mitigate the renewable production uncertainties. As alluded to earlier, a viable option to counter the renewable uncertainty is to utilize energy storage systems, such as BESs. However, in the system planning stage, if the power capability and the energy capacity of the BESs are not properly designed, it may lead to inefficient techno-economic operation of the system. Under the RESs production uncertainty, it is extremely challenging to decide how much power will be needed from the BESs at a future time to avoid both over and underproduction scenarios. In addition, to be able to store (or provide) energy, whenever required, it is essential to ensure the BESs have sufficient energy capacity. Therefore, the key challenges of the MGs are:

- (1) How to decide the amount of power to be dispatched from the BESs, at a future time, given the RESs uncertainty and the load demand, to guarantee demand-supply balance without over and underproduction?
- (2) What is the minimum required BESs energy capacity, to ensure that the BESs

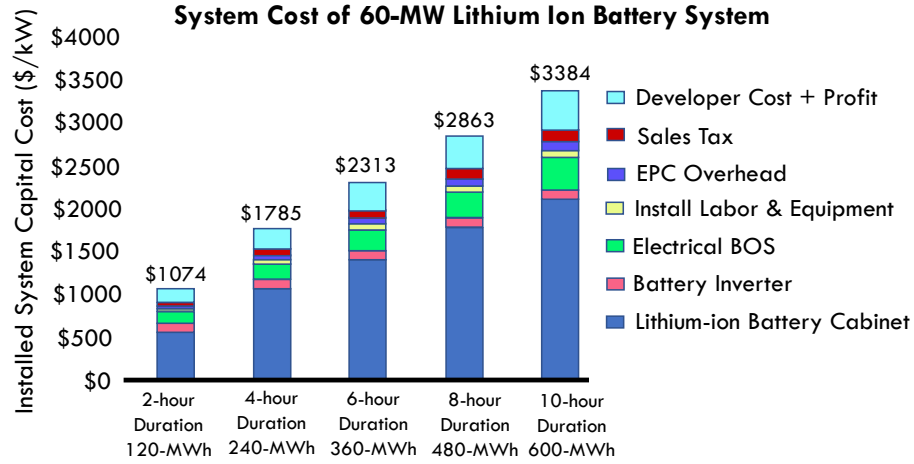


Figure 1.5: Utility scale 60 MW Li-ion Battery Storage Cost for 2 – 10 hours duration

operate within their specified operating range at all times, given the generation and demand uncertainty and the operation time horizon?

Another important aspect of a distribution system with multiple MGs is the interconnection between them. Here, the facility to share power has the potential to harness the benefits of spatial generation diversity to minimize the needed BESs capacity. However, even if the BESs capacity can be reduced by interconnecting the MGs, the economic benefits of such a reduction may be nullified by the cost associated with the transmission lines connecting the MGs. As the transmission line installation and maintenance cost primarily depends on the flow capacity, it is essential to analyze the required capacity of both BESs and transmission lines in unison. In this context, the challenges are to decide:

(3) What is the possible reduction of the required BESs capacity if transmission lines, connecting the MGs, can be installed?

(4) What should be the capacity of the power lines? How does it relate to the required BESs capacity and the operation time horizon? How can we reduce the power line capacity without affecting the system's reliability?

Such design challenges are crucial factors in quantifying and establishing the benefits of RES-powered multi-MG systems.

1.3 System Monitoring and Cyber-attack Resiliency

The proliferation of MGs necessitates a real-time situational awareness of the distribution network due to their spatial diversity and operational flexibility. To efficiently operate the system, the distribution network operator needs real-time measurements of voltages, branch status, etc., from different parts of the network. For a reliable operation of the system, it is required to infer the system status such as contingencies, faults, etc. from the measurements. Such requirement is fulfilled with the deployment of Information and Communication Technology (ICT) infrastructure. Hence the monitoring and control capability of the network operator, intertwined with the ICT, must be enhanced to support the expansion of the MGs in a seamless manner. However, the wireless nature of the ICT renders the system vulnerable to cyber-attacks. Fig. 1.6 illustrates the trend in the electricity disturbance events, categorized by the causes, from 2021 to 2023 [4]. From the figure, it is evident that the three primary causes of electricity outages are weather, attacks, and operational failures. In this context, there are three major requirements to enhance the reliability and resiliency of the distribution system: (i) real-time information on the system topology including the fault status, (ii) cyber-security, and (iii) a monitoring and control infrastructure.

System topology is a crucial information for efficient system operation. The topology information is utilized for various techno-economic purposes such as economic dispatch, unit commitment, fault localization and restoration etc. In general, the network operators obtain the topology information from prior knowledge. However, the flexible operation of the renewable-powered MGs, enabling plug-and-play operation of the same, can change the topology from time to time. Further, extreme weather events can result in branch faults which in turn alters the topology structure. Therefore, it is of extreme importance to infer the topology in real-time from the available field measurements.

The reliance on the measurements, to efficiently operate the system, is further exacerbated by cyber-attacks. The most common type of cyber-attack is False Data Injection (FDI) where the attacker manipulates the measurements transmitted wirelessly to the control center. In the last decade, several security breaches have been reported across the globe. For example, in 2015, a cyber-attack on an electrical power station in Ukraine impacted around 80,000 people causing a blackout [6]. Similarly, the ‘Stuxnet’ malware attack in Iran in 2010 exposed the vulnerability of the energy sector to the cyber-attacks. In 2007, the US Department of Homeland Security (DHS), North American Electric Reliability Corporation (NERC), and the Electricity Information Sharing and

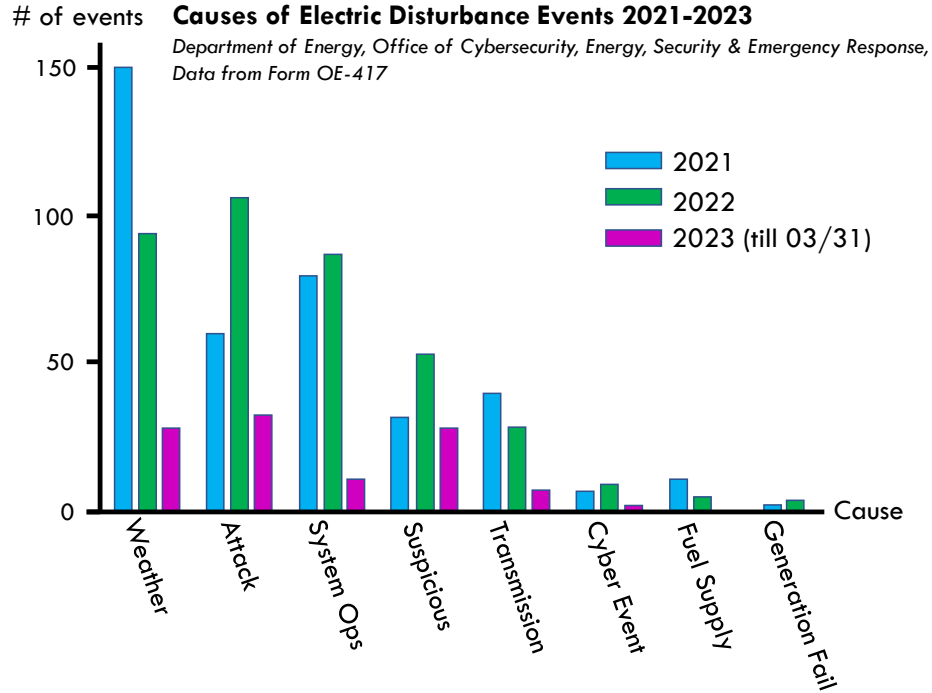


Figure 1.6: Energy storage capacity is expected to grow to about 200 GW by 2050, deploying a range of durations. This translates to about 1,200 gigawatt-hours (GWh) of stored energy.

Analysis Center (E-ISAC) demonstrated the impact of FDI and Denial-of-Service (DoS) attack on a diesel generator operation. In the demonstration, code-named Aurora, the physical destruction, as a result of the cyber-attack, of the generator was showcased. To this end, it is evident that establishing a resilient network monitoring and control framework is the need of the hour.

1.3.1 Situational Awareness Challenges

The situational awareness of a power distribution network provides a holistic view of the network topology, real-time system status, such as branch faults, restoration, etc., and an ICT framework immune to cyber-attacks. In this context, the major challenges are as follows:

- (1) How to infer the system topology and detect changes in the same caused by branch fault (or restoration) from the field measurements?

(2) How to make a resilient system monitoring and control infrastructure that is shielded from malicious cyber-attacks, such as false data injection attacks?

Innovations toward this end require a comprehensive test suite to validate the utility of the same before it is deployed in the system. Here, a cyber-physical testbed capable of integrating different components of a monitoring and control framework, such as data aggregation tools and database, hardware and controller-in-the-loop integration, data-driven algorithm execution infrastructure, system visualization, etc., is required to be developed. Such cyber-physical testbeds should be flexible in terms of supporting various system resources and measurement devices, and scalable.

In summary, a methodology to estimate system topology and topology changes, a cyber-secured measurement and control framework, along with a cyber-physical testbed to validate the aforementioned strategies are essential to guarantee real-time situational awareness of the power distribution systems.

1.4 Contributions and Overview of the Thesis

1.4.1 Uncertainty Mitigation of Renewable-Powerd Microgrids

Section 1.2.3 discusses the major challenges that renewable-based microgrids experience today due to the inherent uncertainty of renewable power generation. Clearly, it is crucial to design and operate the system in a manner that enhances reliability and self-sufficiency by countering the renewable uncertainties. This thesis contributes towards achieving these objectives by proposing energy resource power capability and energy capacity design methodologies for RES-based multi-microgrid systems.

■ In this thesis, **Chapter 2** contributes towards the development of a dynamic battery allocation and control strategy to mitigate the RESs production uncertainty. A power distribution system with multiple microgrids and a distribution network operator (DNO) is considered. Each microgrid has an RES which is uncertain in nature, and is modeled by geometric Brownian motion. The validity of such a model is justified by real-life renewable power generation data from the northeast part of the US. Each microgrid also postulates a critical demand at a predetermined future time. The primary challenge here is to control the battery power output capability in such a way that the demands of all the microgrids, at the requested future time, are met with probability one without any over or underproduction. A review of existing works in this context is

presented to highlight the limitations of the state-of-the-art. A novel electricity market operation is proposed where the RESs and battery resources can be traded. The buyer (or seller) of the RESs and batteries reserves the right to buy (or sell) the power produced by the same at a future time. Under the stochastic model of the RESs production, we propose a dynamic feedback control strategy of the RESs and batteries, via the contracts in the electricity market, to guarantee that the set of realizations of RESs generation, for which there is over or underproduction at a future time when the demand is requested, has probability zero. Thus almost surely we guarantee that any realization of renewable generation will not have over or under-production for the given demand. We demonstrate the utility and scalability of our algorithm by instantiating it to a modified IEEE-33 bus network with two microgrids, and a 225-bus network with 10 microgrids, on a real-time OPAL-RT based hardware-in-the-loop platform.

■ **Chapter 3** contributes towards developing a battery energy capacity design methodology to avoid both generation and load curtailment under renewable uncertainties. To limit load curtailment, the battery capacity should be such that it does not get completely discharged during the operating time horizon. Similarly, generation curtailment is avoided by ensuring that the battery does not hit the maximum allowable charge limit at any time. Utilizing a stochastic control framework, an explicit analytical expression of the minimum required battery energy capacity for MGs, with RESs as the main power source, is derived that provides probabilistic guarantees of limiting generation and load curtailment. For multi-microgrid systems, the facility to share power has the potential to harness the benefits of spatial diversity to further minimize needed BESs capacity. In this context, a stochastic optimization framework to determine how the microgrids should share power to minimize the battery capacity requirement while respecting the transmission line capacity is proposed. To provide a comprehensive system design guideline, a theoretical analysis of the trade-offs between transmission line capacity and battery energy capacity is presented. It is established that for a large operation time horizon, it is even possible to reduce the transmission line capacity while maintaining the least amount of battery.

1.4.2 Situational Awareness and Cyber-Attack Resiliency

Section 1.3.1 outlines the major challenges in detecting power network topology changes in real-time from the field measurements, which is essential to achieve situational awareness of the system.

■ **Chapter 4** contributes towards developing a data-driven methodology to detect and localize changes in a power network topology. The proposed strategy utilizes only bus voltages and phase angles to detect any change and further localize the change. The methodology is built upon machine-learning models and is capable of detecting multiple topology changes (branch outages or restoration) occurring simultaneously or over a time horizon. Further, the time instants of such changes are estimated to aid the system operators in obtaining complete information about the system. Real-time hardware-in-the-loop experiments on a modified IEEE 33-bus, 226-bus, and 1262-bus network, emulated on OPAL-RT real-time simulator, are conducted to demonstrate the efficacy of the proposed methodology. The 1262-bus distribution network comprises of a part the University of Minnesota campus and surrounding suburbs. The detailed experiments, as discussed in Chapter 4 of this thesis, establish the utility of the proposed methodology in large power distribution networks.

■ **Chapter 5** contributes towards developing strategies in making a power distribution system resilient to cyber-attacks. FDI attacks are investigated in two contexts: (i) cyber-topology attacks to disrupt electricity market operation and (ii) measurement and actuator control signal manipulation to destabilize a microgrid. In real-time electricity markets, the electricity prices at different buses in the network are decided based on the locational marginal price (LMP). LMP is derived from the solution to DC optimal power flow which further depends on the grid topology. Cyber-topology attacks, targeting breaker status measurements to manipulate the branch status to make a branch appear offline while it is operational, can falsify the topology information and thus can induce errors in the LMP calculation. The first part of Chapter 5 analyzes such vulnerabilities by identifying the critical breakers in the power network that, if attacked, can lead to a large deviation in LMP from the actual value and disrupt the electricity market. A reinforcement learning-based strategy is proposed to analyze the aforementioned vulnerabilities. As an outcome, it aids the system operators in identifying the critical branches in a network that are to be protected. The proposed strategy is validated on IEEE 33-bus and IEEE 300-bus distribution networks. The second part of Chapter 5 focuses on FDI attacks on state feedback-based load frequency control of microgrids. Load frequency control is required to compensate for frequency deviation in a microgrid caused by renewable generation and/or demand uncertainties. FDI attacks are considered in both measurements and control commands in demonstrating the impact of the same on system stability. A defense mechanism, incorporated in the state feedback controller, is proposed. The methodology is instantiated in a microgrid with electric

vehicles, diesel generators, and loads, emulated inside OPAL-RT real-time simulator. The experimental results corroborate the efficacy of the proposed method in maintaining the stability of the system and in enhancing resiliency against FDI attacks.

■ **Chapter 6** illustrates an end-to-end cyber-physical testbed infrastructure for grid control and monitoring. The testbed is completely developed in the lab and has four key elements: (1) a system layer that embeds a physical or emulated power system network, (2) a data analysis layer to execute real-time data-driven grid analysis algorithms, (3) backend layer with a generic data storage framework which supports multiple databases with functionally different architectures, and (4) visualization layer where multiple customized or commercially available user interfaces can be deployed concurrently for grid control and monitoring. These four layers are interlinked via bi-directional communication channels. Such a flexible and scalable framework provides a cohesive environment to enhance smart grid situational awareness.

Chapter 2

Optimal Multi-microgrid Feedback Policies: Guaranteed Demand Fulfillment under Renewable Uncertainty

2.1 Gist of the Chapter

With increased penetration of Renewable Energy Sources (RESs), the conventional distribution grid is advancing towards interconnected multi-microgrid (IMMG) systems supervised by a Distribution Network Operator (DNO). However, the inherent uncertainty of RESs poses a challenge in meeting the power demand in the microgrids unless sufficient battery storage is maintained. Although the problem of RESs uncertainty and battery storage procurement are partially addressed by the day-ahead, real-time, and ancillary service markets, the existing market mechanisms cannot guarantee demand-supply balance without over and underproduction. In this chapter, we propose a dynamic battery allocation and control strategy, under a novel electricity market mechanism where the RES and battery assets can be traded. To ensure that critical demand is met with a provable guarantee, by a combination of the power output from the battery and RESs, the power capability of the battery reserve is controlled by exchanging the battery units with RESs in the market. Our solution is built upon stochastic feedback control techniques. Under our proposed scheme, the DNO responds

to the evolving uncertainty by dynamically balancing the RESs and battery resources, via contracts made in the electricity market, and eliminates the risk of over or under-production. We derive battery power capability control strategy for multi-microgrid systems in *two* settings: when microgrids *can* (interconnected) and, *cannot* (individualized) share power amongst each other. We present examples under different scenarios with detailed comparison of performance of the proposed algorithm for individualized and shared settings. In particular, the advantages of interconnecting microgrids through savings in battery power requirements under IMMIG over individualized microgrids are quantified. We demonstrate the utility and scalability of our algorithm by instantiating it to a modified IEEE-33 bus network with two microgrids, and a 225-bus network with 10 microgrids, on a real-time OPAL-RT based simulation platform.

2.2 State-of-the-art on Load Demand Fulfillment under Renewable Uncertainty

In recent times, the proliferation of distributed energy resources (DER), propelled by a mounting interest towards the generation and utilization of clean renewable energy, has fostered a paradigm change in the operation and control of power distribution networks. Renewable energy sources (RESs), like solar and wind energy systems, reduce the adverse environmental effects of traditional energy sources; integration of DERs also offers improved grid assistance, efficiency and reliability. As renewables are expected to contribute 60% of total generation by 2050 [7], a rapid growth of ‘prosumers’, equipped with distributed RESs, is envisaged to transform the conventional energy systems to a substantial ecosystem of multiple individual microgrids (MGs). However, due to the intermittent nature of RESs, the MGs are exposed to the risk of underproduction if extra measures such as mechanisms for ancillary services are not present [8]. Similarly, uncertainty of RESs, exacerbated by excess provisioning of ancillary services, such as energy storage, can result in overproduction which increases the operating cost. Usually in power distribution systems, a Distribution Network Operator (DNO) supervises the MGs and provides support to maintain demand-supply balance. To mitigate the production uncertainty, currently the DNO participates in the electricity market where a fixed amount of immediate or future power [9] is exchanged for money. While prices in the futures (day ahead) market are more stable, the prices in the real-time market are more volatile and sensitive to current production levels [10, 11]. Thus, to avoid

the price volatility of the real-time markets for meeting the future demand of the MGs with absolute guarantee, DNO purchases power from the day-ahead market. However, if at the time of realization, there is excess generation, the DNO either has to sell the excess power in the real-time market at a lower price, thus incurring operational losses, or store it locally in batteries, necessitating related capital expenses. In this chapter, we propose a novel real-time electricity market mechanism, where RESs and battery reserves can be traded. The buyer (or seller) of the RESs and batteries reserves the right to buy (or sell) the power produced by the same at a future time. We provide a stochastic differential equation model of RES production, which we validate using field data. Under the stochastic model of the RESs production, we propose a dynamic feedback control strategy of the RESs and batteries, via the contracts in the electricity market, to guarantee that the set of realizations of RESs generation, for which there is over or underproduction at a future time when the demand is requested, has probability zero. Thus almost surely we guarantee that any realization of renewable generation will not have over or under-production for the given demand.

2.2.1 Prior Works

In most of the existing literature, the problem of load demand fulfillment under renewable uncertainties is primarily addressed with the objective of minimizing the operation cost of the system and cost of energy procurement from the electricity market [12]. In this context, schemes based on chance-constrained techniques [13, 14, 15, 16, 17], stochastic programming [18, 19], model-predictive control [20], and various optimization techniques [21, 22, 23, 24] are proposed. In [13], a battery dispatch pattern is decided to minimize battery maintenance cost, RESs generation cost, and load and generation curtailment cost with supply-demand balance constraints, and a chance-constraint on RESs curtailment amount. Similar approaches are presented in [14, 15, 16, 17] in the context of optimal power flow with the assurance of meeting demand-supply balance with a high probability. Such approaches minimize generation or load curtailment to reduce the system operational cost, however, probabilistic guarantees of ensuring zero curtailment amount are not achieved. Another popular methodology to minimize system operation cost by intelligent generation dispatch scheduling, to counter RESs uncertainty, is probabilistic unit commitment [25, 26, 27]. However in existing unit commitment models, loss of load is still allowed, though with high penalty cost, and thus such models are difficult to deploy in a system with critical facilities where load curtailment is prohibited.

Apart from probabilistic unit commitment models, usually considered in ancillary service markets, various stochastic optimization methods are devised to minimize energy procurement expenditure in day-ahead and real-time markets [28, 29, 30]. However, such strategies also cannot guarantee zero over and underproduction. In summary, the methodologies in [14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 25, 26, 27, 24, 28, 29, 30] have the following limitations: (1) minimizes over or underproduction to reduce system operational cost, however, cannot achieve zero over or underproduction simultaneously, allowing additional power transaction with external resources (such as grid in the case of grid-connected systems or grid-forming resources in the case of islanded systems etc.), (2) theoretical guarantees of demand-supply balance with probability *one*, while ensuring zero over or underproduction, are not present.

To prioritize the problem of over or underproduction, a battery charging and discharging policy is proposed in [31], to reduce the power imbalance due to RESs uncertainty with explicit stochastic model for RESs. However, no provable guarantee of meeting the demand is established. Further, analytical approaches [32, 33], data-driven approaches [34], game theoretic approaches [35, 36] are proposed in prior works to improve power balance in systems with multiple MGs, however, self-sufficiency is not guaranteed, that is demands are met with high probability but not with absolute guarantee while ensuring minimum amount of power is drawn from (or fed to) the grid to minimize operational cost. In the case of grid-independent MGs, the power imbalance due to over or under-production is addressed by grid-forming resources. In this context, [37, 38] propose model-predictive control frameworks, however, power curtailments are considered a viable option and theoretical analysis of minimization of the power drawn from the grid-forming resources are not present. Apart from self-sufficiency, there is another potential advantage embedded in an ecosystem comprising of multiple MGs that needs to be studied. In traditional Individualized Multi-microgrid (IDMMG) systems, energy sharing among MGs is not allowed which leads to large energy reserve requirement. In contrast, in IMMIG, collaborative operation across multiple facilities allows for risk pooling between RESs and reduces the storage needed to maintain the guarantee of demand-supply balance. However, the additional benefits of IMMIG over IDMMG must be quantified to justify the utility of IMMIG without compromising the self-sufficiency of the entire system. In this context, a power management technique to improve frequency regulation of a multi-microgrid system, considering dynamic generation control based on electricity price, is devised in [39]. Further, [40, 41, 42] propose various power sharing methodologies among multiple MGs, however, it is assumed that

the centralized grid can assist with over or underproduction of each MG in the entire system. Such methodologies address the complex problem of interconnection between multiple MGs to enhance cost-effectiveness, however, they cannot guarantee that for almost all realizations of the stochastic RESs generation, there will be no over and underproduction. In summary, to the best of the authors' knowledge, solutions that simultaneously guarantee zero overproduction and underproduction for every realization of stochastic RESs generation, are not explored in existing works. In this chapter, we explore such solutions and establish that our proposed methodology achieves demand-supply balance, without over and underproduction, with probability one; that is, there will be no over and underproduction at a future time, when the demand is requested, for almost all realizations of stochastic RESs generation.

2.2.2 Key Contributions of the Chapter

Consider an Interconnected Multi-microgrids (IMMG) framework consisting of centralized battery storage and multiple MGs with critical infrastructures that require a *guarantee* of supplying the demand requested at a pre-specified future time. Each MG has uncertain RES, thus posing a risk of being unable to meet the local critical demand. To mitigate this uncertainty, the DNO, which supervises the IMMG, participates in an electricity market and controls the battery storage, via the contracts in the electricity market, to support the MGs. The focus of the chapter is on the operational stage where battery power capability is controlled by the DNO to meet the power demand of each MG at the requested future time, without over or underproduction. The major contributions are summarized below:

- (i) We propose an electricity market operation, to counter the RESs uncertainty, where the contracts are made in terms of battery energy storage and RES assets to be utilized at a future time when the demand is requested. Under our proposed market mechanism, after the initial contracts are purchased, generating a portfolio of fixed power (henceforth battery power) and uncertain power (from RES assets), additional future contracts are dynamically procured by the DNO as RESs generation varies with time. The battery power capability is changed to maintain budget neutrality. The associated problem of how much battery to provision initially to ensure that the precisely needed capacity at the time of use is available without under or overproduction is also addressed.
- (ii) Our solution is built-upon stochastic feedback control policy where the power capability of the battery storage is controlled in a constrained manner (discussed in detail

in Section 2.4), once the initial power provisioning is decided. We establish that the optimal initial power provisioning of battery storage decided by our algorithm is such that (i) any smaller provisioning will lead to scenarios where demand cannot be met, (ii) any larger provisioning will lead to scenarios where generation exceeds the demand. We emphasize that our stochastic feedback policy, allows the DNO to avoid both cases (i) and (ii) above *almost surely*, i.e., with probability *one*. We show that the solutions to the probabilistic chance-constrained optimization problems (surplus and deficit problems, discussed in detail in Section 2.4), usually considered in the literature for load/generation curtailment reduction up to a small non-zero probability, converge to the solution to a stochastic differential equation (SDE) in the limit. Thus, we provide a limit of performance, in terms of minimum initial provisioning required to eliminate the risks of underproduction and overproduction at the future pre-specified time, when critical demand is requested, with provable guarantee.

(iii) To deploy our methodology in a realistic power distribution framework, we provide a practical algorithm for discrete-time battery power control. We demonstrate the utility of our methodology by instantiating it to a 33-bus power distribution network, with two MGs, emulated in a real-time OPAL-RT based simulation platform. Further, to showcase the scalability of our proposed technique, we conduct a real-time OPAL-RT based simulation with 10 MGs in a 225-bus distribution network, powered by Xcel Energy in the city of Minneapolis, USA.

2.3 System Description

We consider a distributed framework composed of N_m geographical regions, where each region has a wind farm and an MG. Each MG has an RES to supply its local demand, whereas the RESs in the wind farms are connected to the upstream grid. The MGs, and a centralized battery storage [43] are connected to a main substation which is managed by a DNO. The main substation is further connected to the upstream grid, thus creating a vertically integrated multi-microgrid power distribution network. Fig. 2.1 shows the hierarchical architecture of the system under consideration. Even if the MGs have RESs to supply their demands, the uncertain nature of RESs poses the risk of either power deficit, when generation is less than demand, or excess power that forces curtailment, when generation exceeds demand. In IDMMG, where the MGs do not share power with each other, each MG independently can minimize such risk by utilizing the centralized battery storage, maintained by the DNO, from the main substation. However, such an

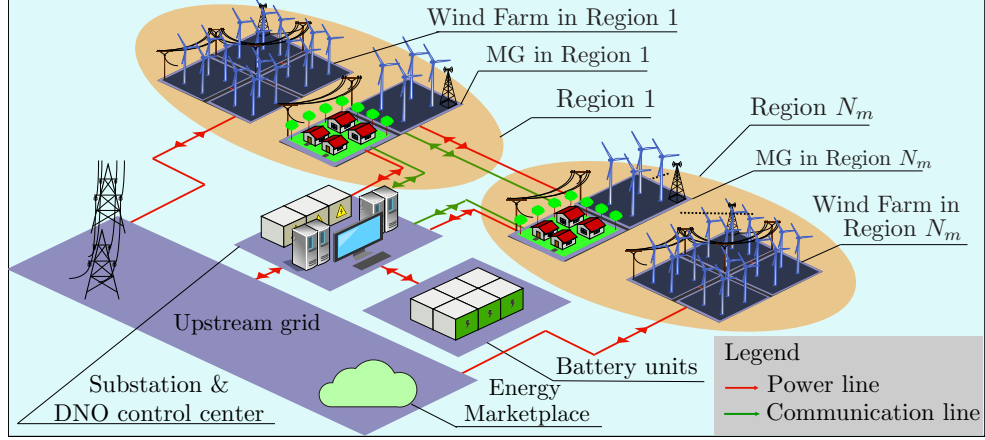


Figure 2.1: IMMIG architecture with centralized BESS and microgrids with RESs.

operation requires the DNO to maintain a large battery energy storage system (BESS) [44]. In contrast, an IMMIG, as shown in Fig. 2.1, enables MGs to share power among each other to maintain demand-supply balance, where excess generation from one MG can be utilized to minimize the deficit of another, instead of power curtailment, thus reducing the power drawn (or fed to) the BESS in the main substation. Without loss of generality, we consider that the BESS, maintained by the DNO, is composed of multiple smaller batteries, which we term as battery units for ease of exposition. Therefore, we consider that the DNO is able to vary the power capability of the BESS by varying the number of battery units utilized. Note that, irrespective of the operational architecture of the system, DNO must ensure the BESS at the main substation has sufficient power capability to support the downstream MGs. If not, DNO must utilize the RESs of the wind farms, located across N_m regions, to provide further support to the MGs via the upstream grid. Here, we consider an energy marketplace where the DNO can buy (sell) power from (to) the wind farms. A communication layer is used to transmit real-time power measurements of the RESs from each region to aid the DNO for efficient battery resource allocation. To minimize the dependency on the upstream grid, the DNO needs to estimate the optimal (minimum) power capability of the battery storage that ensures load demands of all the MGs are met, irrespective of the irregularity of the RESs. In the subsequent sections we formally introduce the problem and provide a solution along with a comparison between IMMIG and IDMMIG, in terms of battery requirement without affecting self-sufficiency of the system.

2.4 Problem formulation

Let $\mathcal{I} = \{1, 2, \dots, N_m\}$ denote the set of regions present in the distribution network. Each RES in the region $i \in \mathcal{I}$ produces $P_{gi}(t)$, units of power at time t , which is stochastic in nature and is modeled by Geometric Brownian Motion (GBM) given as:

$$dP_{gi}(t) = \mu_{gi}P_{gi}(t)dt + \sigma_{gi}P_{gi}(t)dW_{ti}, \quad (2.1)$$

with an initial condition, $P_{gi}(0) > 0$, for all $i \in \mathcal{I}$. The constants μ_{gi} and σ_{gi} are the drift and volatility terms respectively and W_{ti} denotes a Wiener process. We empirically verify the appropriateness of (2.1) with RESs generation data in Section 2.6. The Wiener processes W_{ti}, W_{tj} are correlated according to $dW_{ti}dW_{tj} = \rho_{ij}dt$, $i, j \in \mathcal{I}$, where ρ_{ij} is the correlation coefficient between RESs output of closely sited regions i and j [45, 46]. Applying Ito's lemma [47], it follows that,

$$dP_{gi}dP_{gj} = \sigma_{gi}\sigma_{gj}\rho_{ij}P_{gi}P_{gj}dt, \quad i, j \in \mathcal{I}. \quad (2.2)$$

Each MG also postulates a (constant) critical power demand of D_{ci} units, $i \in \mathcal{I}$, at a future time denoted by T_f . In the case $P_{gi}(T_f) \geq D_{ci}$, MG i can ensure, via its local generation, sustained operation of its own load demand. In addition, the excess generation of $D_{ci} - P_{gi}(T_f)$ units can be used to supply the load demands of other MGs. Note that, in IDMMG, this excess generation would lead to an undesirable overproduction due to no cooperation between the MGs. On the other hand if $P_{gi}(T_f) < D_{ci}$, the deficit can be provided by excess RESs generation at other MGs in IMMIG and/or by the battery storage maintained by the DNO (see Fig. 2.1). To avail these benefits of IMMIG, MGs participate in a contract with the DNO, which controls the power output of the battery to make the system as self-sufficient as possible. At time T_f , assume that a total of m , with $0 \leq m \leq N_m$, MGs will have enough individual generation to supply their local demand. We denote the set of indices of the MGs with sufficient generation at time T_f by $\mathcal{I}_s$, with $|\mathcal{I}_s| = m$, and let $\mathcal{I}_d$ be the set of indices of the MGs not capable of supplying their own local demand by RESs generation individually at T_f ; thus $|\mathcal{I}_d| = N_m - m$. We have $P_{gi}(T_f) \geq D_{ci}$ if $i \in \mathcal{I}_s$ and $P_{gi}(T_f) < D_{ci}$ if $i \in \mathcal{I}_d$. The DNO is responsible for supporting MG j , $j \in \mathcal{I}_d$, by utilizing the surplus power (if available) from other MGs, and/or centralized battery units. Therefore, a focus on the time evolution of the RESs from $t = 0$ till the time $t = T_f$ is required to guarantee that the demand is met at the requested time T_f . At time T_f , there are two possibilities:

(1) $\sum_{i=1}^{N_m} P_{gi}(T_f) \geq \sum_{i=1}^{N_m} D_{ci}$: In this case, collectively the microgrids have sufficient RESs generation to supply critical load demands of each individual microgrid. Hence, no additional power from battery units of DNO, is required. (2) $\sum_{i=1}^{N_m} P_{gi}(T_f) < \sum_{i=1}^{N_m} D_{ci}$: Here, the total deficit of $\sum_{i=1}^{N_m} [D_{ci} - P_{gi}(T_f)]$ units of power is required from the battery units of DNO to support the critical load demand of the MGs. Note that, if MG $j \in \mathcal{I}_d$ also has a local battery storage, capable of supplying b_j kW of power, it can adjust its local demand from D_{cj} kW to $(D_{cj} - b_j)$ kW, which would be requested from the DNO, and the MGs with surplus power. However, to address both the possibilities, DNO is at risk of over or underproduction, as the amount of RESs generation at T_f is not deducible *a priori* due to inherent RESs uncertainty. In particular, if the number of battery units, maintained by the DNO, has more power capability than required by the MGs, it will lead to resource wastage, and if the same cannot produce the power required by the MGs, the system will have net power deficit. To mitigate this risk, DNO participates in an energy marketplace to make contracts with the wind farms. Here we consider an energy marketplace where the RESs and battery units are traded. In particular, the contracts made in such a marketplace specify the number of RESs and the battery units to be bought (sold) at a pre-specified future time, and the parties involved in the contracts are obliged to buy (sell) the power produced by the aforementioned number of resources (RESs and/or battery units). In this chapter, we show that such an energy marketplace can harness the benefits of zero over and underproduction, for the DNO, with probability one. At $t=0$, DNO receives the measurement of RESs power output, $P_{gi}(0)$, from each wind farm i . Upon receiving the measurements, it signs a contract, specifying the number of RESs, denoted by $a_i(0, P_{gi}(0))$, to be bought (sold) from (to) the i -th wind farm at the future time T_f , to support the MGs. Along with that, it decides the number of battery units, denoted by $b(0, P_{g1}(0), \dots, P_{gN_m}(0))$ each producing constant P_b amount of power, to be utilized at T_f . To enjoy the guaranteed support from DNO, the MGs are required to sponsor the initial allocation, $a_i(0, P_{gi}(0)), b(0, P_{g1}(0), \dots, P_{gN_m}(0))$. In return, DNO needs to ensure that at the future time T_f , the combination of the power bought (sold) from (to) the marketplace, which is governed by the total number of RESs under the contracts, and the power produced by the battery units, will be sufficient to supply the MGs, if required, without any over or underproduction. Naturally, the amount of power bought (or sold) from (or to) the marketplace, will depend on RESs generation of the regions and needs to be readjusted if RESs generation changes with time. Thus, DNO needs to solve a resource control problem, by signing additional contracts with the wind farms

at each time $t \in (0, T_f)$, to update the number of RESs it needs to buy (sell) from (to) the marketplace and by updating the number of battery units it needs to maintain, to support the MGs without any over or underproduction of power. Here, we propose that DNO maintains a power portfolio given by,

$$\begin{aligned} \Gamma(P_{g1}(t), \dots, P_{gN_m}(t), t) = & \sum_{i=1}^{N_m} [a_i(t, P_{gi}(t)) P_{gi}(t)] \\ & + b(t, P_{g1}(t), \dots, P_{gN_m}(t)) P_b, \end{aligned} \quad (2.3)$$

where $b(\cdot)$ is the number of battery units, each producing constant P_b amount of power and $a_i(\cdot), i \in \mathcal{I}$, is the number of RESs to be bought (sold) from (to) the wind farm i as per the contracts, at time t . For brevity of notation we denote the values of $a_i(t, P_{gi}(t))$ and $b(t, P_{g1}(t), \dots, P_{gN_m}(t))$ by a_{it} and b_t respectively. Note that the initial allocation, a_{i0}, b_0 , determines the initial value of the portfolio. As mentioned earlier, as per the contracts with the wind farms, the DNO agrees to buy (sell) the power produced by the initially allocated amount of RESs at time T_f , which changes due to the change in the RESs production with time. Further, under our proposed market mechanism, once the initial allocation amount is decided at $t=0$, at any future time t , the number of RESs and battery units are traded in an equal ratio of the power produced by them at t . Thus, after the initial allocation, if DNO voluntarily changes the amount of power bought (sold) from (to) the wind farms, by changing a_{it} , at a future time t , the power change must be compensated by exchanging with the battery units in the portfolio of the DNO. Therefore, DNO has the following constraint,

$$\sum_{i=1}^{N_m} [da_{it}(P_{gi}(t) + dP_{gi}(t))] + db_t P_b = 0, \text{ for } t \in (0, T_f). \quad (2.4)$$

We emphasize that, once the initial allocations are made at $t=0$, the constraint (2.4) ensures that the change in the amount of power available in the portfolio is only due to the change in the RESs generation. Such constraint ensures the transactions in the power portfolio remain commensurate with the initial allocations and thus no exogenous amount of power is infused in the power portfolio at any time $t \in (0, T_f)$. DNO needs to determine a_{it}, b_t for all $t \in (0, T_f)$ and $i \in \mathcal{I}$ such that at $t=T_f$, it can always support the MGs, without any over or underproduction, from its power portfolio $\Gamma(\cdot)$. The DNO

solves the following problem: Given, $P_{g1}(0), \dots, P_{gN_m}(0)$, under the model (2.3),

$$\text{find } (\{a_{it}\}, b_t)_{t \in [0, T_f]}, i \in \mathcal{I} \quad (2.5)$$

$$\text{such that, } \mathbb{P} \left[\sum_{i=1}^{N_m} [a_{iT_f} P_{gi}(T_f)] + b_{T_f} P_b \right.$$

$$\left. - r_i \sum_{i=1}^{N_m} [D_{ci} - P_{gi}(T_f)] = 0 \right] = 1,$$

$$P_{gi}(t), t \in (0, T_f] \text{ satisfies (2.1) and (2.2),}$$

$$\sum_{i=1}^{N_m} [da_{it}(P_{gi}(t) + dP_{gi}(t))] + db_t P_b = 0, \text{ for } t \in [0, T_f],$$

where, $r_i = 1$ if $\sum_{i=1}^{N_m} [D_{ci} - P_{gi}(T_f)] > 0$, and 0 otherwise. Let $(\{a_{it}^*\}, b_t^*)_{t \in [0, T_f]}$ denote the solution to (2.5). In the sequel, we show that the initial allocation a_{i0}^*, b_0^* obtained by the DNO by solving problem (2.5) is the minimum provisioning required to meet the demand of the MGs at time $t = T_f$ with probability one. In particular, this provides a limit of performance for any policy taken by the DNO under the model (2.1)-(2.3) and constraint (2.4) for always meeting the demand of the MGs. In the subsequent analysis, we present two sub-problems (surplus and deficit problems) to address over and underproduction respectively. In Proposition 1 we show that our proposed solution to (2.5) provides a limit of performance, achieving zero over and under-production simultaneously with probability one. Let $\mathbb{T}_k := \{0, \Delta_k, 2\Delta_k, \dots, T_f\}$ denote a discrete time partition sequence of the interval $[0, T_f]$, with $\Delta_k := \frac{T_f}{2^k}, k \in \mathbb{N}$, being the time discretization. Further, let $\varepsilon_k := \frac{\epsilon}{2^k} > 0$, where $\epsilon \in (0, 1), k \in \mathbb{N}$. The following problems are of interest:

1. Surplus problem: Provides maximum allowable initial provisioning such that supply at time $t = T_f$ is not greater than the required power demand with high probability. In

particular, given, $P_{g1}(0), \dots, P_{gN_m}(0)$, and probability ε_k ,

$$\xi_k := \underset{(\{a_{it}\}, b_t)_{t \in \mathbb{T}_k}}{\text{maximize}} \quad \sum_{i=1}^{N_m} [a_{i0} P_{gi}(0)] + b_0 P_b \quad (2.6)$$

subject to $P_{gi}(t + \Delta_k) \sim F(P_{gi}(t), \Delta_k), \forall t \in \mathbb{T}_k$

$$(a_{i(t+\Delta_k)} - a_{it}) P_{gi}(t + \Delta_k) \quad (2.7)$$

$$+ (b_{(t+\Delta_k)} - b_t) P_b = 0, \forall i, t \in \mathbb{T}_k$$

$$\mathbb{P} \left[\left(\sum_{i=1}^{N_m} [a_{iT_f} P_{gi}(T_f)] + b_{T_f} P_b \right. \right. \\ \left. \left. - r_i \sum_{i=1}^{N_m} [D_{ci} - P_{gi}(T_f)] \right) > 0 \right] \leq \varepsilon_k,$$

where, $F(P_{gi}(t), \Delta_k)$ is the probability distribution of $P_{gi}(t+\Delta_k)$ according to (2.1), (2.2) and (2.7) is the discrete-time version of (2.4). We use the notation $P_{gi}(t + \Delta_k) \sim F(P_{gi}(t), \Delta_k)$ to denote that the random variable $P_{gi}(t + \Delta_k)$ follows the probability distribution $F(P_{gi}(t), \Delta_k)$. A related problem is the deficit problem formulated as,

2. Deficit problem: Given, $P_{g1}(0), \dots, P_{gN_m}(0)$, and probability ε_k ,

$$\Psi_k := \underset{(\{a_{it}\}, b_t)_{t \in \mathbb{T}_k}}{\text{minimize}} \quad \sum_{i=1}^{N_m} [a_{i0} P_{gi}(0)] + b_0 P_b \quad (2.8)$$

subject to $P_{gi}(t + \Delta_k) \sim F(P_{gi}(t), \Delta_k), \forall t \in \mathbb{T}_k$

$$(a_{i(t+\Delta_k)} - a_{it}) P_{gi}(t + \Delta_k)$$

$$+ (b_{(t+\Delta_k)} - b_t) P_b = 0, \forall i, t \in \mathbb{T}_k$$

$$\mathbb{P} \left[\left(\sum_{i=1}^{N_m} [a_{iT_f} P_{gi}(T_f)] + b_{T_f} P_b \right. \right. \\ \left. \left. - r_i \sum_{i=1}^{N_m} [D_{ci} - P_{gi}(T_f)] \right) < 0 \right] \leq \varepsilon_k.$$

The deficit problem provides the minimum initial allocation such that with high probability, the supply available at T_f is at least the required power.

Proposition 1. *Let ξ_k and Ψ_k be the optimal value of problems (2.6) and (2.8) respectively. Then, $\Psi_k \leq \xi_k$ for all k . Further, $\lim_{k \rightarrow \infty} \Psi_k = \sum_{i=1}^{N_m} [a_{i0}^* P_{gi}(0)] + b_0^* P_b = \lim_{k \rightarrow \infty} \xi_k$.*

Proof. Please see Appendix A.1. □

Section 2.5 of this chapter presents the proposed strategy of the DNO to find the control policy $(\{a_{it}\}, b_t), t \in [0, T_f]$ under both IDMMG and IMMIG framework. In Section 2.6,

we compare the solutions to the surplus and deficit problems with our proposed solution to establish the benefits of our scheme.

2.5 Solution Methodology

2.5.1 Individualized Multi-microgrid System

First, we consider the case of IDMMG where power sharing among the MGs is not possible and each MG is treated independently by the DNO *i.e.* available power in the portfolio maintained by the DNO for each microgrid $i \in \mathcal{I}$ is:

$$\gamma_i(P_{gi}(t), t) := a_i(t, P_{gi}(t))P_{gi}(t) + b_i(t, P_{gi}(t))P_b, \quad (2.9)$$

for $0 \leq t \leq T_f$, where, $a_i(\cdot)$, $b_i(\cdot)$, decide the number of RESs allocated from the wind farm i and the number of battery units, respectively, for MG $i \in \mathcal{I}$. For brevity of notation we denote the values of $a_i(t, P_{gi}(t))$ and $b_i(t, P_{gi}(t))$ by a_{it} and b_{it} respectively. Then the constraint (2.4) becomes:

$$da_{it}(P_{gi}(t) + dP_{gi}(t)) + db_{it}P_b = 0, \text{ for all } i \in \mathcal{I}, \quad (2.10)$$

and the total power required from the portfolios of DNO is:

$$\hat{\Gamma}(P_{g1}(t), \dots, P_{gN_m}(t), t) = \sum_{i=1}^{N_m} \gamma_i(P_{gi}(t), t), \quad 0 \leq t \leq T_f. \quad (2.11)$$

To ensure the power demand of MG i is met at the requested time T_f , DNO needs to supply the deficit of $D_{ci} - P_{gi}(T_f)$ units of power if the MG is not capable of supplying its own demand from the RESs generation, and zero otherwise. Thus the i^{th} power portfolio should satisfy

$$\gamma_i(P_{gi}(T_f), T_f) = \begin{cases} 0 & \text{if } P_{gi}(T_f) \geq D_{ci} \\ D_{ci} - P_{gi}(T_f) & \text{if } P_{gi}(T_f) < D_{ci}. \end{cases} \quad (2.12)$$

As the amount of renewable generation at T_f is not deducible and also not known a priori due to inherent uncertainty [48], we provide a strategy to determine the numbers a_{it}, b_{it} for all $t \in [0, T_f)$, satisfying the terminal condition (2.12), such that the critical demand of D_{ci} , $i \in \mathcal{I}$, units is met *almost-surely* at time T_f in the following theorem.

Theorem 1. Under the constraint (2.10), suppose a_{it} and b_{it} in case of IDMMG are given by:

$$a_{it} = -\Phi \left[\frac{\ln \left(\frac{D_{ci}}{P_{gi}(t)} \right) - \frac{\sigma_{gi}^2}{2} (T_f - t)}{\sigma_{gi} \sqrt{(T_f - t)}} \right],$$

$$b_{it} = \frac{D_{ci}}{P_b} \Phi \left[\frac{\ln \left(\frac{D_{ci}}{P_{gi}(t)} \right) + \frac{\sigma_{gi}^2}{2} (T_f - t)}{\sigma_{gi} \sqrt{(T_f - t)}} \right], \quad t \in [0, T_f]$$

where, $\Phi(\cdot)$ is the standard normal cumulative distribution function, then the terminal condition specified by (2.12) is satisfied almost surely at time T_f .

Proof. Please see Appendix A.2. □

It follows that, in case of IDMMG, the total power from the battery units, $\hat{b}(t) := \sum_{i=1}^{N_m} b_{it}$, for all $t \in [0, T_f]$ can be determined by evaluating the numbers as per Theorem 1 and utilizing the power portfolio given in (2.9) and (2.11), while the total power dispatched from power portfolios at $t = T_f$ is:

$$\hat{\Gamma}(P_{g1}(T_f), \dots, P_{gN_m}(T_f), T_f) = \sum_{i=1}^{N_m} \max(D_{ci} - P_{gi}(T_f), 0).$$

2.5.2 Interconnected Multi-microgrid System

In this section we consider the case of IMMIG where the power portfolio exercised by the DNO is given by (2.3). Here a single power portfolio is utilized in contrast to the N_m independent ones considered in IDMMG. The IMMIG needs to ensure that the DNO is able to make up for the deficit $D_{ci} - P_{gi}(T_f)$, $i \in \mathcal{I}_d$ at $t = T_f$ from the surplus generation $P_{gj}(T_f) - D_{cj}$, $j \in \mathcal{I}_s$, thus requiring less amount of power from the centralized battery units. In the following sections, we provide a strategy of determining the numbers $a_{it}, b_t, i \in \mathcal{I}$ for all $t \leq T_f$, that ensures that the power deficit is optimally met with guarantees. For brevity of notation, let the value of power portfolio, $\Gamma(\cdot)$ at any time $t \in [0, T_f]$ be denoted by Γ_t . Similar to the terminal condition (2.12) for IDMMG, the terminal condition for IMMIG is given by,

$$\Gamma_{T_f} = \max(\sum_{i=1}^{N_m} (D_{ci} - P_{gi}(T_f), 0)). \quad (2.13)$$

To determine a_{it}, b_t , the DNO first needs to find a solution to Γ_t , under constraint (2.4), for all $t \in [0, T_f]$ that satisfies (2.13) at $t = T_f$. However, finding a solution analytically

is intractable due to higher dimensionality (see Appendix A.3). Hence, we provide a numerical solution in the following sections.

Let $\mathbf{R} := [\frac{\mu_{g1}}{\sigma_{g1}} \dots \frac{\mu_{gN_m}}{\sigma_{gN_m}}]^T$ and $\boldsymbol{\rho}$ be the correlation matrix of W_{ti} , and define $\boldsymbol{\eta}_g := [\eta_{g1} \dots \eta_{gN_m}]^T = \mathbf{L}^{-1}\mathbf{R}$, where $\mathbf{L}$ is a lower triangular matrix with real and positive diagonal elements defining the Cholesky factorization of $\boldsymbol{\rho}$, i.e. $\boldsymbol{\rho} = \mathbf{L}\mathbf{L}^T$. We define, $Z_t := e^{-\sum_{i=1}^{N_m} S_i W_{ti} - \frac{1}{2} \sum_{i=1}^{N_m} \eta_{gi}^2 t}$, where $[S_1 \dots S_{N_m}] = \boldsymbol{\eta}_g^T \mathbf{L}^{-1}$. Let W_{ti} be defined on the probability space $(\Sigma, \mathcal{F}, \mathbb{P})$, then, we can transform the probability measure from $\mathbb{P}$ to $\mathbb{Q}$ in the following way: $\mathbb{Q}(A) = \int_A Z_t(w) d\mathbb{P}(w)$, for $A \in \mathcal{F}$ [47]. Let us denote the expected value of a random variable under this transformed probability measure by $\bar{\mathbb{E}}[\cdot]$ and let $\mathbf{P}_g(t) := \{P_{g1}(t), \dots, P_{gN_m}(t)\}$. The following theorem provides the solution to Γ_t for all $t \in [0, T_f]$ under this transformed probability measure.

Theorem 2. *Under constraint (2.4), the solution to Γ_t under the transformed probability measure $\mathbb{Q}$ is given by,*

$$\Gamma_t = \bar{\mathbb{E}}[\max(\sum_{i=1}^{N_m} (D_{ci} - P_{gi}(T_f), 0)) | \mathbf{P}_g(t)]. \quad (2.14)$$

Proof. Please see Appendix A.4. □

From Theorem 2, under the transformed probability measure, the value of Γ_t for any $t \leq T_f$ is the expected value of Γ_{T_f} , which can be determined numerically as described below, to find the numbers, $a_{it}, b_t, i \in \mathcal{I}$ for all $t \in [0, T_f]$.

Let the time horizon $[0, T_f]$ be partitioned into N equal intervals $0 = t_0 < t_1 < \dots < t_N = T_f$ such that $t_n = n\Delta t, n \in \{0, 1, \dots, N\}$ where $\Delta t = T_f/N$ denotes the length of each interval. The vector of discretized RESs generation values at $t = t_n$ is denoted by $\mathbf{P}_g^{\Delta t}(t_n) = [P_{g1}^{\Delta t}(t_n) \dots P_{gN_m}^{\Delta t}(t_n)]^T$ and the corresponding power portfolio is denoted by $\Gamma_{t_n}^{\Delta t}$. We assume that at the end of each time interval, each RES output can either increase or decrease by a constant factor of $u_i (\geq 1)$, $d_i (< 1)$ respectively, compared to the current values, with certain probabilities. Here, $\mathbf{P}_g^{\Delta t}(t_{n+1})$ can take one of the 2^{N_m} possible values with corresponding probabilities $P_k \in \mathbf{P} \in \mathbb{R}^{2^{N_m} \times 1}$ and $\sum_{k=1}^{2^{N_m}} P_k = 1$. The time evolution of $\mathbf{P}_g^{\Delta t}(t_n)$ is shown in Fig. 2.2 for two consecutive time intervals and $N_m = 2$ for brevity of exposition. Without loss of generality, imposing the constraint, $u_i d_i = 1, i \in \mathcal{I}$, we can estimate P_k, u_i, d_i , given $\mathbf{P}_g^{\Delta t}(t_n), \mu_{gi}, \sigma_{gi}, \rho_{ij}, i, j \in \mathcal{I}$, by comparing expectation, variance and covariance of the stochastic processes, under the transformed probability measure, with that from the discrete N_m -ary tree. Let $\mathbf{U} \in \mathbb{R}^{2^{N_m} \times N_m}$ denote a matrix where k^{th} row of $\mathbf{U}$ contains the movement factors of $\mathbf{P}_g^{\Delta t}(t_n)$

associated with probability $P_k, k \in \{1, 2, \dots, 2^{N_m}\}$. Once P_k, u_i, d_i are determined, we divide the algorithm in two sequentially executed parts as explained below.

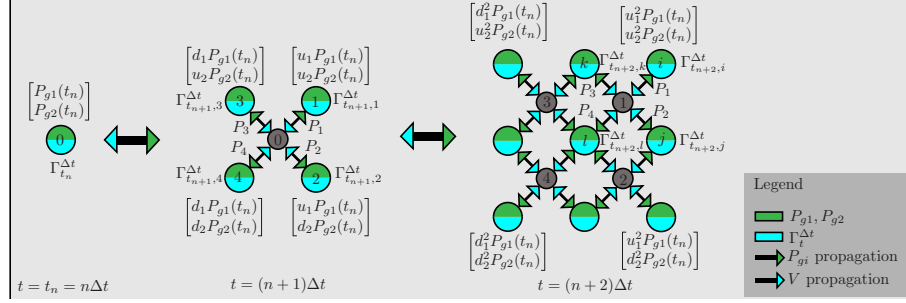


Figure 2.2: Forward and backward propagation to calculate Γ_{t_n} with 2 MGs

Forward propagation

Our objective is to find $a_i(t_n), i \in \mathcal{I}$ and $b(t_n)$ where $t_n < t_N = T_f$. Given the current value of the RESs generation vector, $\mathbf{P}_g^{\Delta t}(t_n)$, first we estimate all possible values of $\mathbf{P}_g^{\Delta t}(t_N)$ with corresponding probabilities as shown in Fig. 2.2. The general algorithm is given in Algorithm 2.

Backward propagation

Using the results obtained in the forward propagation and $D_{ci}, i \in \mathcal{I}$ values, $\Gamma_{t_N}^{\Delta t}$ can be computed using (2.13) for all possible points at $t = t_N$. To estimate $\Gamma_{t_n}^{\Delta t}$, the power portfolio value is back propagated from t_N till t_n , using (2.14) at each $t_k, k \in \{N-1, N-2, \dots, n\}$. For example, in case of $N_m = 2$, as shown in Fig. 2.2, $\Gamma_{t_n}^{\Delta t} = \sum_{i=1}^4 P_i \Gamma_{t_{n+1}, i}^{\Delta t}$, where $\Gamma_{t_{n+1}, i}^{\Delta t}$ are estimated from preceding back propagation steps. The backpropagation algorithm is given in Algorithm 3.

The entire procedure is summarized in Algorithm 1. Each node in the implementation of tree data structure, shown in Fig. 2.2, has elements $\mathbf{P}_g^{\Delta t}$, having RES output associated with the node, $\Gamma^{\Delta t}$, to store power portfolio value of the node, p , which contains probability of encountering the node starting from initial time t_n , ohp , containing one hop probability of running into the node from its parent and a unique ID, id , assigned to each node. We run Algorithm 1 continuously at an interval of Δt . At the beginning of each interval, RESs generation is measured which is fed back to generate the power portfolio values. However, the error in computing these power portfolio values according

Algorithm 1: power portfolio Value Determination

```

Function getCF ( $\mathbf{P}_g^{\Delta t}(t_n) = \{P_{gi}^{\Delta t}(t_n)\}_{i=1}^{N_m}$ ,  $\mathbf{D}_c = \{D_{ci}\}_{i=1}^{N_m}$ ,  $T_f$ ,  $N$ ,  $w$ ,
 $\mathbf{P}, \mathbf{U}, \mathbf{a} = \{a_i(t_{n-1})\}_{i=1}^{N_m}, P_b$ ):
    /* Initialization */
     $\Delta t \leftarrow T_f/N; n \leftarrow (t_n - w)/\Delta t; \mathbf{L}, \mathbf{L}_n \leftarrow []; \mathbf{v} \leftarrow \vec{0} \in \mathbb{R}^{2^{N_m} \times 1};$ 
     $\mathbf{r} \leftarrow \vec{0} \in \mathbb{R}^{(N_m+1) \times 1}; \mathbf{A} \leftarrow \vec{0} \in \mathbb{R}^{2^{N_m} \times (N_m+1)}; \mathbf{A}[:, N_m+1] \leftarrow P_b;$ 
    /* Forward Propagation */
     $\mathbf{L} \leftarrow \text{FP}(N - n, \mathbf{U}, \mathbf{P}, \mathbf{P}_g^{\Delta t}(t_n));$ 
    /* Backpropagation */
     $\Gamma_{t_n}^{\Delta t}, \mathbf{L}_n \leftarrow \text{BP}(\mathbf{L}, \mathbf{D}_c);$ 
    if  $|\mathbf{L}_n| \neq 1$  then
        for  $j = 1$  to  $2^{N_m}$  do
             $c \leftarrow \mathbf{L}_n[j]; \mathbf{v}[j] \leftarrow c \cdot \Gamma_{t_n}^{\Delta t}; \mathbf{A}[j, 1:N_m] \leftarrow (c \cdot \mathbf{P}_g^{\Delta t})^T;$ 
        end
         $\mathbf{r} \leftarrow \mathbf{A}^\dagger \mathbf{v};$ 
    else
         $\mathbf{r}[1:N_m] \leftarrow \mathbf{a}; \mathbf{r}[N_m + 1] \leftarrow \Gamma_{t_n}^{\Delta t} - \mathbf{r}[1:N_m]^T \mathbf{P}_g^{\Delta t}(t_n);$ 
    end
return  $\Gamma_{t_n}^{\Delta t}, \mathbf{r};$ 

```

to Algorithm 1 depends on the chosen time step Δt . The following theorem provides a bound on this error.

Theorem 3. Let $\epsilon(t_n) := |\mathbb{E}[\Gamma_{T_f}^{\Delta t} - \Gamma_{T_f} | \mathbf{P}_g(t_n)]|$ denote the absolute error in calculating the power portfolio value, at $t_n = n\Delta t, n \in \{0, 1, \dots, N\}$, due to time discretization with a time step of $\Delta t = T_f/N$. Then, $\epsilon(t_n) \leq L\Delta t$, where L is a constant independent of Δt .

Proof. Please see Appendix A.5. □

Next, we provide experimental results of our methodology by instantiating it to benchmark power distribution networks.

2.6 Experimental results

2.6.1 Model assumption verification

First we analyse the empirical wind power time series data to validate the model assumption (2.1) and estimate the parameters of the stochastic process. Wind generation data from the northwestern part of USA is taken from [49] for the month of June, 2020 at an interval of 5 minutes. As typical wind generation dynamics vary over different hours of a day [50], it is necessary to modify the parameters of (2.1) according to period

Algorithm 2: Forward Propagation

```

Function FP ( $nStep, U, P, P_g^{\Delta t}$ ):
  create root node with  $id = 0, p = 1, P_g^{\Delta t} = P_{gI}^{\Delta t}$ ;
   $L_f \leftarrow [root]$ ;  $pid \leftarrow 0$ ;
  for  $i = 1$  to  $nStep$  do
    for  $j = 1$  to  $(2^{N_m})^i$  do
       $parent \leftarrow$  get node with  $id = pid$ ;
       $pr \leftarrow parent.p; C_G \leftarrow parent.P_g^{\Delta t}$ ;
      for  $k = 1$  to  $N_m$  do
         $child \leftarrow$  new node;  $child.P_g^{\Delta t} \leftarrow C_G \otimes U[k, :]$ ;
         $child.p \leftarrow pr \otimes P[k]$ ;  $child.id \leftarrow (2^{N_m} \times pid) + k$ ;
         $child.ohp \leftarrow P[k]$ ; append  $child$  to  $parent$ ;
        if  $i = nStep$  then
          | append  $child$  to  $L_f$ ;
        end
      end
       $pid \leftarrow pid + 1$ ;
    end
  end
  return  $L_f$ ;

```

of interest during a day. In our case, wind data from 10 a.m.-5 p.m. is chosen and maximum likelihood estimator is used to find drift and volatility parameters. The estimated drift and volatility parameters are 0.007 and 0.027 respectively with chi-square goodness of fit score of 18.86 including 13 degrees of freedom. The p -value, calculated from the goodness of fit test with significance level $\alpha = 0.05$, is 0.128 which justifies the validity of GBM model of wind power dynamics [51].

2.6.2 Sample size determination

For the simulation, two regions with RESs generations, $P_{g1}(t)$ and $P_{g2}(t)$ with parameters $\mu_{g1} = 0.006, \sigma_{g1} = 0.03$ and $\mu_{g2} = 0.005, \sigma_{g2} = 0.04$ respectively, are considered and realized using discrete approximation of (2.1). We also assume that the generations are correlated with a correlation coefficient of $\rho_{12} = 0.6$ [50]. We first focus on the number of samples needed to arrive at meaningful statistical inferences. Fig. 2.3 shows the density histogram plot of $P_{g1}(t)$ for different time instants for two sample sets each containing 10000 samples of $P_{g1}(t)$ drawn from numerical simulation of the dynamic equation (2.1) with $\mu_{g1} = 0.006, \sigma_{g1} = 0.03$ and initial condition, $P_{g1}(0) = 20$ kW. Both sample sets consist of samples of $P_{g1}(t)$ with the terminal case of $P_{g1}(T_f) > D_{c1}$. We compare the distribution of the samples, at $t = 1, 2, \dots, 5$, in the two sample sets, using Two-sample Kolmogorov–Smirnov (KS) test [52] which produces KS statistics values

Algorithm 3: Backpropagation

```
Function  $BP(\mathbf{L}, D_c)$ :  
  foreach  $c \in \mathbf{L}$  do  
     $\mathbf{p}_g^{\Delta t} \leftarrow$  member  $\mathbf{P}_g^{\Delta t}$  of node  $c$ ;  
     $tv \leftarrow \max(\sum_{i=1}^{N_m} [D_c - \mathbf{p}_g^{\Delta t}][i], 0)$ ;  
    update value of member  $\Gamma^{\Delta t}$  of  $c$  with  $tv$ ;  
  end  
   $\mathbf{L}_c, \mathbf{L}_n \leftarrow \mathbf{L}$ ;  $end \leftarrow |\mathbf{L}_c|$ ;  $ev \leftarrow tv$ ;  
  while  $end \neq 1$  do  
     $\mathbf{L}_p \leftarrow []$ ;  $numNodes \leftarrow end$ ;  
    while  $numNodes > 0$  do  
       $\mathbf{s} \leftarrow 2^{N_m}$  siblings from  $\mathbf{L}_c$ ;  
       $\mathbf{v} \leftarrow \Gamma^{\Delta t}$  values of each node in  $\mathbf{s}$ ;  
       $\mathbf{p}_s \leftarrow ohp$  values of each node in  $\mathbf{s}$ ;  
       $parent \leftarrow$  parent node of siblings in  $\mathbf{s}$ ;  
       $parent.\Gamma^{\Delta t} \leftarrow \sum_{i=1}^{2^{N_m}} \mathbf{v}[i]\mathbf{p}_s[i]$ ;  
      append  $parent$  to  $\mathbf{L}_p$ ;  
       $numNodes \leftarrow numNodes - 2^{N_m}$ ;  
    end  
     $\mathbf{L}_c \leftarrow \mathbf{L}_p$ ;  $end \leftarrow |\mathbf{L}_c|$ ;  
    if  $end = 2^{N_m}$  then  
      |  $\mathbf{L}_n \leftarrow \mathbf{L}_c$ ;  
    end  
  end  
  return  $ev, \mathbf{L}_n$ ;
```

0.008, 0.01, 0.0062, 0.0079 and 0.0094 respectively. For sample size of 10000 in each set, the critical value of KS statistics is 0.019. The lower values of KS statistics than the critical value indicates that simulation of our algorithm for 10000 independent runs can produce highly stable results. Similar conclusions can be drawn for other terminal cases of $P_{g1}(T_f)$ and $P_{g2}(t)$, such as, $P_{g1}(T_f) \leq D_{c1}$, $P_{g2}(T_f) > D_{c2}$ and $P_{g2}(T_f) \leq D_{c2}$.

2.6.3 Simulation Results: Algorithmic Efficacy

We assume $P_b = 1$ kW. The critical load demands, $D_{c1} = 20$ kW and $D_{c2} = 25$ kW, of the two MGs respectively, are to be met at $T_f = 5$ hrs. We use Algorithm 1 to compute optimal allocation of initial resources and policy updates at each time instant, under the constraint 2.4, that satisfies power balance at T_f . We assume that the feedback policy is adjusted at an interval of one hour. To establish the efficacy of our proposed algorithm, three terminal cases: (a) Case 1: $P_{g1}(T_f) \geq D_{c1}$, $P_{g2}(T_f) \geq D_{c2}$, (b) Case 2: $P_{g1}(T_f) \geq D_{c1}$, $P_{g2} < D_{c2}$, and (c) Case 3: $P_{g1}(T_f) < D_{c1}$, $P_{g2} < D_{c1}$ are simulated in Python 3.8. Fig. 2.4 shows the simulation results for 10000 runs, for Case 2 where MG 1 is capable of supplying its load demand of $D_{c1} = 20$ kW but RESs generation of MG

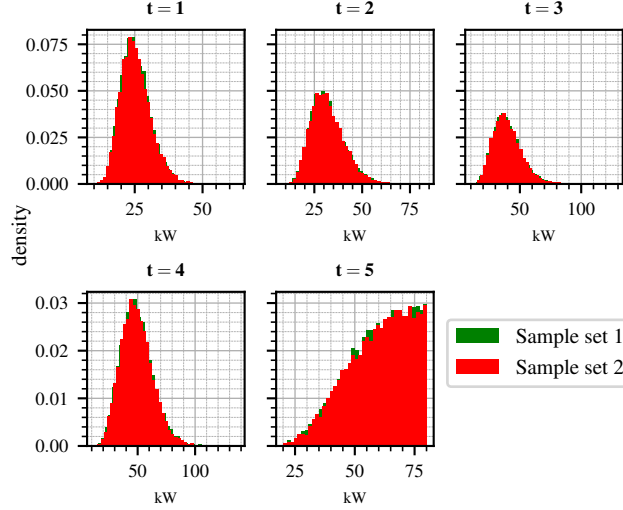


Figure 2.3: Histograms of $P_{g1}(t)$ for $t \in \{1, 2, \dots, 5\}$ with two sample sets each containing 10000 samples of $P_{g1}(t)$ with terminal condition $P_{g1}(T_f) > D_{c1}$.

2 is not sufficient to meet $D_{c2} = 25$ kW at $t = T_f$ (the terminal state is not available *a priori* at any time $t < T_f$). For RESs generations, the mean values are plotted along with standard deviations in shaded regions. For all other resources, mean values along with 95% bootstrap confidence interval (CI) around the mean, with 10000 bootstrap samples, are plotted. Here the power deficit of MG 2 is more than the surplus generated from MG 1. The plot shows that at the end of the time horizon, IMMIG enables DNO to utilize the surplus thus requiring no power from the portfolio at $t = T_f$ whereas in case of IDMMG, average power from the portfolio is 6 kW to guarantee that the demand is met. Similar plots can be obtained for Case 1 and 2 also, which we omit here to conserve space. For all three cases, the average reduction in the required power from the portfolio, for the entire time horizon, are shown in Table 2.1 and Fig. 2.5. The narrow 95% CI around the mean for power portfolio reduction, as seen from Fig. 2.5, indicate high confidence of the outcomes of our proposed algorithm.

Next we demonstrate the optimality of the initial amount of battery allocation derived by our proposed algorithm. Fig. 2.6 illustrates the scenario, under case 3 in IMMIG, when initial battery allocation is not optimal as derived in Section 2.5.2. In Fig. 2.6, $\bar{\Gamma}_u^{\Delta t}(t)$ denotes the power portfolio trajectory when initial battery allocation is 20% more than optimal and $\Gamma^{\Delta t}(t)$ denotes the optimal power portfolio value. We observe that, at the end of the horizon, DNO has overproduction. $\bar{\Gamma}_d^{\Delta t}(t)$ denotes the scenario when

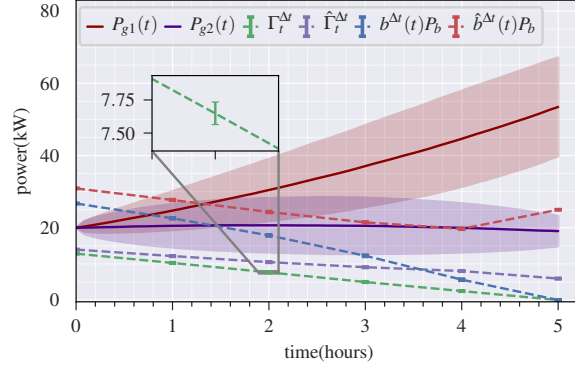


Figure 2.4: Excess generation from MG 1 is utilized to supply demand of MG 2 in IMM at T_f , whereas in IDMMG, average power from portfolio is 6 kW to guarantee the demand is met. Power portfolio value is lower in IMM than IDMMG. Zoomed inset, showing an example of 95% CI around the mean of power portfolio for IMM at 2 hrs, indicates high confidence in our scheme.

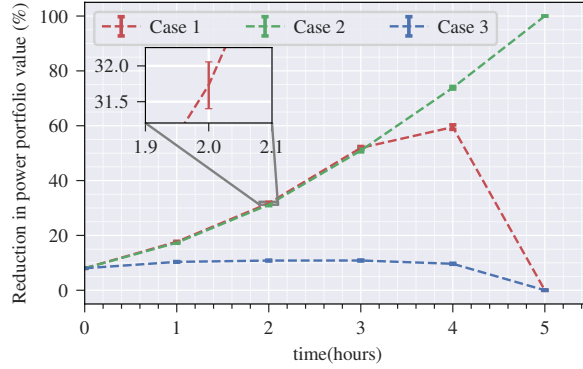


Figure 2.5: Reduction in power portfolio value in IMM as a percentage of the same in IDMMG for three cases. For all $t < T_f$, in IMM, DNO requires less amount of power from portfolio compared to IDMMG

initial battery allocation is 20% less than optimal. Here, DNO cannot supply the critical loads at the end of horizon, as the battery power output is not sufficient to support the combined production deficit of the MGs. Similar plots can be shown for case 1 and 2, which we omit to conserve space. Thus our algorithm provides assurance of meeting the demand of the MGs, without over or underproduction.

In the subsequent paragraphs, we consider the chance-constrained problems (2.6), (2.8) introduced in Section 2.4, and compare the performance of their optimal solution with

Table 2.1: Average reduction (%) in portfolio value in IMM for Case 1,2,3

Time (hrs.)	$P_{g1}(T_f) \geq D_{c1}$	$P_{g1}(T_f) \geq D_{c1}$	$P_{g1}(T_f) < D_{c1}$
	$P_{g2}(T_f) \geq D_{c2}$	$P_{g2}(T_f) < D_{c2}$	$P_{g2}(T_f) < D_{c2}$
0	8.03	8.03	8.03
1	17.68	17.32	10.35
2	31.73	31.19	10.83
3	51.91	50.94	10.85
4	59.49	73.86	9.65
5	0	100	0
Overall	28.14	46.89	8.29

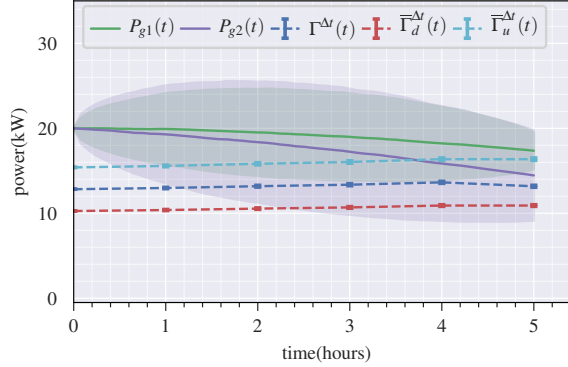


Figure 2.6: Incorrect initial battery allocation leads to risk of over or underproduction. $\bar{\Gamma}_u^{\Delta t}(t)$ denotes the power portfolio trajectory when initial battery allocation is 20% more than optimal, $\Gamma^{\Delta t}(0)$, derived in Case 3. Here, at the end of the horizon, DNO has overproduction. $\bar{\Gamma}_d^{\Delta t}(t)$ shows the scenario when initial battery allocation is 20% less than optimal. Here, DNO cannot produce required power to support the loads at requested time T_f .

our proposed algorithm. Let $\tilde{\Gamma}^{\Delta t,s}$ and $\tilde{\Gamma}^{\Delta t,d}$ denote the portfolio value of the DNO under the solution to the surplus problem (2.6) and deficit problem (2.8) respectively with deficit and surplus probabilities $\epsilon_k = 0.02$. In Table 2.2, we present the statistics of surplus or deficit amount, in the power portfolio, at the end of the horizon. For surplus problem, for 1.7% of 1000 realizations of $P_{g1}(t)$ and $P_{g2}(t)$, the DNO ends up

with surplus power in its portfolio. The maximum surplus amount encountered is 15.6 kW. Such large surplus amount can be detrimental for the entire system. For the deficit problem, 1.9% of 1000 RESs realizations lead to underproduction with a maximum deficit amount of 23.1 kW. However, in our proposed algorithm, only 0.1% of RESs realizations lead to surplus or deficit, while the maximum amount of surplus and deficit are 3.4 kW and 0.5 kW respectively. Note that, in our proposed scheme, such deviation from the optimal portfolio value arises due to the error introduced in discrete time simulation. Nevertheless, such improvement is extremely beneficial for multi-microgrid systems with critical infrastructures that focus on minimization of grid dependency. Similar results can be shown for case 1 and 3 which we omit here to conserve space.

Table 2.2: Surplus/deficit comparison for 1000 RESs realizations, for case 2

Methodology	realizations with surplus/ deficit (%)	Average surplus/deficit (kW)	Max. surplus (kW)	Max. deficit (kW)
Surplus problem (2.6)	1.7	9.8	15.6	0
Deficit problem (2.8)	1.9	-16.2	0	23.1
Our solution	0.1	1.6	3.4	0.5

2.6.4 Real-time Simulation on a Distribution Network

In this section, we demonstrate the applicability and utility of our proposed methodology in real-time operation of power distribution networks considering all power network constraints, such as line flow, generation limit etc. The power networks under study consists of (i) a battery controlled in a Grid Following (GFL) mode, which operates according to our proposed strategy, (ii) another battery controlled in a Grid Forming (GFM) mode [53], connected to the feeder to run the system stably (emulating a grid), (iii) wind farms and loads deployed at various parts of the networks. We set a time window of length equal to T_f and run Algorithm 1 within the window with a time step of Δt . The power portfolio values are updated at an interval of Δt and the same is dispatched as the power reference set-point to the GFL battery at T_f . The entire process is repeated at the end of each time window. To demonstrate the utility as well as the scalability of our proposed algorithm under real-time operation, we first conduct a real-time OPAL-RT-based simulation on a modified IEEE-33 bus distribution network

with two microgrids, and further extend it to a 225-bus distribution network with 10 microgrids. The details of our experimental setup are described in the subsequent sections.

IEEE-33 bus network

A modified IEEE-33 bus distribution network with a total load of 850 kW is taken as the system under study. A GFM battery is connected to bus 1 and two wind generators are connected to bus 4 and 26 respectively. The wind generators produce $P_{g1}(t)$ and $P_{g2}(t)$ kW of power respectively. The GFL battery, which responds to the power set-points generated by our resource allocation algorithm, is connected to bus 5. The power network is emulated inside OP5700 real-time simulator box, manufactured by OPAL-RT. The RT simulation is conducted using eMEGASIM-based EMT-simulation platform of RTLAB with simulation step size of $50\mu s$. The window length is chosen to be 5 minutes and the power portfolio values are updated at an interval of 1 minute. The experiment is conducted for 3 hours in real-time and the result is shown in Fig. 2.7. It is observed that the RESs uncertainties are completely mitigated by the GFL battery, injecting $P_{GFL}(t)$ kW to the system based on the set-points generated by our algorithm. Hence, the impact of $P_{g1}(t)$ and $P_{g2}(t)$ variations on the GFM battery is negligible as seen by the minimal variation of the GFM battery power output, $P_{GFM}(t)$. The GFM battery mostly provides only the losses in the system, and thus the entire system is minimally dependent on the same in terms of maintaining demand-supply balance under RESs uncertainty. To establish a high confidence in the outcomes of our proposed algorithm, a similar experiment is conducted on the same network in PYPOWER for 10,000 realizations of $P_{g1}(t)$ and $P_{g2}(t)$ and the results are shown in Fig. 2.8. The mean values of all the quantities are plotted in solid lines while the standard deviations are plotted in shaded regions. The narrow standard deviation of the GFM battery power output, $P_{GFM}(t)$, signifies that it is not impacted by the RESs generation uncertainties as the GFL battery, producing $P_{GFL}(t)$ kW power as per our proposed algorithm, completely mitigates the risk of meeting the demands of the MGs.

225 bus distribution network

To demonstrate the scalability of our algorithm, we deploy it on a 225-bus distribution network, with 10 microgrids. The 225-bus network entails a part of University of Minnesota campus, and surrounding city areas, that are powered by Xcel Energy. The total

load in the network is 5.12 MW. We deployed 10 wind generators at different locations spanning the university campus and the city of Minneapolis, and connected a GFM battery near to the main feeder, to emulate the operation of upstream grid. The power network is provided by Xcel Energy and University of Minnesota Energy Management Service and emulated inside OP5700 real-time simulator box. The simulation setup is similar to the IEEE-33 bus case and the result is shown in Fig.2.9. It is observed that the GFL battery responds to the RESs variation of the MGs in such a way that the uncertainty is completely mitigated by the GFL battery, and the GFM battery provides only the network losses. This indicates that the GFL battery, controlled as per our proposed algorithm, minimizes the dependency on the GFM resources in maintaining demand-supply balance under RESs uncertainties.

The results corroborate the efficacy and utility of our electricity market mechanism and the underlying real-time battery power control strategy in IMMIG while providing provable guarantees of meeting the power needs at the end of the prescribed horizons, which is an indispensable requirement for critical facilities. Therefore, it can be concluded that our proposed algorithm enables DNO to mitigate the uncertainty of supplying the demand of the microgrids at the requested time while reducing the dependency on the grid, thus enhancing self-sufficient and reliable operation of multi-microgrids.

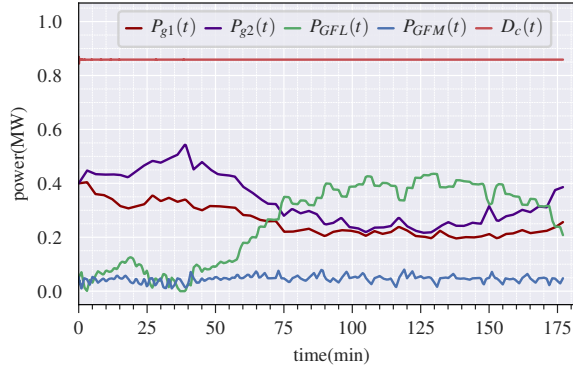


Figure 2.7: Real-time simulation on 33 bus network: The variation of power output of the GFM battery is minimal denoting negligible impact of renewable uncertainties on it. Most of the uncertainties are mitigated by the GFL battery, delivering power as per the reference set-points generated by our algorithm.

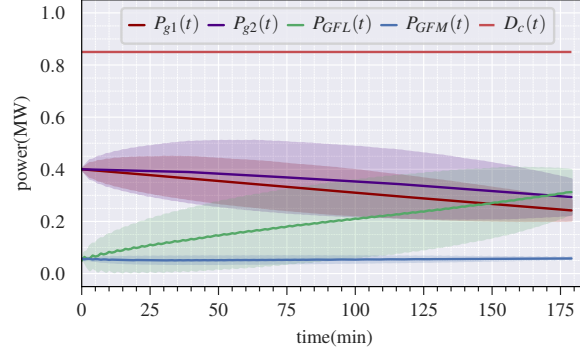


Figure 2.8: Simulation on 33 bus network in PYPOWER for 10000 realizations of $P_{g1}(t)$, $P_{g2}(t)$: The narrow standard deviation of power output from the GFM battery indicates high confidence in the outcomes of the algorithm.

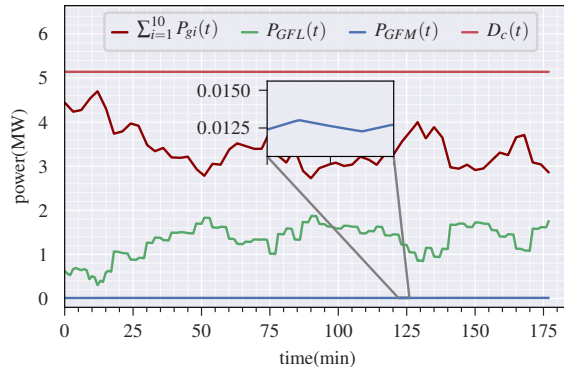


Figure 2.9: Real-time simulation on 225 bus network with 10 MGs: The variation of power output of GFM battery is minimal. Most of the uncertainties are mitigated by the GFL battery controlled as per our algorithm.

Chapter 3

Guaranteeing Service in Connected Microgrids: Storage Planning and Optimal Power Sharing Policy

3.1 Gist of the Chapter

The integration of renewable energy sources (RES) into power distribution grids poses challenges to system reliability due to the inherent uncertainty in their power production. To address this issue, battery energy sources (BESs) are being increasingly used as a promising solution to counter the uncertainty associated with RES power production. During the overall system planning stage, the optimal capacity of the BES has to be decided. In the operational phase, policies on when to charge the BESs and when to use them to support loads must be determined so that the BES remains within its operating range, avoiding depletion of charge on one hand and remaining within acceptable margins of maximum charge on the other. In this paper, a stochastic control framework is used to determine battery capacity, for microgrids, which ensures that during the operational phase, BESs' operating range is respected with pre-specified high probability. We provide an explicit analytical expression of the required BESs energy capacity for a single microgrid with RES as the main power source. Leveraging insights from the single microgrid case, the article focuses on the design and planning of BESs for

the two-microgrid scenario. In this setting, microgrids are allowed to share power while respecting the capacity constraints imposed by the power lines. We characterize the optimal power transfer policy between the microgrids and the optimal BES capacity for multiple microgrids. This provides the BES savings arising from connecting the microgrids.

3.2 State-of-the-art: Energy Storage Planning and Power Sharing for Multi-microgrid Systems

In recent years, there has been a tremendous growth in the penetration level of renewable energy sources (RES) in the traditional power distribution networks to address environmental issues, such as carbon emission, and techno-economic issues, such as conventional energy resource depletion, and increasing consumer demand. It is projected that RES will contribute 90% of the total energy production by 2050, and approximately 1 trillion USD per year will be invested in the development of renewable energy sectors [7]. However, despite the environmental benefits renewable penetration poses a major challenge to the reliable operation of the distribution system as RES are inherently uncertain. Battery energy sources (BESs), provide an attractive option to counter RES uncertainty with the ability to store excess energy when supply exceeds demand and to provide energy when demand exceeds RES-based energy. A key challenge is to determine the capacity of the BES and the associated strategy of when to charge the battery and when to use it to service loads. The problem is difficult because of the stochasticity of RES. Moreover, in the multi-microgrid scenario the facility to share power has the potential to provide benefits of spatial diversity for minimizing needed BES capacity. However, the related problem for determining capacity and charging/discharging policy incorporating the power-sharing ability while respecting constraints of power lines and ensuring BES charge state within specified limits is formidable [54]. Note that, while BES capacity governs the resource installation cost of the system, the flow capacity of the lines determines the line installation cost. Thus, a proper characterization of both is extremely important to efficiently determine the total installation cost of the system. In this article, we describe a methodology to determine the BES capacity and power-sharing policy respecting the flow capacity. Summarizing, two tasks posed by RES penetration into the grid are addressed in this article:

- (i) How much BES energy capacity each microgrid (MG) should have to ensure the BESs are operated within the specified operating range at all times?
- (ii) What should be the power-sharing policy amongst the microgrids?

3.2.1 Prior Works

In prior state of the art, the problem of BESs size determination is approached with two primary objectives: (1) operational cost minimization to address techno-economic goals, (2) enhancement of system reliability. Most of the existing works focus on BESs capacity determination with the aim of optimizing installation and operation costs. In this context, several strategies are proposed to find the optimal battery size for both grid-connected and grid-isolated systems [55, 25, 56]. In [55, 25, 57], battery sizing strategies, based on cost optimization in probabilistic unit commitment settings, are proposed. A stochastic programming approach, to decide the optimal size of battery storage by minimizing the cost of purchasing energy from the grid, is proposed in [58]. In [56], a chance-constrained approach to minimize the system operation cost, considering a probabilistic constraint on the amount of underproduction, is synthesized. However, overproduction scenarios are not addressed. Furthermore, various optimization techniques [59, 60, 8] are proposed for the optimal allocation of BESs to enable cost-efficient operation of renewable-powered microgrids. However, in critical application scenarios, such as hospitals, reliability of the system is more important compared to cost optimization. For improving system reliability, various techniques in the state-of-the-art include chance-constrained optimization to minimize power curtailment [61], battery charge-discharge control [62], data-driven approaches [63], load-shaping techniques [64]. Further, optimization methods to find optimal energy storage size for load loss minimization are devised [65]. However, such methodologies consider either generation or load curtailment probability as the reliability index and thus are difficult to deploy in critical applications where minimizing curtailment of both generation and load is of importance. In [66], a numerical analysis with joint optimization of generation and load curtailment is proposed. However, no guarantees of demand fulfillment are provided. Despite the importance, works on determining optimal BESs capacity to minimize both generation and load curtailment without degradation of the BESs, are limited in the existing literature. On the other hand, various data-driven power-sharing strategies to improve multi-microgrid operation are discussed in [67, 68, 69]. However, to the best of the author's knowledge, existing works on analytically tractable power-sharing policies,

in the case of multiple microgrids, and an analysis of how the transmission line capacity affects the choice of battery capacity is scarce.

3.2.2 Key Contributions of the Chapter

We propose a chance-constrained BESs sizing methodology and power-sharing policy, for multi-microgrid systems, to improve system reliability by minimizing both generation and load curtailment. Our primary contributions are summarized below:

1. We propose an analytical method to determine the optimal BES energy capacity to minimize both load and generation curtailment for a finite time horizon. Our proposed method provides probabilistic guarantees on the amount of curtailment and ensures the available energy in the BESs remains within the operating range throughout the time horizon with high probability.
2. We extend our BES energy capacity design methodology to a system having two MGs that can share power with each other via a connecting power line. Here, we provide a power transfer policy that achieves the minimum BES energy capacity, required to be installed in each of the MGs, while ensuring the BESs are operated within specified energy limits at all times. We also quantify the BES investment savings arising from connected operation of two MGs.

3.3 Battery Storage Sizing for a Single Microgrid System

3.3.1 System Description

We consider an MG having local RES, loads, and N number of BESs. The RES and the BESs are utilized to supply the loads. The system architecture is illustrated in Fig. 3.1 which shows a microgrid network receiving power from RES and BESs while servicing loads. We denote the power output of the renewables, load demand, and the total power output of the BESs by $P_g(t)$, $D_c(t)$, and $b(t)$ respectively. We consider that each BES is identical and has a maximum capacity of $B_{max} \in \mathbb{R}^+$ kWh. The objective of the microgrid is to determine the number of BESs, N , to sustain the system for a given finite time horizon with probabilistic guarantees on the amount of generation and load curtailment by ensuring that the energy available in the BESs remains within acceptable limits. Here we focus on determining optimal BESs capacity which satisfies the following: (i) if the RES produces more power than what is demanded by the load, then the battery can be charged with the surplus generation and (ii) when the RES

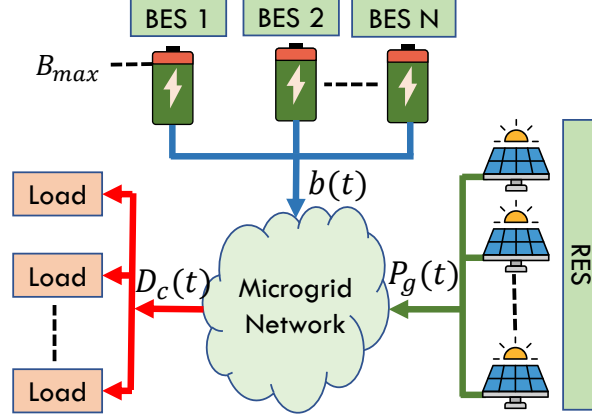


Figure 3.1: Microgrid architecture

production is less than the required load, the battery has sufficient storage to supply the deficit power. The above conditions are satisfied if the maximum capacity of BESs is not reached and the BESs never get depleted under the uncertainty presented by the RES. In the next section, we make the problem description precise.

3.3.2 Problem Formulation

We consider that the load demand (with power $D_c(t)$ demanded at time t) in the system is required to be supplied continuously for a time horizon denoted by T_f . Note that, the total energy provided by N number of BESs, till time $t \geq 0$ is given by $\int_0^t b(s)ds = \int_0^t (D_c(s) - P_g(s))ds$. We denote $E_i^{p.u.} = \frac{1}{B_{max}} \int_0^t b(s)ds$ as the per unit (p.u.) energy supplied by N number of BESs. Further, let the net p.u. energy available from the BESs, at any time $t \geq 0$, be denoted by $X(t)$. Then, we have,

$$X(t) = X(0) - \frac{1}{B_{max}} \int_0^t b(s)ds = X(0) + \frac{1}{B_{max}} \int_0^t (P_g(s) - D_c(s))ds,$$

where $X(0)$ denotes the initially available p.u. energy in the BESs. Further, let $B_0 \in [0, B_{max}]$ kWh at time $t = 0$ be the initial charge held by each of the N BESs, and let $\alpha := B_0/B_{max}$. Thus, the net initial p.u. energy available in the BESs is $X(0) = N \frac{B_0}{B_{max}} = \alpha N$. For notational convenience, we define $\bar{B} := \frac{1}{B_{max}}$. The load demand and the RES power generation are both uncertain. To characterize the uncertainty we assume that the net p.u. surplus energy, $\bar{B} \int_0^t (P_g(s) - D_c(s))ds$, follows a Brownian motion characterized by $\sigma W(t)$ where $\sigma \neq 0$ models the volatility, and $W(t)$ is a Wiener

process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Thus, $X(t)$ can be written as:

$$X(t) = \sigma W(t) + \alpha N. \quad (3.1)$$

Note that, to avoid curtailment of RESs power, the Distribution System Operator (DSO) has to ensure that all the batteries have charge below the designated maximum $B_{\max}$ (or equivalently 1 p.u.) for all times $t \in [0, T_f]$. Similarly, to reduce load curtailment, BESs should have positive charge for all times $t \in [0, T_f]$. Thus to reduce both generation and load curtailment, the DSO is responsible for sustaining the batteries throughout the time interval $[0, T_f]$ without discharging the batteries completely or charging them to the maximum capacity. Let $X_t^{sup} := \sup_{s \in [0, t]} X(s)$ and $X_t^{inf} := \inf_{s \in [0, t]} X(s)$ denote the maximum and minimum p.u. energy available in the BESs at any time $s \in [0, t]$ respectively. Therefore, the DSO needs to ensure the following:

$$\mathbb{P}[X_{T_f}^{sup} < N, X_{T_f}^{inf} > 0] \geq 1 - \delta, \quad (3.2)$$

where $\delta > 0$ is a small pre-specified tolerance in the probability of violating the conditions on the charge of the BESs. In the above setup, the decision variables include the initial p.u. charge $\alpha = \frac{B_0}{B_{max}}$, and the number N of batteries to be used. The problem translates to: given σ, B_{max}, δ , decide the minimum number, N , of batteries required, and the initial p.u. charge α , such that (3.2) and (3.1) are satisfied. Thus the following chance-constrained problem needs to be solved:

$$\min_{\alpha, N} N \quad (3.3)$$

subject to $N \geq 0$,

$$0 \leq \alpha \leq 1,$$

$$X(t) = \sigma W(t) + \alpha N,$$

$$\mathbb{P}[X_{T_f}^{sup} < N, X_{T_f}^{inf} > 0] \geq 1 - \delta. \quad (3.4)$$

In the planning stage, the DSO solves (3.3) to determine the required capacity of BESs. In the next section, we describe the solution methodology for the DSO.

3.3.3 Solution Methodology

In this section, we describe our proposed methodology to solve the problem of the DSO as described in the previous section. We emphasize that obtaining an analytical solution to (3.3) remains intractable. Hence, we first analyze the chance-constraint (3.4) to derive a tractable approximation of (3.3). Further, we derive the corresponding value of α that leads to an explicit solution to the approximated problem.

Analysis of the chance constraint

Note that, $\mathbb{P}[X_{T_f}^{sup} < N, X_{T_f}^{inf} > 0] = 1 - \mathbb{P}[\{X_{T_f}^{sup} \geq N\} \cup \{X_{T_f}^{inf} \leq 0\}]$. Therefore, the constraint (3.4) can be equivalently written as $\mathbb{P}[\{X_{T_f}^{sup} \geq N\} \cup \{X_{T_f}^{inf} \leq 0\}] \leq \delta$. Further, for $a \in \mathbb{R}$, let

$$T(a) := \inf(t \geq 0, X(t) = a)$$

denote the first time the battery energy level reaches a p.u. Then we have, $X_{T_f}^{sup} \geq N$ if and only if $T(N) \leq T_f$, and $X_{T_f}^{inf} \leq 0$ if and only if $T(0) \leq T_f$ [70]. Thus, the constraint (3.4) can be written as $\mathbb{P}[\{T(N) \leq T_f\} \cup \{T(0) \leq T_f\}] \leq \delta$. Consider the following problem:

$$\min_{\alpha, N} N \tag{3.5}$$

$$\text{subject to } N \geq 0, 0 \leq \alpha \leq 1,$$

$$X(t) = \sigma W(t) + \alpha N,$$

$$\mathbb{P}[T(N) \leq T_f] + \mathbb{P}[T(0) \leq T_f] \leq \delta. \tag{3.6}$$

Note that a feasible solution to (3.5) is also a feasible solution to (3.3), as $\mathbb{P}[\{T(N) \leq T_f\} \cup \{T(0) \leq T_f\}] \leq \mathbb{P}[T(N) \leq T_f] + \mathbb{P}[T(0) \leq T_f]$. The solution to (3.5) provides an upper bound of the solution to (3.3). In the subsequent development, we derive a tractable upper bound to the chance constraint (3.6), which we further utilize to find a theoretical solution to (3.5).

Battery Capacity Determination

In this section, we derive a tractable approximation to the chance-constraint (3.6) to obtain an analytical solution to (3.5). Here, first we provide an upper bound to the

chance-constraint (3.6) as presented in the following theorem:

Lemma 1. *Let*

$$f(N, \alpha) := e^{-\frac{N^2(1-\alpha)^2}{2\sigma^2 T_f}} + e^{-\frac{N^2\alpha^2}{2\sigma^2 T_f}}, \quad (3.7)$$

where $\alpha \in [0, 1]$ and $N \geq 0$. Then, we have,

$$\mathbb{P}[T(N) \leq T_f] + \mathbb{P}[T(0) \leq T_f] \leq f(N, \alpha), \quad (3.8)$$

for all N and α .

Proof. Please see Appendix B.1. □

Further note, for any $N \geq 0$, we have $f(N, \alpha) > 1$ if $\alpha=1$ or 0 . However, a feasible solution to (3.5) requires $f(N, \alpha) \leq 1$ as $\delta \in (0, 1)$. Now, consider the following problem:

$$\tilde{N} := \min_{\alpha, N} N \quad (3.9)$$

$$\text{subject to } N \geq 0, \alpha \in (0, 1),$$

$$f(N, \alpha) \leq \delta. \quad (3.10)$$

From Lemma (1), we derive that any feasible solution to (3.9) is also a feasible solution to (3.5). Thus the solution to (3.9) provides an upper bound of the solution to (3.5). In the subsequent development, we show how an explicit solution to $\tilde{N}$ is obtained. Let $N(\alpha)$ denote the solution to (3.9) for a given $\alpha \in (0, 1)$, that is,

$$N(\alpha) := \min_N N \quad (3.11)$$

$$\text{subject to } N \geq 0, f(N, \alpha) \leq \delta.$$

Then we have,

$$\tilde{N} = \min_{\alpha} N(\alpha) \quad (3.12)$$

$$\text{subject to } \alpha \in (0, 1).$$

We present the following theorem to provide an explicit solution to (3.9) by considering the sub-problems (3.11) and (3.12).

Theorem 4. *Let*

$$N_o := \sqrt{8\sigma^2 T_f \ln(2/\delta)}. \quad (3.13)$$

Then, for $\alpha = \frac{1}{2}$, the optimal value of (3.11) is given by $N(\frac{1}{2}) = N_o$. Further, for all $\alpha \in (0, 1) \setminus \{\frac{1}{2}\}$, we have, $N(\frac{1}{2}) \leq N(\alpha)$. Thus optimal value of (3.12), or equivalently the optimal value of the problem (3.9), is $\tilde{N} = N(\frac{1}{2}) = N_o$.

Proof. Please see Appendix B.2. □

Therefore, given σ, B_{max}, δ and T_f , an optimal strategy is to initially charge the BESs to half of their maximum capacities, B_{max} (or equivalently $\frac{1}{2}$ p.u.). Further, the DSO can derive the optimal number of required BESs using (3.13) such that the batteries can sustain the demand throughout the time interval $[0, T_f]$ without getting discharged completely or charged to the maximum capacity with a probability of $1 - \delta$.

In the next section, we extend our analysis to a system having two microgrids connected to each other and derive the battery capacity for the entire system. Further, we show how the flow capacity of the power line connecting the MGs affects the required battery capacity and provide a comprehensive system design guideline.

3.4 Battery Sizing and Power Sharing Policy for Connected multi-microgrids

3.4.1 Problem formulation

In this section, we extend our battery capacity determination methodology, described in Section 3.3.3, to an interconnected two-microgrid system, as shown in Fig. 3.2. Each MG has its local RES generation and loads and further, they can share power with each other. Let $P_{12}(t)$ denote the amount of power flow from MG-1 to MG-2 at time $t \in [0, T_f]$. Further, let $P_{gi}(t), D_{ci}(t), b_i(t)$ denote the RES power output, load, and power supplied by the BESs of MG- $i \in \{1, 2\}$ respectively. Hence, in each MG, the total energy supplied by N number of BESs, till time $t \geq 0$, is given by:

$$\begin{aligned} \int_0^t b_1(s) ds &= \int_0^t (D_{c1}(s) - P_{g1}(s)) ds + \int_0^t P_{12}(s) ds, \\ \int_0^t b_2(s) ds &= \int_0^t (D_{c2}(s) - P_{g2}(s)) ds - \int_0^t P_{12}(s) ds. \end{aligned}$$

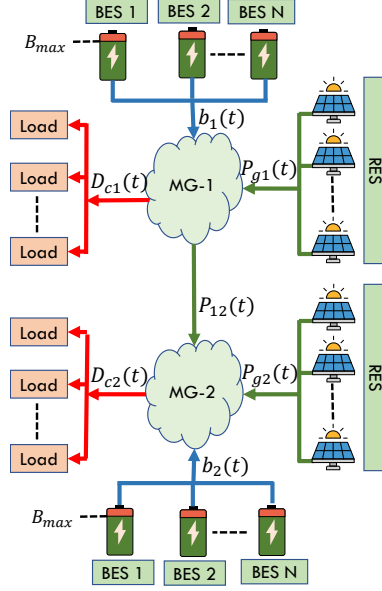


Figure 3.2: Connected two microgrid architecture

Denoting $X_i(t), i \in \{1, 2\}$ as the net p.u. energy available from the BESs of MG- $i, i \in \{1, 2\}$ at time $t \geq 0$, we have, $X_i(t) = X_i(0) - \bar{B} \int_0^t b_i(s) ds$, where $X_i(0)$ denotes the initially available p.u. energy in the BESs. Similar to the case of single MG, let N denote the number of BESs to be installed at each MG- $i, i = 1, 2$, and let $B_0 \in [0, B_{max}]$ kWh be the initial charge held by each of the N BESs at time $t = 0$. Further, let $\alpha := B_0/B_{max}$ denote the initial p.u. energy of each of the BES. Then, we have, $X_i(0) = \alpha N, i \in \{1, 2\}$. Similar to the case of a single microgrid, here, we assume that the net p.u. surplus energy, $\bar{B} \int_0^t (P_{gi}(s) - D_{ci}(s)) ds$, follows a Brownian motion characterized by $\sigma W_i(t), t \in [0, T_f]$, where $\sigma \neq 0$ models the volatility, and $W_i(t)$ is a Weiner process. Further assume $W_i(t)$ is independent of $W_j(t), i \neq j, i, j \in \{1, 2\}$. Here, we have the following:

$$\begin{aligned} X_1(t) &= X_1(0) - \bar{B} \int_0^t b_1(s) ds = \alpha N + \sigma W_1(t) - \bar{B} \int_0^t P_{12}(s) ds, \text{ and,} \\ X_2(t) &= X_2(0) - \bar{B} \int_0^t b_2(s) ds = \alpha N + \sigma W_2(t) + \bar{B} \int_0^t P_{12}(s) ds. \end{aligned} \quad (3.14)$$

Our problem is divided into two stages; in the planning stage, the DSO determines the number of BESs (or equivalently the BESs capacities), N , along with their initial p.u. charge α . Once the batteries are deployed, the DSO controls the power transfer amount,

$P_{12}(t), t \in [0, T_f]$, between the MGs in the operational stage. Let the maximum power flow limit of the line, connecting MG-1 and MG-2, is given by $\bar{P}$. Here, we define the power transfer policy, to be determined by the DSO, by $\pi: [0, T_f] \rightarrow \mathbb{R}$ and let $\Pi(\bar{P}) := \{\pi: [0, T_f] \rightarrow \mathbb{R} | \pi(t) \in [-\bar{P}, \bar{P}] \text{ for all } 0 \leq t \leq T_f\}$ denote the set of permissible policies adhering to the given power line capacity. In the subsequent analysis, we suppress the explicit dependency of $\pi(t)$ on t for the brevity of notation.

Let $X_{i,T_f}^{sup} := \sup_{t \in [0, T_f]} X_i(t)$ and $X_{i,T_f}^{inf} := \inf_{t \in [0, T_f]} X_i(t), i=1, 2$ denote the maximum and minimum p.u. energy available, respectively, at any time $t \in [0, T_f]$ in the BESs of MG- i . Note that, the available energy in the BESs at any time depends on the number N , initial p.u. charge α , and the power transfer policy π . Hence, let the probability that the BESs, given an initial charge of $X_i(0) = \alpha N$, do not get fully charged or emptied any time during the time horizon $[0, T_f]$ be denoted by

$$P(N, \alpha, \pi) := \mathbb{P}[X_{1,T_f}^{sup} < N, X_{1,T_f}^{inf} > 0, X_{2,T_f}^{sup} < N, X_{2,T_f}^{inf} > 0].$$

Here, the objective of the DSO is to determine a suitable control strategy π and initial p.u. charge α , that minimizes the number of BESs, N , and ensures with a given high probability of $1 - \delta$, that the BESs are not fully charged or empty during the time horizon $[0, T_f]$. Therefore, DSO solves the following chance-constrained problem: Given $\bar{P}, \delta, T_f$,

$$\begin{aligned} & \min_{\alpha, N, \pi} N & (3.15) \\ & \text{subject to (3.14), } N \geq 0, 0 \leq \alpha \leq 1, \\ & \pi \in \Pi(\bar{P}), P(N, \alpha, \pi) \geq 1 - \delta. \end{aligned}$$

In the planning stage, the DSO solves (3.15) to decide the number of BESs to be installed in both the MGs, their initial charges, and the power transfer policy. The power transfer policy is followed throughout the time horizon $[0, T_f]$ in the operational stage, once the BESs are deployed. In Section 3.4.2, we present an analytical solution to (3.15). Further note that, the power line capacity $\bar{P}$ affects the solution to (3.15); In the following sections, we characterize such dependency of the optimal solution of (3.15), characterizing the number of batteries required, on the power line capacity.

3.4.2 Solution methodology

Given $\bar{P}, \delta, T_f$, let the solution set of (3.15) be denoted by,

$$\begin{aligned} S_1 &:= \operatorname{argmin}_{N, \alpha, \pi} \{N : (N, \alpha, \pi) \in F_1\}, \text{ where,} \\ F_1 &:= \{(N, \alpha, \pi) : (3.14) \text{ is satisfied}, 0 \leq \alpha \leq 1, N \geq 0, \\ &\quad \pi \in \Pi(\bar{P}), P(N, \alpha, \pi) \geq 1 - \delta\}. \end{aligned}$$

Suppose (N^*, α^*, π^*) solves the problem (3.15), that is $(N^*, \alpha^*, \pi^*) \in S_1$ and $N^* = \min_{N, \alpha, \pi} \{N : (N, \alpha, \pi) \in F_1\}$. In the following theorem, we show how an equivalent tractable problem can be constructed to obtain N^* , the optimal value of (3.15).

Theorem 5. *Consider the following problem:*

$$\begin{aligned} \nu &:= \max_{N, \pi} \{P(N, \alpha^*, \pi) : (N, \pi) \in F_2\}, \text{ where,} \\ F_2 &:= \{(N, \pi) : (3.14) \text{ with } \alpha = \alpha^* \text{ is satisfied}, \\ &\quad \pi \in \Pi(\bar{P}), N \leq N^*\}, \end{aligned} \tag{3.16}$$

and let the solution set of (3.16) be denoted by,

$$S_2 := \operatorname{argmax}_{N, \pi} \{P(N, \alpha^*, \pi) : (N, \pi) \in F_2\}.$$

Suppose $(\hat{N}, \hat{\pi}) \in S_2$, then $\hat{N} = N^*$ and $(\hat{N}, \alpha^*, \hat{\pi}) \in S_1$.

Proof. Please see Appendix B.3. □

Remark: Thus if $(\hat{N}, \hat{\pi})$ solves problem (3.16) then $(\hat{N}, \alpha^*, \hat{\pi})$ also solves problem (3.15) and $\hat{N}$ will be equal to N^* which is the optimal value of (3.15).

Note that, as $(\hat{N}, \alpha^*, \hat{\pi})$ obtained from the solution to (3.16), solves the problem (3.15) as well, we first seek to derive $\hat{\pi}$ from (3.16). In this context, let $\pi'(N, \alpha)$ denote a power transfer policy that maximizes $P(N, \alpha, \pi)$ for a given $N > 0$ and $\alpha \in [0, 1]$, that is,

$$\begin{aligned} \pi'(N, \alpha) &:= \operatorname{argmax}_{\pi} P(N, \alpha, \pi) \\ &\text{subject to (3.14), } \pi \in \Pi(\bar{P}). \end{aligned} \tag{3.17}$$

In Appendix B.4, we derive the policy $\pi'(N, \alpha)$ and show that the policy does not depend on N and α and rather depends on the available energy in the batteries of the MGs. In particular, denoting the optimal policy by $P'_{12}(t) = \pi'(N, \alpha)(t)$, we show that $P'_{12}(t)$ is as shown below:

$$P'_{12}(t) = \begin{cases} \bar{P}, & \text{if } X_1(t) > X_2(t), \\ -\bar{P}, & \text{if } X_1(t) < X_2(t), \\ 0, & \text{if } X_1(t) = X_2(t) \end{cases} \quad (3.18)$$

In the next section, we analyze the required battery capacity under the power transfer policy (3.18) and discuss the trade-offs between the power line capacity, $\bar{P}$, and the battery capacity.

3.5 Trade-offs Between Battery Storage Energy Capacity and Transmission Line Capacity

3.5.1 Limiting Cases of Transmission Line Capacity

First, consider the scenario when $\bar{P} = 0$, that is, the MGs are not connected to each other. In this case, each MG can be treated as a single MG, and the number of BESs required in each can be obtained from the solution methodology for a single MG as discussed in Section 3.3.3. With $\bar{P} = 0$, we have $X_1(t) = \alpha N + \sigma W_1(t)$, and $X_2(t) = \alpha N + \sigma W_2(t)$. Here, the chance-constraint

$$\mathbb{P}[X_{1,T_f}^{sup} < N, X_{1,T_f}^{inf} > 0, X_{2,T_f}^{sup} < N, X_{2,T_f}^{inf} > 0] \geq 1 - \delta, \quad (3.19)$$

can be written as

$$\mathbb{P}[\{X_{1,T_f}^{sup} \geq N\} \cup \{X_{1,T_f}^{inf} \leq 0\} \cup \{X_{2,T_f}^{sup} \geq N\} \cup \{X_{2,T_f}^{inf} \leq 0\}] \leq \delta. \quad (3.20)$$

Since $W_1(t)$ and $W_2(t)$ are independent of each other, it can further be written as

$$\begin{aligned} & \mathbb{P}[\{X_{1,T_f}^{sup} \geq N\} \cup \{X_{1,T_f}^{inf} \leq 0\} \cup \{X_{2,T_f}^{sup} \geq N\} \cup \{X_{2,T_f}^{inf} \leq 0\}] \\ &= \mathbb{P}[\{X_{1,T_f}^{sup} \geq N\} \cup \{X_{1,T_f}^{inf} \leq 0\}] + \mathbb{P}[\{X_{2,T_f}^{sup} \geq N\} \cup \{X_{2,T_f}^{inf} \leq 0\}] \leq \delta. \end{aligned}$$

In this case, we can treat each MG separately and find the minimum number of BESs N , such that

$$\begin{aligned}\mathbb{P}[\{X_{1,T_f}^{sup} \geq N\} \cup \{X_{1,T_f}^{inf} \leq 0\}] &\leq \frac{\delta}{2}, \text{ and,} \\ \mathbb{P}[\{X_{2,T_f}^{sup} \geq N\} \cup \{X_{2,T_f}^{inf} \leq 0\}] &\leq \frac{\delta}{2}.\end{aligned}$$

Let $\tilde{N}_0$ denote the required number of BESs in each MG. Following the single MG methodology of Section 3.3.3, we can obtain

$$\tilde{N}_0 = 2\sqrt{2}\sigma\sqrt{\ln(4/\delta)}\sqrt{T_f}. \quad (3.21)$$

Next, consider the scenario when the power line capacity is infinite. In this case, the combination of the two MGs can be effectively treated as a single MG. Here, the total p.u. energy available from the BESs, at time $t \in [0, T_f]$, is given by $X_1(t) + X_2(t) = \alpha(2N) + \sigma(W_1(t) + W_2(t))$. Further, by definition ([71]), we have $(W_1(t) + W_2(t)) \sim \sqrt{2}W_c(t)$, where $W_c(t)$ is a Wiener process. Denote $\sigma_c = \sqrt{2}\sigma$ and let $\tilde{N}_\infty$ denote the required number of BESs in each MG. Following the methodology for single MG, we have,

$$\begin{aligned}2\tilde{N}_\infty &= 2\sqrt{2}\sigma_c\sqrt{\ln(2/\delta)}\sqrt{T_f} \\ \implies \tilde{N}_\infty &= 2\sigma\sqrt{\ln(2/\delta)}\sqrt{T_f}.\end{aligned} \quad (3.22)$$

Let, $c_\infty = 2\sigma\sqrt{\ln(2/\delta)}$. Then, consolidating (3.21) and (3.22), we have the following

$$\begin{aligned}\tilde{N}_\infty &= c_\infty\sqrt{T_f} \\ \tilde{N}_0 &= K_0 c_\infty\sqrt{T_f}, \text{ where } K_0 = \sqrt{\frac{2\ln(4/\delta)}{\ln(2/\delta)}}.\end{aligned} \quad (3.23)$$

Note that, $\frac{\tilde{N}_0}{\tilde{N}_\infty} = K_0 > 1$. Thus, as the power line capacity is increased, the required number of BESs decreases up to a factor of K_0 . However, a higher power line capacity increases the investment cost of the transmission lines, and thus potentially can nullify the benefits of reducing the required number of BESs. Here, it is pertinent to answer the question if the chance-constraint (3.20) can be met with $\tilde{N}_\infty$ number of BESs with a finite power line capacity. In this context, we present our analysis in the next section.

3.5.2 Necessary and Sufficient Condition on Transmission Line Capacity

To reduce the operational and investment cost of the multi-MG system, it is beneficial to install the least amount of BESs that can meet the probability constraint (3.19). Hence, as discussed in Section 3.5.1, installing $\tilde{N}_\infty$ number of BESs is desirable for the DSO. Note that, to reduce the overall system costs it is also required to have low investment in the power lines. In this section, we discuss the possibility of meeting the probability constraint with $\tilde{N}_\infty$ number of BESs while having a finite amount of power line capacity. In particular, considering $\bar{P} = c_p T_f^\xi$, we show that it is sufficient to have $\xi \in (-\frac{1}{2}, 0)$ while ensuring the probability constraint (3.19) is met for any $\delta \in (0, 1)$ with $\tilde{N}_\infty$ number of BESs. Thus, the DSO can reduce the transmission line capacity as the time horizon T_f increases. In the subsequent paragraphs, we present our analysis in detail.

Consider that N number of BESs is installed in each MG and each BES is initially charged to half of its maximum p.u. capacity, that is the initially available p.u. energy of the BESs is $\frac{1}{2}N$. Suppose, $\bar{P} = c_p T_f^\xi$, where $\xi \in \mathbb{R}$, and $c_p > 0$ is a constant independent of T_f , and let $X_1^{\pi'}(t)$ and $X_2^{\pi'}(t)$ denote the total available p.u. energy in the BESs of MG-1 and MG-2 respectively, following the policy (3.18) at time $t \in [0, T_f]$. Here, from (3.14), we have the following:

$$\begin{aligned} X_1^{\pi'}(t) &= \frac{1}{2}N + \sigma W_1(t) - \bar{B} \int_0^t P'_{12}(s) ds, \text{ and,} \\ X_2^{\pi'}(t) &= \frac{1}{2}N + \sigma W_2(t) + \bar{B} \int_0^t P'_{12}(s) ds, \end{aligned} \quad (3.24)$$

where $P'_{12}(t)$ follows (3.18). We define $X_{i,T_f}^{\pi',sup} := \sup_{0 \leq t \leq T_f} X_i^{\pi'}(t)$, and $X_{i,T_f}^{\pi',inf} := \inf_{0 \leq t \leq T_f} X_i^{\pi'}(t)$, $i = 1, 2$. For a given N , we define the following sets of realizations:

$$\begin{aligned} A_1 &:= \{X_{1,T_f}^{\pi',sup} \geq N\}, \quad A_2 := \{X_{1,T_f}^{\pi',inf} \leq 0\}, \text{ and,} \\ A_3 &:= \{X_{2,T_f}^{\pi',sup} \geq N\}, \quad A_4 := \{X_{2,T_f}^{\pi',inf} \leq 0\}. \end{aligned} \quad (3.25)$$

Here, from (3.19), that the chance-constraint can be equivalently written as

$$\mathbb{P}[A_1 \cup A_2 \cup A_3 \cup A_4] \leq \delta. \quad (3.26)$$

First, we show that if $\xi < -\frac{1}{2}$ and $N = \tilde{N}_\infty$, the DSO cannot always meet the chance-constraint (3.26) for small values of δ . Thus, as the time horizon increases, the power

line capacity cannot be less than $\frac{c_p}{\sqrt{T_f}}$ to ensure the chance-constraint is met for small values of δ with $\tilde{N}_\infty$ number of BESs. To this end, we present the following theorem.

Theorem 6. *Let A_1, A_2, A_3, A_4 be as defined in (3.25) with $N = \tilde{N}_\infty$. Let $\bar{P} = c_p T_f^\xi$, where $\xi < -\frac{1}{2}$, and c_p is a positive constant independent of T_f . Then there exists a $\delta^* \in (0, 1)$, such that for all $0 < \delta \leq \delta^*$, $\lim_{T_f \rightarrow \infty} \mathbb{P}[A_1 \cup A_2 \cup A_3 \cup A_4] > \delta$.*

Proof. Please see Appendix B.8. □

Upon obtaining the necessary condition on ξ to meet the constraint (3.26), next, we show that if $\xi \in (-\frac{1}{2}, 0)$, then for all $\delta \in (0, 1)$, DSO can meet the chance-constraint, with $\tilde{N}_\infty$ number of BESs. Here, we present the following theorem.

Theorem 7. *Let A_1, A_2, A_3, A_4 be as defined in (3.25) with $N = \tilde{N}_\infty$. Given $\delta \in (0, 1)$, and $\bar{P} = c_p T_f^{-\frac{1}{2} + \epsilon}$, where $c_p > 0$ and $0 < \epsilon < \frac{1}{2}$ are constants, we have,*

$$\lim_{T_f \rightarrow \infty} \mathbb{P}[A_1 \cup A_2 \cup A_3 \cup A_4] \leq \delta.$$

We remark that such a result is extremely important for the DSO to assess the installation cost of the power lines. Recall that $\tilde{N}_\infty = c_\infty \sqrt{T_f}$, and hence, as the time horizon increases, the required number of BESs increases resulting in a higher BES investment cost. However, as the power line capacity can be decreased in an order of $\frac{1}{\sqrt{T_f}}$, DSO can reduce the investment cost for the power lines. Such trade-offs between the BESs and transmission line investment costs ensure systematic economic planning of multi-MG systems.

We provide the proof of Theorem 7 utilizing some intermediate results that are presented in Proposition 2, Lemma 2, Lemma 3, and Lemma 4. In the following sections, we present the latter first followed by the proof of Theorem 7.

Let $X_c^{\pi'}(t) := X_1^{\pi'}(t) + X_2^{\pi'}(t)$ and $X_d^{\pi'}(t) := X_2^{\pi'}(t) - X_1^{\pi'}(t)$, $t \in [0, T_f]$. Then, from (3.24), we have the following:

$$\begin{aligned} X_c^{\pi'}(t) &= N + \sigma(W_1(t) + W_2(t)) \\ X_d^{\pi'}(t) &= \sigma(W_2(t) - W_1(t)) + 2\bar{B} \int_0^t P'_{12}(s) ds. \end{aligned} \tag{3.27}$$

Note that, $(W_1(t) + W_2(t)) \sim \sqrt{2}W_c(t)$ ([70]) where $W_c(t)$ is a Wiener process. Hence,

$X_c^{\pi'}(t)$ can be constructed from the process,

$$X_c^{\pi'} = N + \sigma_c W_c(t), \quad (3.28)$$

where $\sigma_c = \sqrt{2}\sigma$. Similarly, $(W_2(t) - W_1(t)) \sim \sqrt{2}W_d(t)$ ([70]) where $W_d(t)$ is a Wiener process. Further, from (3.18), $P'_{12}(t)$ can be written as

$$P'_{12}(t) = -\bar{P} \text{sgn}(X_d^{\pi'}(t)) \times 1 \text{ kW}, \quad (3.29)$$

where, the $\text{sgn}(x)$ function is given as

$$\text{sgn}(x) = \begin{cases} 1, & \text{if } x > 0, \\ -1, & \text{if } x < 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Hence, $X_d^{\pi'}(t)$ can be constructed from the process,

$$X_d^{\pi'} = \sigma_d W_d(t) - 2\bar{B}\bar{P} \int_0^t \text{sgn}(X_d^{\pi'}(s)) ds, \quad (3.30)$$

where $\sigma_d = \sqrt{2}\sigma$. Note that, $X_c^{\pi'}(t)$ does not depend on the power line capacity $\bar{P}$, whereas $X_d^{\pi'}(t)$ does. Furthermore, $X_c^{\pi'}(t)$ and $X_d^{\pi'}(t)$ are one-dimensional processes. Therefore, a tractable analysis of the chance-constraint (3.26) can be done by analyzing the processes $X_c^{\pi'}(t)$ and $X_d^{\pi'}(t)$. We define

$$\begin{aligned} X_{c,T_f}^{\pi',sup} &:= \sup_{t \in [0, T_f]} X_c^{\pi'}(t), \\ X_{c,T_f}^{\pi',inf} &:= \inf_{t \in [0, T_f]} X_c^{\pi'}(t). \end{aligned} \quad (3.31)$$

In the next proposition, we first show how the probability $\mathbb{P}[A_1 \cup A_2 \cup A_3 \cup A_4]$ can be upper bounded in terms of probabilities involving $X_{c,T_f}^{\pi',sup}$, $X_{c,T_f}^{\pi',inf}$ and $X_d^{\pi'}(t)$.

Proposition 2. *Let $X_d^{\pi'}(t)$, and $X_{c,T_f}^{\pi',sup}$, $X_{c,T_f}^{\pi',inf}$ be as defined in (3.30) and (3.31) respectively. Then, for a given N and $0 < \beta < N$, we have*

$$\begin{aligned} &\mathbb{P}[A_1 \cup A_2 \cup A_3 \cup A_4] \\ &\leq \mathbb{P}[X_{c,T_f}^{\pi',sup} \geq 2N - \beta] + \mathbb{P}[X_{c,T_f}^{\pi',inf} \leq \beta] \\ &+ \mathbb{P}[\sup_{0 \leq t \leq T_f} |X_d^{\pi'}(t)| \geq \beta]. \end{aligned}$$

Proof. Please see Appendix B.7. $\square$

We remark that Proposition 2 plays a key role in proving the Theorem 7. Here, as stated in Theorem 7, in the case of $\bar{P} = c_p T_f^{-\frac{1}{2}+\epsilon}$, $0 < \epsilon < \frac{1}{2}$, and $N = \tilde{N}_\infty = c_\infty \sqrt{T_f}$, we choose $\beta = T_f^{(\frac{1}{2}-\psi)}$, where $0 < \psi < \epsilon$. We first analyze the quantity $\mathbb{P}[\sup_{0 \leq t \leq T_f} |X_d^{\pi'}(t)| \geq \beta]$. In particular, we establish that for $\beta = T_f^{(\frac{1}{2}-\psi)}$, and $\bar{P} = c_p T_f^{-\frac{1}{2}+\epsilon}$, $\lim_{T_f \rightarrow \infty} \mathbb{P}[\sup_{0 \leq t \leq T_f} |X_d^{\pi'}(t)| \geq \beta] = 0$. This is established in Lemma 4 via the intermediate Lemmas 2-3, which we present next.

Note that,

$$\begin{aligned} \mathbb{P}[\sup_{0 \leq t \leq T_f} |X_d^{\pi'}(t)| \geq \beta] &= \mathbb{P}[\{\sup_{0 \leq t \leq T_f} X_d^{\pi'}(t) \geq \beta\} \cup \{\inf_{0 \leq t \leq T_f} X_d^{\pi'}(t) \leq -\beta\}] \\ &\leq \mathbb{P}[\sup_{0 \leq t \leq T_f} X_d^{\pi'}(t) \geq \beta] + \mathbb{P}[\inf_{0 \leq t \leq T_f} X_d^{\pi'}(t) \leq -\beta], \end{aligned} \quad (3.32)$$

where the last inequality follows from the union bound on probabilities. To show that $\lim_{T_f \rightarrow \infty} \mathbb{P}[\sup_{0 \leq t \leq T_f} |X_d^{\pi'}(t)| \geq \beta] = 0$, we show $\lim_{T_f \rightarrow \infty} \mathbb{P}[\sup_{0 \leq t \leq T_f} X_d^{\pi'}(t) \geq \beta] = 0$ and $\lim_{T_f \rightarrow \infty} \mathbb{P}[\inf_{0 \leq t \leq T_f} X_d^{\pi'}(t) \leq -\beta] = 0$, and utilize the inequality (3.32).

Let

$$\hat{\tau}_\beta := \inf\{t \geq 0 : X_d^{\pi'}(t) = \beta\},$$

denote the first time the process $X_d^{\pi'}(t)$ reaches the level β . Here we have

$$\mathbb{P}[\sup_{0 \leq t \leq T_f} X_d^{\pi'}(t) \geq \beta] = \mathbb{P}[\hat{\tau}_\beta \leq T_f].$$

Consider an intermediate level $0 < \hat{\beta} < \beta$. Note that to reach the level β , the process $X_d^{\pi'}(t)$ must reach the intermediate level $\hat{\beta}$ first. Thus, an upper bound to $\mathbb{P}[\hat{\tau}_\beta \leq T_f]$ can be obtained by analyzing the number of times the process $X_d^{\pi'}(t)$ reaches the intermediate level $\hat{\beta}$ starting from the value 0, within the time horizon $[0, T_f]$ followed by determining an upper bound on the probability of reaching β from the intermediate level $\hat{\beta}$, for each of those aforementioned times. To formalize the notion of the number of times the process $X_d^{\pi'}(t)$ reaches the intermediate level $\hat{\beta}$ from the value 0, we first define epochs related to the process $X_d^{\pi'}(t)$.

Consider the process $X_d^{\pi'}(t)$ starts at $t = 0$, with a value of $X_d^{\pi'}(0) = 0$ and reaches the level $\hat{\beta}$, for the first time at $\hat{\tau}_{\hat{\beta},0} > 0$. Further, let $\hat{\tau}_{\hat{\beta},0}^+ > \hat{\tau}_{\hat{\beta},0}$ denote the first time the process returns to zero after $\hat{\tau}_{\hat{\beta},0}$. We define the interval $[0, \hat{\tau}_{\hat{\beta},0}^+]$ as epoch 0. To

define the successive epochs $i \in \mathbb{N}/\{0\}$, we introduce the following:

$$\begin{aligned}\hat{\tau}_{\hat{\beta},-1}^+ &:= 0 \\ \hat{\tau}_{\hat{\beta},i} &:= \inf\{t \geq \hat{\tau}_{\hat{\beta},i-1}^+ : X_d^{\pi'} = \hat{\beta}\} \\ \hat{\tau}_{\hat{\beta},i}^+ &:= \inf\{t \geq \hat{\tau}_{\hat{\beta},i} : X_d^{\pi'} = 0\}.\end{aligned}\tag{3.33}$$

Let the start time of any epoch i be when the previous epoch ends, that is $\hat{\tau}_{\hat{\beta},i-1}^+$. Further, $\hat{\tau}_{\hat{\beta},i}$ denotes the first time $X_d^{\pi'}(t)$ reaches the intermediate level $\hat{\beta}$ after the epoch starts. $\hat{\tau}_{\hat{\beta},i}^+$ denotes the first time $X_d^{\pi'}(t)$ returns to zero after $\hat{\tau}_{\hat{\beta},i}$. We define the interval $[\hat{\tau}_{\hat{\beta},i-1}^+, \hat{\tau}_{\hat{\beta},i}^+]$ as epoch i . The epochs are graphically shown in Fig. 3.3 for a sample path of $X_d^{\pi'}(t)$.

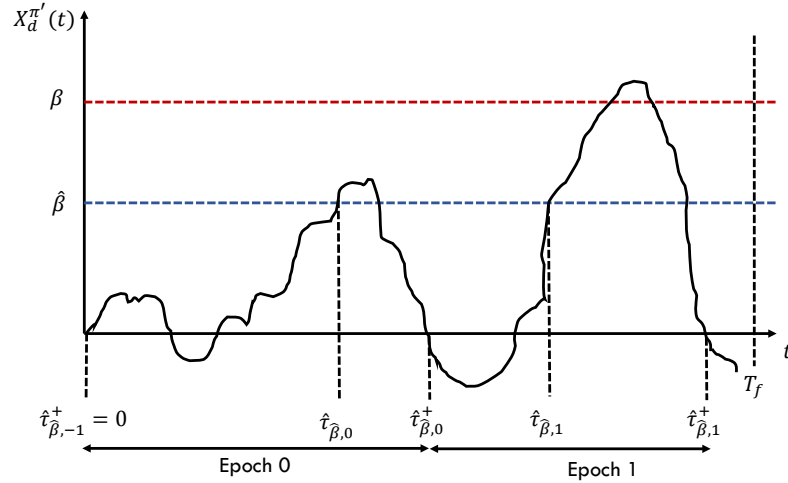


Figure 3.3: Illustration of epochs for a sample trajectory of $X_d^{\pi'}(t)$

Recall that, we wish to find an upper bound to $\mathbb{P}[\hat{\tau}_{\beta} \leq T_f]$ by analyzing the number of times the process $X_d^{\pi'}(t)$ reaches the level $\hat{\beta}$ starting from the value 0, within the time horizon $[0, T_f]$ followed by the probability of reaching β from the level $\hat{\beta}$, for each of those aforementioned times. Here, upon defining the epochs, we proceed as follows: we first choose a time T , and for each epoch, analyze the probability of $X_d^{\pi'}(t)$ reaching $\hat{\beta}$ by the time T after the epoch starts, that is $\mathbb{P}[\hat{\tau}_{\hat{\beta},i} \leq \hat{\tau}_{\hat{\beta},i-1}^+ + T]$. In Lemma 2, we provide an upper bound to the same. Here random variable $\hat{\tau}_{\hat{\beta},i} - \hat{\tau}_{\hat{\beta},i-1}^+$ denotes the time taken by $X_d^{\pi'}(t)$ to reach $\hat{\beta}$ for the first time in epoch i . Since the process $X_d^{\pi'}(t)$ needs to return

to zero, after $\hat{\tau}_{\hat{\beta},i}$, to mark the end of an epoch, the length of an epoch is larger than the time $\hat{\tau}_{\hat{\beta},i} - \hat{\tau}_{\hat{\beta},i-1}^+$. Note that, the length of the epochs determines the number of epochs (or equivalently, the number of times the process $X_d^{\pi'}(t)$ reaches the level $\hat{\beta}$ for the first time starting from the value 0) occurring within the horizon $[0, T_f]$. As in Lemma 2, we provide an upper bound to $\mathbb{P}[\hat{\tau}_{\hat{\beta},i} \leq \hat{\tau}_{\hat{\beta},i-1}^+ + T]$, or equivalently, $\mathbb{P}[\hat{\tau}_{\hat{\beta},i} - \hat{\tau}_{\hat{\beta},i-1}^+ \leq T]$, we can analyze the number of epochs occurring within $[0, T_f]$ by comparing the length of each epoch with T . Here, denoting the random variable K as the number of epochs occurring within the time interval $[0, T_f]$, and $K_{max} := \frac{T_f}{T}$, we provide an upper bound to the probability $\mathbb{P}[K > K_{max}]$ in Corollary 1. Further, we show that as the horizon T_f increases the above-mentioned upper bound on $\mathbb{P}[K > K_{max}]$ decreases. Thus we establish that, for large T_f , the number of epochs occurring within the horizon $[0, T_f]$ is at most K_{max} with high probability. Here, we formally present the Lemma 2 and Corollary 1.

Lemma 2. *Let c_p, ϵ be positive constants, independent of T_f , and let the power line capacity, $\bar{P}$, be given by $\bar{P} = c_p T_f^{-\frac{1}{2} + \epsilon}$. Given $T > 0$, and $\hat{\beta} > 0$, for any epoch $i \in \mathbb{N}$ with $\hat{\tau}_{\hat{\beta},i}, \hat{\tau}_{\hat{\beta},i-1}^+$ defined as in (3.33), we have*

$$\mathbb{P}[\hat{\tau}_{\hat{\beta},i} \leq \hat{\tau}_{\hat{\beta},i-1}^+ + T] \leq e^{-\left(\frac{\hat{\beta}^2}{2\sigma_d^2 T} - \frac{2\bar{B}\hat{\beta}c_p T_f^{-\frac{1}{2} + \epsilon}}{\sigma_d^2}\right)}. \quad (3.34)$$

Proof. Please see Appendix B.9. □

Corollary 1. *Let the random variable K denote the number of epochs occurring within the time interval $[0, T_f]$, and $c_p, \epsilon, \bar{P}$ be as defined in Lemma 2. Given $T > 0$, and $\hat{\beta} > 0$, define $K_{max} := \frac{T_f}{T}$. Then we have,*

$$\mathbb{P}[K > K_{max}] \leq K_{max} e^{-\left(\frac{\hat{\beta}^2}{2\sigma_d^2 T} - \frac{2\bar{B}\hat{\beta}c_p T_f^{-\frac{1}{2} + \epsilon}}{\sigma_d^2}\right)}. \quad (3.35)$$

Proof. Please see Appendix B.10. □

Upon analyzing the number of epochs occurring within the time interval $[0, T_f]$, we analyze the probability of reaching β from the intermediate level $\hat{\beta}$ before the epoch i ends, and the probability of epoch i ending before T_f . In particular, in the following

Lemma, we analyze $\mathbb{P}[\hat{\tau}_{\beta,i} \leq \hat{\tau}_{\hat{\beta},i}^+, \hat{\tau}_{\hat{\beta},i}^+ \leq T_f]$, where,

$$\hat{\tau}_{\beta,i} := \inf\{t \geq \hat{\tau}_{\hat{\beta},i} : X_d^{\pi'}(t) = \beta\}, \quad (3.36)$$

denotes the first time $X_d^{\pi'}(t)$ reaches the level β in epoch i , and $\hat{\tau}_{\hat{\beta},i}$ is as defined in (3.33).

Lemma 3. *Let c_p, ϵ and $\bar{P}$ be as defined in Lemma 2, and let $\beta > \hat{\beta} > 0$, be given. Then for any epoch $i \in \mathbb{N}$, with $\hat{\tau}_{\hat{\beta},i}^+$, and $\hat{\tau}_{\beta,i}$ defined as in (3.33), (3.36) respectively, we have*

$$\begin{aligned} \mathbb{P}[\hat{\tau}_{\beta,i} \leq \hat{\tau}_{\hat{\beta},i}^+, \hat{\tau}_{\hat{\beta},i}^+ \leq T_f] &\leq e^{-\frac{2\bar{B}c_p T_f^{-\frac{1}{2}+\epsilon}(\beta-\hat{\beta})}{\sigma_d^2}}, \\ \mathbb{P}[\hat{\tau}_{\beta,i} \leq T_f, \hat{\tau}_{\hat{\beta},i}^+ > T_f] &\leq e^{-\frac{2\bar{B}c_p T_f^{-\frac{1}{2}+\epsilon}(\beta-\hat{\beta})}{\sigma_d^2}} \end{aligned} \quad (3.37)$$

Proof. Please see Appendix B.11. □

Finally, with the results of Corollary 1 and Lemma 3 in hand, we choose the quantities $T, \hat{\beta}, \beta, K_{max}$ in the following way

$$\begin{aligned} T &= T_\gamma := \frac{1}{4\bar{B}c_p} T_f^{2\gamma-(\epsilon-\psi)}, \\ K_{max} &= K_{max,\gamma} := \frac{T_f}{T} = 4\bar{B}c_p T_f^{1-2\gamma+(\epsilon-\psi)}, \\ \hat{\beta} &= T_f^\gamma, \text{ and,} \\ \beta &= T_f^{(\frac{1}{2}-\psi)}, \end{aligned} \quad (3.38)$$

where $0 < \psi < \epsilon < \frac{1}{2} - \gamma$. We show that as T_f approaches ∞ , the probability of $X_d^{\pi'}(t)$ reaching the level $T_f^{(\frac{1}{2}-\psi)}$ by the time T_f , that is $\mathbb{P}[\hat{\tau}_{T_f^{(\frac{1}{2}-\psi)}} \leq T_f]$, goes to zero. Specifically, we note that the event of $X_d^{\pi'}(t)$ reaching the level $T_f^{(\frac{1}{2}-\psi)}$ from the intermediate level T_f^γ happens in at least in one of the epochs between $[0, T_f]$, and the number of such epochs being greater than $K_{max,\gamma}$ goes to zero (using Corollary 1). Further in any epoch between $[0, T_f]$, the probability of reaching $T_f^{(\frac{1}{2}-\psi)}$ from the intermediate level T_f^γ before the epoch ends, goes to zero (using Lemma 3). Here we present the following Lemma.

Lemma 4. *Let $X_d^{\pi'}(t)$ and $\epsilon, c_p, \bar{P}$ be as defined in (3.30) and Lemma 2 respectively.*

Then for any given $T > 0$ and $\beta, \hat{\beta}$ satisfying $\beta > \hat{\beta} > 0$, we have

$$\begin{aligned} & \mathbb{P}[\sup_{0 \leq t \leq T_f} X_d^{\pi'}(t) \geq \beta] \\ & \leq K_{max} \left[e^{-\left(\frac{\hat{\beta}^2}{2\sigma_d^2 T} - \frac{2\bar{B}\hat{\beta}c_p T_f^{-\frac{1}{2}+\epsilon}}{\sigma_d^2} \right)} \right] + (K_{max} + 1) \left[e^{-\frac{2\bar{B}c_p T_f^{-\frac{1}{2}+\epsilon}(\beta-\hat{\beta})}{\sigma_d^2}} \right], \end{aligned} \quad (3.39)$$

where $K_{max} = \frac{T_f}{T}$. Further if $\hat{\beta} = T_f^\gamma, \beta = T_f^{(\frac{1}{2}-\psi)}$, where $0 < \psi < \epsilon < \frac{1}{2} - \gamma$, and $T = T_\gamma = \frac{1}{4\bar{B}c_p} T_f^{2\gamma-(\epsilon-\psi)}$, $K_{max,\gamma} = \frac{T_f}{T_\gamma}$, then,

$$\begin{aligned} \mathbb{P}[\sup_{0 \leq t \leq T_f} X_d^{\pi'}(t) \geq T_f^{(\frac{1}{2}-\psi)}] & \leq (2K_{max,\gamma} + 1) e^{-\frac{2\bar{B}c_p}{\sigma_d^2} (T_f^{\epsilon-\psi} - T_f^{-\frac{1}{2}+\gamma+\epsilon})} \\ & = (8\bar{B}c_p T_f^{1-2\gamma+(\epsilon-\psi)} + 1) e^{-\frac{2\bar{B}c_p}{\sigma_d^2} (T_f^{\epsilon-\psi} - T_f^{-\frac{1}{2}+\gamma+\epsilon})}, \end{aligned}$$

and hence,

$$\lim_{T_f \rightarrow \infty} \mathbb{P}[\sup_{0 \leq t \leq T_f} X_d^{\pi'}(t) \geq T_f^{(\frac{1}{2}-\psi)}] = 0.$$

Proof. Please see Appendix B.12. $\square$

A symmetric treatment for $\mathbb{P}[\inf_{0 \leq t \leq T_f} X_d^{\pi'}(t) \leq -\beta]$ can be done following the Lemmas 2-4 to show that,

$$\begin{aligned} & \mathbb{P}[\inf_{0 \leq t \leq T_f} X_d^{\pi'}(t) \leq -\beta] \\ & \leq K_{max} \left[e^{-\left(\frac{\hat{\beta}^2}{2\sigma_d^2 T} - \frac{2\bar{B}\hat{\beta}c_p T_f^{-\frac{1}{2}+\epsilon}}{\sigma_d^2} \right)} \right] + (K_{max} + 1) \left[e^{-\frac{2\bar{B}c_p T_f^{-\frac{1}{2}+\epsilon}(\beta-\hat{\beta})}{\sigma_d^2}} \right]. \end{aligned} \quad (3.40)$$

Further, as indicated in Lemma 4, considering $\hat{\beta} = T_f^\gamma, \beta = T_f^{(\frac{1}{2}-\psi)}$, and $\psi < \epsilon < \frac{1}{2} - \gamma, T = T_\gamma = \frac{1}{4\bar{B}c_p} T_f^{2\gamma-(\epsilon-\psi)}$, we can show that $\lim_{T_f \rightarrow \infty} \mathbb{P}[\inf_{0 \leq t \leq T_f} X_d^{\pi'}(t) \leq -\beta] = 0$. Therefore, from the inequality (3.32), it follows that

$$\lim_{T_f \rightarrow \infty} \mathbb{P}[\sup_{0 \leq t \leq T_f} |X_d^{\pi'}(t)| \geq \beta] = 0. \quad (3.41)$$

We now present the proof of Theorem 7 where we show that, with $\beta = T_f^{(\frac{1}{2}-\psi)}$, $\lim_{T_f \rightarrow \infty} (\mathbb{P}[X_{c,T_f}^{\pi',sup} \geq 2\tilde{N}_\infty - \beta] + \mathbb{P}[X_{c,T_f}^{\pi',inf} \leq \beta]) \leq \delta$, and further combine the latter with the results from Lemma 4 and Proposition 2 to conclude that with $N = \tilde{N}_\infty$, and

$\bar{P} = c_p T_f^{-\frac{1}{2} + \epsilon}$, where $0 < \epsilon < \frac{1}{2}$, $\lim_{T_f \rightarrow \infty} \mathbb{P}[A_1 \cup A_2 \cup A_3 \cup A_4] \leq \delta$. Thus, as the horizon T_f increases, the DSO can reduce the power line capacity $\bar{P}$ in an order of $\frac{1}{\sqrt{T_f}}$, and yet meet the chance-constraint (3.20) with $\tilde{N}_\infty$ number of BESs. The formal proof of the Theorem 7 is presented in Appendix B.13.

With the analysis of the trade-offs between BESs and transmission line capacities for large time horizons (T_f approaching ∞) in hand, we now present the BESs and transmission line design methodology for a finite time horizon.

Trade-offs Between BESs and Line Capacities for a Finite Time Horizon

In this section, we discuss the trade-off between the BESs and transmission line capacities for a finite time horizon utilizing the results obtained in the previous section. Recall that, the DSO has to ensure $\mathbb{P}[A_1 \cup A_2 \cup A_3 \cup A_4] \leq \delta$ by a proper design of the number of BESs, N , and the transmission line capacity $\bar{P}$, while following the power transfer policy (3.29). Further, recall from Proposition 2, that an upper bound to $\mathbb{P}[A_1 \cup A_2 \cup A_3 \cup A_4]$ is obtained in terms of the probabilities involving the processes $X_d^{\pi'}(t)$ and $X_d^{\pi'}(t)$. In particular, for a given N and $0 < \beta < N$, we have $\mathbb{P}[A_1 \cup A_2 \cup A_3 \cup A_4] \leq \mathbb{P}[X_{c,T_f}^{\pi',sup} \geq 2N - \beta] + \mathbb{P}[X_{c,T_f}^{\pi',inf} \leq \beta] + \mathbb{P}[\sup_{0 \leq t \leq T_f} |X_d^{\pi'}(t)| \geq \beta]$. Here, we proceed as follows: we choose a $\Delta \in (0, 1)$, and find N, β , and $\bar{P}$, that ensure

$$\begin{aligned} \mathbb{P}[X_{c,T_f}^{\pi',sup} \geq 2N - \beta] + \mathbb{P}[X_{c,T_f}^{\pi',inf} \leq \beta] &\leq (1 - \Delta)\delta, \text{ and,} \\ \mathbb{P}[\sup_{0 \leq t \leq T_f} |X_d^{\pi'}(t)| \geq \beta] &\leq \Delta\delta, \end{aligned}$$

thereby ensuring $\mathbb{P}[A_1 \cup A_2 \cup A_3 \cup A_4] \leq \delta$. In the following Lemmas, we provide such solutions to N, β , and $\bar{P}$, and establish that a choice of smaller Δ leads to a lower number of BESs, N , however, increases the required transmission line capacity $\bar{P}$.

Lemma 5. *Let $X_{c,T_f}^{\pi',sup}$ and $X_{c,T_f}^{\pi',inf}$ be as defined in (3.28), and $\Delta \in (0, 1)$, $0 < \psi < \frac{1}{2}$, $c_\beta > 0$ are given constants. Further, for a given $\delta \in (0, 1)$, let $N = K\tilde{N}_\infty$, where*

$$K = \sqrt{1 + \frac{\ln(1/(1-\Delta))}{\ln(2/\delta)}} + \frac{\Delta T_f^{-\psi} + c_\beta T_f^{-\frac{1}{2}}}{2\sigma\sqrt{\ln(2/\delta)}}. \quad (3.42)$$

Define $\beta_{T_f} := \Delta T_f^{(\frac{1}{2} - \psi)} + c_\beta$. Then we have,

$$\mathbb{P}[X_{c,T_f}^{\pi',sup} \geq 2N - \beta_{T_f}] + \mathbb{P}[X_{c,T_f}^{\pi',inf} \leq \beta_{T_f}] \leq (1 - \Delta)\delta.$$

Proof. Please see Appendix B.14. $\square$

Lemma 6. *Let Δ, ψ be as defined in Lemma 5. For a given $\delta \in (0, 1)$, and T_f , suppose γ is chosen such that $0 < \gamma < \frac{1}{2}$, and $T_f^{(\frac{1}{2}-\psi)} > T_f^\gamma$. Let the transmission line capacity be given as $\bar{P} = c_p T_f^{-\frac{1}{2}+\epsilon}$, where $c_p > 0$ is a constant, and $\epsilon \in \mathbb{R}$ satisfies*

$$(8\bar{B}c_p T_f^{1+(\epsilon-\psi)-2\gamma} + \Delta) e^{-\frac{2\bar{B}c_p \Delta [T_f^{-\psi} - T_f^{-\frac{1}{2}+\gamma}] T_f^\epsilon}{\sigma_d^2}} = \frac{\Delta^2 \delta}{2} \quad (3.43)$$

Then, with $X_d^{\pi'}(t)$ as defined in (3.30) and β_{T_f} as defined in Lemma 5, we have

$$\mathbb{P}[\sup_{0 \leq t \leq T_f} |X_d^{\pi'}(t)| \geq \beta_{T_f}] \leq \Delta \delta.$$

Proof. Please see Appendix B.15. $\square$

Combining Lemma 5 and Lemma 6 with Proposition 2, we conclude that, with $N = K\tilde{N}_\infty$, where K is given as in (3.42), and $\bar{P} = c_p T_f^{-\frac{1}{2}+\epsilon}$, where ϵ satisfies (3.43),

$$\begin{aligned} & \mathbb{P}[A_1 \cup A_2 \cup A_3 \cup A_4] \\ & \leq \mathbb{P}[X_{c,T_f}^{\pi',sup} \geq 2N - \beta_{T_f}] + \mathbb{P}[X_{c,T_f}^{\pi',inf} \leq \beta_{T_f}] \\ & + \mathbb{P}[\sup_{0 \leq t \leq T_f} |X_d^{\pi'}(t)| \geq \beta_{T_f}] \\ & = (1 - \Delta)\delta + \Delta\delta = \delta. \end{aligned}$$

Thus, DSO can meet the chance-constraint. We emphasize that the choice of Δ determines the trade-offs between the required BESs capacity and the transmission line capacity. Given the horizon of operation T_f and δ , note from (3.42) that, as Δ decreases, K decreases, thus reducing the required BESs capacity to meet the chance constraint. However, from (3.43), we can observe that a choice of smaller Δ leads to a higher value of ϵ , thus increasing the required transmission line capacity.

3.6 Experimental Results

In this section, we demonstrate the applicability and utility of our proposed BES capacity design and power transfer policy determination methodology. First, we consider the case of a single microgrid where a modified IEEE-33 bus distribution network

with a total load of 15 kW is taken as the system under study. A battery, operated in Grid Forming Mode (GFM), is connected to bus 1 and a wind farm is connected to bus 4. The parameters, considered, for the real-time simulation, are as follows: $\sigma = 1, \delta = 0.02, B_{max} = 1 \text{ kWh}, T_f = 5 \text{ hours}$. The power network is emulated inside OP5700 real-time simulator box, manufactured by OPAL-RT. The RT simulation is conducted using eMEGASIM-based EMT-simulation platform of RTLAB with a simulation step size of $50\mu s$. Using Theorem 4, GFM battery capacity is set to 13 kWh with initial battery charge set to 6.5 kWh. The system is run for 5 hours in real-time. The wind and load variation and the power provided by the GFM battery are shown in Fig. 3.4a. The battery State-of-the Charge (SoC) is shown in Fig. 3.4b. It can be seen that, throughout the time horizon of 5 hours, the battery SoC neither hits the maximum capacity of 13 kWh, nor gets completely depleted, establishing the utility of our proposed battery sizing methodology.

Next, we consider a system of two MGs, connected to each other via a power line. The parameters, considered for the simulation, are as follows: $\sigma = 1, \delta = 0.02, B_{max} = 1 \text{ kWh}, \bar{P} = 15 \text{ kW}, T_f = 5 \text{ hours}$. The time step, Δt , for the simulation is chosen to be 30 seconds. Using the methodology described in Section 3.4, the BES capacity of each MG is set to 10 kWh. The power transfer policy, (3.18), is employed. Fig. 3.5 shows the simulation results for 5000 runs, where we plot the available energy in the BES of each MG. It is observed that in only 0.4% of the total number of realizations, either the SoC of MG-1 battery or MG-2 battery hits the upper limit of 10 kWh or hits the lower limit of 0 kWh, which is well within the chosen limit of 2% ($\delta=0.02$). Next, we empirically analyze the effect of power line capacity, $\bar{P}$ under our proposed power transfer policy, on the required BES capacity to ensure that both the battery SoC remains within the operating range with a probability of at least $(1 - \delta) = 0.98$. Here, in Fig. 3.6, we plot the required BES capacity, averaged over 5000 runs, as we vary the power line capacity $\bar{P}$. It is observed that under our proposed power transfer policy, with the increase in line capacity, the MGs can share more power with each other, which in turn reduces the required BES capacity. Thus our proposed BES sizing and power transfer policy can provide a system design guideline for the distribution system operators.

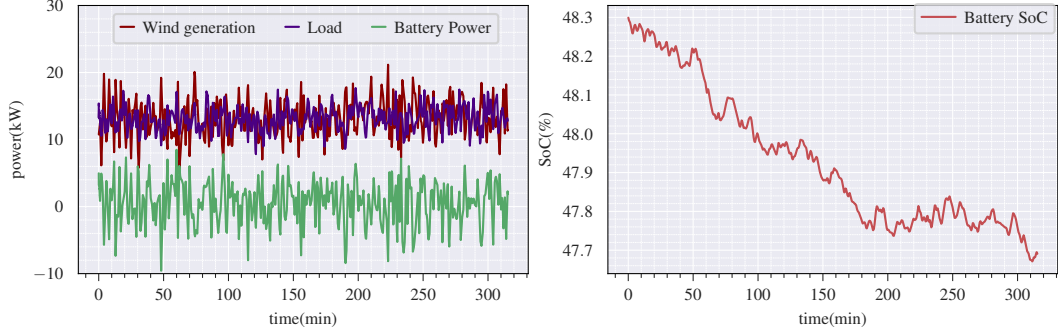


Figure 3.4: (a) Wind, load variation, and GFM battery power for 5 hours in real-time in the IEEE-33 bus single MG system. (b) The battery does not get fully charged or get depleted throughout the time horizon.

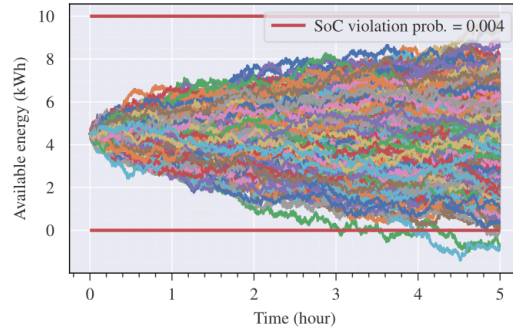


Figure 3.5: Two MG system: Battery capacity of each MG is 10 kWh with a power line capacity of 15 kW. Out of 5000 realizations of the available energy in the batteries of MG-1 and MG-2, only 0.4% realizations, hits either the upper limit of 10 kWh or the lower limit of 0 kWh.

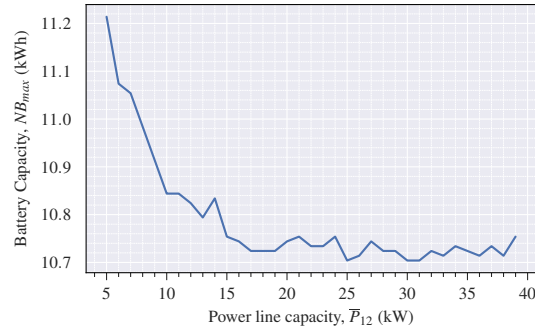


Figure 3.6: Required BES capacity for each MG decreases, under the optimal power transfer control policy, as power line capacity increases.

Chapter 4

Topology Change Estimation in Power Distribution Networks: A Machine Learning Approach

4.1 Gist of the Chapter

The proliferation of distributed energy resources has provided the modern distribution grid with flexibility in terms of operation and control, however, it has necessitated a real-time situational awareness of the entire system. A critical component in grid operations is the knowledge of the distribution network and the capability to detect and localize any topology changes, including line outages and additions, in near real-time. The topology change detection requirement is traditionally achieved with the breaker status measurements. However, the circuit breakers' intrusive nature increases line losses and makes the deployment and repair work difficult. In this context, the utilization of phasor data, consisting of bus voltage and phase angle data, is a robust alternative. In this article, we propose a topology change detection, localization, and estimation technique that only relies on phasor data. In particular, we devise a machine learning methodology to detect line outages and additions, along with the timestamp of such changes. Further, to localize the topology changes, we employ a phasor measurement-based topology-learning algorithm and intertwine the same with our change detection strategy. We perform real-time hardware-in-the-loop experiments on a modified IEEE

33-bus, 226-bus, and 1262-bus network emulated on a real-time simulator. Our experimental validation corroborates the utility and scalability of our proposed methodology.

4.2 State-of-the-art on Topology Change Estimation

In recent years, traditional power distribution systems have witnessed an architectural evolution due to the proliferation of distributed energy resources (DERs) and microgrids. Further, increased penetration of renewable energy sources and electric vehicles has transformed the landscape of grid operation and control strategy, ranging from traditional methodologies such as state estimation, and optimal power flow analysis, etc. to modern ones such as data-driven system control, demand response, transactive energy, etc. A critical component in all such architectures is the knowledge of the underlying grid topology. Usually, the topology information is conveyed to the grid operator during the planning stage, however, the increased frequency of climate-related catastrophes necessitates the real-time knowledge of line outages and corresponding changes in the grid. Apart from that, the operational flexibility of microgrids, and distributed energy prosumers causes frequent changes in the power network. Therefore, it is pertinent to detect changes and estimate topology in real time. Naturally, such a requirement propelled the widespread deployment of Advanced Metering Infrastructure (AMI) and transformed the physical grid into a cyber-physical system (CPS). Therefore, it is crucial to explore how to utilize the innovations in AMI to devise topology change detection and estimation methodologies.

The most common method to estimate and detect topology changes relies on the breaker status and power flow measurements from the field [72]. However, an increase in cyber-attack events on circuit breakers and flow meters [73, 74] in the last decade renders them no longer reliable. Such attacks are not only targeted at the physical layer but also at the communication layer, where an attacker spoofs the breaker status and power flow measurements [75] to manipulate the topology information conveyed to the grid operator. If the system is operated based on falsified topology it could lead to catastrophic system failure. Here, the utilization of phasor measurement units (PMUs) is a viable alternative [76], primarily for two reasons; 1) high-resolution samples provide near real-time monitoring capability (a typical PMU provides 30 measurement samples per second) [77], 2) they are synchronized to global positioning system (GPS) and protected by multiple advanced cyber-security tools, and thus are very difficult to manipulate [78, 79]. Therefore, the possibility of utilizing PMU measurements for

topology change detection and estimation is capable of offering immunity against cyber-attacks. In this context, the following questions arise:

- 1) How to detect any change in the topology (line outage and/or addition) based on PMU measurements only?
- 2) If any change is detected, how to estimate the time instant of such topology change? Note that, such information is crucial to prevent system collapse and to initiate protective measures in a timely manner.
- 3) How to estimate the changed topology? We emphasize that the topology estimation is essential to efficiently operate the system and to detect the location of change, if any.

4.2.1 Prior Works

In prior literature, topology change identification is analyzed both in the context of line faults and cyber-topology attacks. For fault localization, various optimization methods [80, 81, 82, 83], mixed-integer programming [84, 85], and graphical models [86, 87, 88] are explored. In [80], the authors propose a measurement residual-based optimization approach to identify the changes in the grid topology, however, such an approach utilizes prior information about the network. A mixed-integer quadratic programming approach is proposed in [84] that combines PMU and smart meter measurements to identify the topology and line faults in real-time. Here the strategy relies on the smart meter measurements, thus making the methodology vulnerable to cyber-attacks. In this context, a state estimation technique-based algorithm is proposed in [86] that detects and localizes the faults by finding suitable sub-graphs of the power networks from the PMU measurements, assuming that the power network is known and the changes in the network are limited to line faults. However, in the case of a large distribution network such information may not be available to the system operator and line additions may alter the topology as well. To address such issues, in [88] a topology change identification methodology is proposed utilizing the structure of the admittance matrix of power networks and PMUs. However, the changes are limited to a single change at a time which is restrictive. Apart from the classical methods, various data-driven algorithms are proposed for network change detection [89, 90, 91]. In [89, 90] convolutional neural networks are proposed to identify system topology and events from PMU data. Although such approaches reduce the dependency on the complex physical modeling of the distribution system, gathering a significant amount of real topology change data for training purposes is very challenging and suffers from scalability as the network size

increases. Another common methodology to identify topology changes is to estimate the network admittance matrices, with data collected before and after the change event, followed by taking the difference between them. Here, various data-driven algorithms are proposed focusing on topology learning [92, 93, 94] and joint-learning and parameter estimation [95, 96]. However, the common deficiency in this approach is the inherent assumption that the change instant is known. In real scenarios, the system operator needs to detect the changes first before deploying such strategies. Further, most of the existing learning algorithms [92, 93, 94, 95] utilize smart meter data which makes the methodologies vulnerable to cyber-attacks.

In the context of cyber-topology attacks various data-driven algorithms are proposed [75], [97, 98, 99, 100, 101]. The cyber-topology attack mechanisms, where the attacker masks line outages (or additions) by carefully manipulating the smart meters, are illustrated in [75], [97]. Building upon such attacks, in [98, 99, 100, 101], the authors propose various data-driven models to identify cyber-topology attacks. While the methodologies are shown to effectively detect the presence of topology changes under stealthy cyber-attacks, the strategies suffer from the inability to localize the changes. Further, the learning methods proposed in [98, 99, 100, 101] require a large amount of real data, containing different types of network events, for model training, which is sometimes difficult to obtain in real scenarios. Here, building a combined model that can detect and localize the network events simultaneously, without requiring historical data, is of utmost importance. In summary, the topology change identification methodologies proposed in [80, 81, 82, 83, 84, 85, 86, 87] either consider that prior information about the network is available to detect any topology events or utilize both PMU and smart meter data, thus making the algorithms prone to cyber-attacks. In contrast, the existing data-driven algorithms [89, 90, 91], [92, 93, 94, 95, 96] can effectively detect topology changes from field data, however, either suffer from the assumption that the change instant is known apriori or require a large amount of historical data of various change scenarios. Further, the works [92, 93, 94, 95] utilize smart meter data and thus are vulnerable to cyber-attacks. Here, works on network change identification [97, 98, 99, 100, 101] address the possibility of cyber-topology attacks but cannot localize the network changes.

4.2.2 Key Contributions of the Chapter

We propose an end-to-end architecture that detects line outages and/or additions along with the corresponding change time instants and localizes the changes via topology estimation methodology. We emphasize that our methodology relies only on voltage and phase angle measurements, obtained from the PMUs, thus offering robustness to cyber-topology attacks. Upon obtaining the PMU data, we first detect the presence of line outages/additions and estimate the instants of such changes using machine-learning (ML) models. Subsequently, we utilize such information to learn the changed topology and identify the faulty/newly added branches. We instantiate our methodology to three different power grid networks with in-depth performance analysis to corroborate the efficacy of our proposed method. Our contributions are summarized below:

(1) We propose an ML-powered topology change detection technique capable of detecting one or multiple changes from high-fidelity PMU measurements. Our solution is built upon the Expectation-Maximization (EM) algorithm and does not use any prior information about the network.

(2) We provide a solution to estimating the instant of line outage/addition from the timestamps of the PMU data. Our methodology supports the detection of multiple changes occurring at the same instant or multiple instants distributed across time.

(3) We propose a hierarchical architecture that utilizes the PMU data distribution model parameters, estimated from our EM-based change detection technique, to learn the underlying topology and estimate the location of line outages/additions.

(4) We deploy our methodology to the IEEE 33-bus system and a 226-bus system comprised of a part of the University of Minnesota and surrounding suburbs. To demonstrate the scalability of our technique, we instantiate the same on a large 1262-bus distribution network. We perform hardware-in-the-loop experiments on real-time simulator and present a detailed performance analysis to showcase the efficacy of our proposed method.

4.3 Problem Formulation

Consider a power distribution network represented by an undirected graph, $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the set of $N + 1$ buses and $\mathcal{E}$ denotes the set of operational lines present in the network. The linearized AC power flow [102] equation for power injection at bus i

can be written as

$$p_i + \hat{i}q_i = \sum_{j:(ij) \in \mathcal{E}} \frac{(v_i - v_j) - \hat{i}(\theta_i - \theta_j)}{z_{ij}^*}, \quad (4.1)$$

where p_i, q_i, v_i , and θ_i represent active power, reactive power injections, bus voltage magnitude, and bus angle respectively. z_{ij} is the complex impedance of the line (ij) and $\hat{i}^2 = -1$, and z_{ij}^* denotes the complex conjugate of z_{ij} . The nodal injection at the slack (reference) bus is the negative sum of power injection at all other buses. Without loss of generality, in the subsequent analysis, we exclude the slack bus and restrict the power flow equation (4.1) to the N number of non-slack buses. Note that, the power injection at bus i includes load demand at bus i and renewable generation, such as solar power, if available at that bus. Let $\mathbf{p} = [p_1, p_2, \dots, p_N]^T$, $\mathbf{q} = [q_1, q_2, \dots, q_N]^T$, $\mathbf{v} = [v_1, v_2, \dots, v_N]^T$, and $\boldsymbol{\theta} = [\theta_1, \theta_2, \dots, \theta_N]^T$ denote the vector of active, reactive power injections and bus voltages and angles, respectively, of all the buses in the grid excluding the slack bus. It is assumed that, under normal operating conditions of the grid, power injection follows Gaussian distribution [103], where $[\mathbf{p}^T \mathbf{q}^T]^T \sim \mathcal{N}(\boldsymbol{\mu}_{pq}, \boldsymbol{\Sigma}_{pq})$, with $\boldsymbol{\mu}_{pq} \in \mathbb{R}^{2N}$ and $\boldsymbol{\Sigma}_{pq} \in \mathbb{R}^{2N \times 2N}$ representing mean and covariance of power injections. We further assume that $[p_i \ q_i]^T$ and $[p_j \ q_j]^T$ are uncorrelated if $i \neq j$, for all $i, j \in \mathcal{V}$. Let $\mathcal{X} = \{\mathbf{x}^{(n)}\}_{n=1}^S$ denote the set of $S \in \mathbb{N}$ number of i.i.d phasor measurement samples collected from the distribution grid, where $\mathbf{x}^{(n)} = [(\mathbf{v}^{(n)})^T \ (\boldsymbol{\theta}^{(n)})^T]^T \in \mathbb{R}^{2N}$, and $\mathbf{v}^{(n)}, \boldsymbol{\theta}^{(n)}$ indicate n -th phasor measurement sample containing bus voltages and angles respectively. Further, let $t^{(n)}$ denote the timestamp of the n -th phasor measurement sample, $\mathbf{x}^{(n)}$, and let $\mathcal{T} := \{t^{(n)}\}_{n=1}^S$ denote the set of timestamps of the voltage and phase angle samples collected from the distribution grid. We refer to a sample collection window as the time taken to collect S phasor samples, that is $t^{(S)} - t^{(1)}$.

Note that, the set of the operational lines, $\mathcal{E}$, is not known a priori. Moreover, the topology may change multiple times during the sample collection window. Suppose the topology is changed C times, during a sample collection window, at the time instants $c_1, \dots, c_C$, and let K denote the total number of different topologies encountered, where $K = 1$ (or equivalently $C = 0$) indicates no change. Here, C does not need to be equal to K as the topology can switch back and forth multiple times during a sample collection window leading to a higher value of C than K . Further, let $\mathcal{E}_i$ denote the set of operational lines present in topology $i \in \{1, \dots, K\}$. Note that none of the parameters, $C, c_1, \dots, c_C, K$, and $\{\mathcal{E}_i\}_{i=1}^K$ are known a priori. Hence, to identify if there

is any change in the topology, the control center needs to estimate how many different topologies are encountered during a sample collection window. Further, if any change is detected, it needs to estimate the change instants, and the associated topologies before and after each change instant, to localize the changes. Therefore, the problem of the control center is three-fold: Given the parameters $\boldsymbol{\mu}_{pq}, \boldsymbol{\Sigma}_{pq}$ and the voltage magnitude and phase angle samples, $\mathcal{X} = \{\mathbf{x}^{(n)}\}$, and the timestamps of the same, $\mathcal{T} = \{t^{(n)}\}_{n=1}^S$,

(1) *Change detection*: Estimate the number of topologies encountered, K . A value of $K = 1$ indicates no change is detected.

(2) *Change instant detection*: If any change is detected, estimate the total number of changes, C , and the change instants, $\{c_i\}_{i=1}^C$, as a prerequisite for change localization.

(3) *Change localization*: Estimate the topologies, $\{\mathcal{E}_i\}_{i=1}^K$, and for each change instant $c_i, i \in \{1, \dots, C\}$, identify which topologies, out of the K different topologies, are encountered before and after that instant. The difference between them indicates the location of the changes.

In the next section, we describe our proposed solution methodology.

4.4 Solution Methodology

Let the random variable $[\mathbf{v}^T \boldsymbol{\theta}^T]^T$ be represented by $\mathbf{X}$ for ease of exposition. We first derive the probability distribution function (PDF) of $\mathbf{X}$ and in the subsequent paragraphs, show how the parameters of the derived PDF can be utilized to detect topology changes and change instants.

Let $\mathbf{Z} = [z_1 \dots z_K]^T$ be a K -dimensional random vector of indicator variables, satisfying $z_k \in \{0, 1\}$ for all $k \in \{1, \dots, K\}$ and $\sum_{k=1}^K z_k = 1$. Note that, $\mathbf{Z}$ can take the values in $\{\mathbf{e}_1, \dots, \mathbf{e}_K\}$, where $\mathbf{e}_k = [0 \dots 0 \ 1 \ 0 \dots 0]^T \in \mathbb{R}^K$, are the standard basis vectors of $\mathbb{R}^K$ with all elements 0 except the k -th one, which is 1. Suppose the event $\{\mathbf{Z} = \mathbf{e}_k\}$ represents that the underlying topology is $\mathcal{E}_k$, and let $\pi_k := \mathbb{P}[\mathbf{Z} = \mathbf{e}_k], k \in \{1, \dots, K\}$. We first show that, given the information that the underlying topology is $\mathcal{E}_k$ (that is given the event $\{\mathbf{Z} = \mathbf{e}_k\}$), the conditional distribution of $\mathbf{X}$ is a Gaussian distribution.

Let $E_k := \{\alpha : \mathbf{Z}(\alpha) = \mathbf{e}_k\}$. For $\alpha \in E_k$ (that is the underlying topology is $\mathcal{E}_k$), and considering $z_{ij} = \frac{1}{g_{ij} + i\beta_{ij}}$, where g_{ij} and β_{ij} are conductance and susceptance of line

$(ij) \in \mathcal{E}_k$, (4.1) can be written in matrix form [88] as follows:

$$\mathbf{X}(\alpha) = \begin{bmatrix} \mathbf{v} \\ \boldsymbol{\theta} \end{bmatrix}(\alpha) = \mathbf{H}_{\mathcal{E}_k}^{-1} \begin{bmatrix} \mathbf{p} \\ \mathbf{q} \end{bmatrix}(\alpha), \text{ with } \mathbf{H}_{\mathcal{E}_k} = \begin{bmatrix} \mathbf{H}_{\mathcal{E}_k, g} & \mathbf{H}_{\mathcal{E}_k, \beta} \\ \mathbf{H}_{\mathcal{E}_k, \beta}^T & -\mathbf{H}_{\mathcal{E}_k, g} \end{bmatrix}. \quad (4.2)$$

Here $\mathbf{H}_{\mathcal{E}_k, g}$ and $\mathbf{H}_{\mathcal{E}_k, \beta}$ matrices are defined as follows:

$$\mathbf{H}_{\mathcal{E}_k, g}(i, j) = \begin{cases} \sum_{m: (im) \in \mathcal{E}_k} g_{im}, & \text{if } i = j \\ -g_{ij}, & \text{if } i \neq j, (ij) \in \mathcal{E}_k \\ 0, & \text{otherwise} \end{cases}. \quad (4.3)$$

Here, we consider line charging capacitance to be non-zero, that is $\mathbf{H}_{\mathcal{E}_k}(i, i) \neq 0$. $\mathbf{H}_{\mathcal{E}_k, \beta}$ is also defined similarly with g_{ij} replaced by β_{ij} in (4.3). Note that the active and reactive power injections do not depend on the topology, and since $[\mathbf{p}^T \mathbf{q}^T]^T \sim \mathcal{N}(\boldsymbol{\mu}_{pq}, \boldsymbol{\Sigma}_{pq})$, from (4.2), we have that, for $\alpha \in E_k$, the random variable $\mathbf{X}(\alpha) \sim \mathcal{N}(\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)$, where,

$$\begin{aligned} \boldsymbol{\Sigma}_k &= \mathbf{H}_{\mathcal{E}_k}^{-1} \boldsymbol{\Sigma}_{pq} \mathbf{H}_{\mathcal{E}_k}^{-1} \\ \boldsymbol{\mu}_k &= \mathbf{H}_{\mathcal{E}_k}^{-1} \boldsymbol{\mu}_{pq}. \end{aligned} \quad (4.4)$$

Let $\phi_k = (\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)$ and let $f_{\mathbf{X}|\mathbf{Z}, \phi_k}(\mathbf{x}|\mathbf{Z} = \mathbf{e}_k)$ denote the conditional distribution function of $\mathbf{X}$ given $\{\mathbf{Z} = \mathbf{e}_k\}$. Here, we have that $f_{\mathbf{X}|\mathbf{Z}, \phi_k}(\mathbf{x}|\mathbf{Z} = \mathbf{e}_k)$ is a Gaussian distribution parameterized by ϕ_k , which is given by

$$f_{\mathbf{X}|\mathbf{Z}, \phi_k}(\mathbf{x}|\mathbf{Z} = \mathbf{e}_k) = \frac{1}{(2\pi)^N |\boldsymbol{\Sigma}_k|^{1/2}} e^{[-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu}_k)^T \boldsymbol{\Sigma}_k^{-1}(\mathbf{x} - \boldsymbol{\mu}_k)]}. \quad (4.5)$$

In summary, given the event that the underlying topology is $\mathcal{E}_k$, the conditional distribution of $\mathbf{X}$ follows a Gaussian distribution. We now derive the PDF of $\mathbf{X}$ in terms of $f_{\mathbf{X}|\mathbf{Z}, \phi_k}(\mathbf{x}|\mathbf{Z} = \mathbf{e}_k)$ and $\pi_k = \mathbb{P}[\mathbf{Z} = \mathbf{e}_k]$, $k \in \{1, \dots, K\}$.

Let $\boldsymbol{\Phi}_K := (\pi_1, \dots, \pi_K, \phi_1, \dots, \phi_K)$, and let $f_{\mathbf{X}|\boldsymbol{\Phi}_K}(\mathbf{x})$ denote the PDF of $\mathbf{X}$. In Appendix C.1, we show that, $f_{\mathbf{X}|\boldsymbol{\Phi}_K}(\mathbf{x})$ can be written as follows:

$$f_{\mathbf{X}|\boldsymbol{\Phi}_K}(\mathbf{x}) = \sum_{k=1}^K \pi_k f_{\mathbf{X}|\mathbf{Z}, \phi_k}(\mathbf{x}|\mathbf{Z} = \mathbf{e}_k), \quad (4.6)$$

with $\sum_{k=1}^K \pi_k = 1$, $\pi_k \geq 0$ for all $k \in \{1, \dots, K\}$. The distribution model represented by (4.6) is known as Gaussian mixture model where $\pi_k, k \in \{1, \dots, K\}$ are the mixing coefficients [104].

Note that, we have S number of observed samples, $\mathcal{X} = \{\mathbf{x}^{(n)}\}_{n=1}^S$, drawn from the distribution of $\mathbf{X}$ which is given by (4.6). If we can estimate the parameters $\phi_1, \dots, \phi_K$ from the observed sample set $\mathcal{X}$, we can utilize each ϕ_k to learn the k -th topology, $k \in \{1, \dots, K\}$, using the topology learning algorithm described in [88]. However, the estimation of ϕ_k from the observed samples using standard maximum likelihood estimation (MLE) [105] is not feasible as π_k are unknown. Furthermore, the number of different topologies, K , is also not known a priori. Here, we proceed as follows: we iterate over various values of K in $\{1, 2, \dots, \bar{K}\}$, where $\bar{K}$ is a design parameter, and for each value of K , (i) we estimate Φ_K , using Expectation-Maximization (EM) algorithm [106, 107] from the observed sample set $\mathcal{X}$ (described in detail in Section 4.4.1), and (ii) compute a metric (described in Section 4.4.2) indicating goodness of fit of the model (4.6) (with respect to the value of K) for the observed sample set $\mathcal{X}$. The metric penalizes both under-fitting (K value lower than the actual number of different topologies) and over-fitting (K value higher than the actual number of different topologies) of the model (4.6). Once the metric is computed for each value of $K \in \{1, \dots, \bar{K}\}$, we compare the same to obtain the optimal value of K . Let $\hat{\Phi}_K = (\hat{\pi}_1, \dots, \hat{\pi}_K, \hat{\phi}_1, \dots, \hat{\phi}_K)$ be the estimate of the parameters Φ_K for each value of $K \in \{1, \dots, \bar{K}\}$. Here $\hat{\cdot}$ notation is used to denote the estimates of the corresponding quantities. Further, let $\hat{K} \in \{1, \dots, \bar{K}\}$ be the optimal value of K . We use $\hat{\Phi}_{\hat{K}}$ for the subsequent objectives of learning the topologies and the change instants. The workflow is shown graphically in Fig. 4.1.

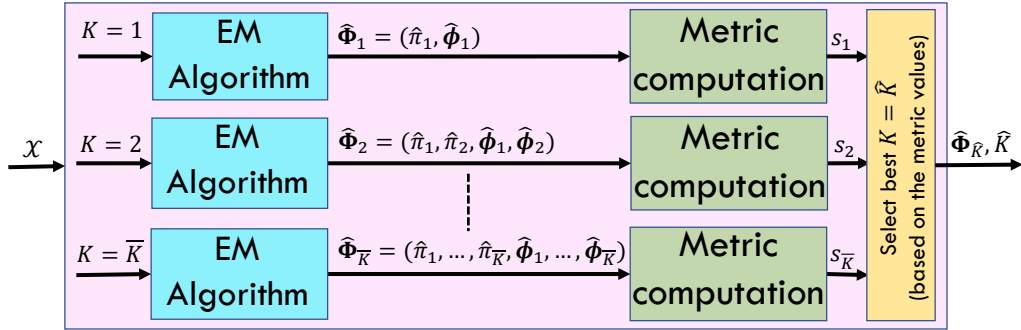


Figure 4.1: Change detection flowchart

Note that, once the parameters $\hat{\Phi}_{\hat{K}}$ are estimated, for each sample $\mathbf{x}^{(n)}$, we can compute the likelihood of the same being generated by k -th topology, $k \in \{1, \dots, \hat{K}\}$

(discussed in detail in Section 4.4.1). The likelihood values are used to classify each sample under one of the $\hat{K}$ number of topologies. In particular, each sample $\mathbf{x}^{(n)}$ is classified to be generated by the topology for which the likelihood value is the highest [106, 107]. Upon classifying each sample in the sample set $\mathcal{X} = \{\mathbf{x}^{(n)}\}_{n=1}^S$, under one of the $\hat{K}$ topologies, we utilize the timestamps of the samples, $\mathcal{T} = \{t^{(n)}\}_{n=1}^S$, to detect the time instants of topology changes, which we describe in detail in Section 4.4.3. Finally, to detect the change locations, we utilize the estimates of $\hat{\boldsymbol{\phi}}_k = (\hat{\boldsymbol{\mu}}_k, \hat{\boldsymbol{\Sigma}}_k)$, $k \in \{1, \dots, \hat{K}\}$, where $\hat{\boldsymbol{\mu}}_k, \hat{\boldsymbol{\Sigma}}_k$ denote the estimates of $\boldsymbol{\mu}_k$ and $\boldsymbol{\Sigma}_k$ respectively. In particular, we learn the topologies from $\hat{\boldsymbol{\Sigma}}_k$ using the topology learning algorithm described in [88]. Thus, the estimates $\hat{K}$ and $\hat{\boldsymbol{\Phi}}_{\hat{K}}$, obtained from the voltage magnitude and phase angle samples, and utilization of the timestamps of the samples enables us to meet all the objectives described in Section 4.3. In the subsequent paragraphs, we describe the entire methodology in detail.

4.4.1 Expectation-Maximization

We first describe, for a given value of K , how $\boldsymbol{\Phi}_K$ can be estimated.

Given K and the observed sample set $\mathcal{X}$ containing S number of i.i.d phasor samples, we define the log-likelihood function as follows:

$$\mathcal{L}(\boldsymbol{\Phi}_K|\mathcal{X}) := \ln [\prod_{n=1}^S f_{\mathbf{X}|\boldsymbol{\Phi}_K}(\mathbf{x}^{(n)})], \quad (4.7)$$

where $f_{\mathbf{X}|\boldsymbol{\Phi}_K}(\cdot)$ is given in (4.6). The estimation of $\boldsymbol{\Phi}_K$ can be formulated as a maximum likelihood estimation problem [105]: Given K , and the probability distribution models (4.5), (4.6),

$$\begin{aligned} \hat{\boldsymbol{\Phi}}_K &= \operatorname{argmax}_{\boldsymbol{\Phi}_K} \mathcal{L}(\boldsymbol{\Phi}_K|\mathcal{X}) \\ \text{such that } &\sum_{k=1}^K \pi_k = 1, \pi_k \geq 0, \text{ for all } k \in \{1, \dots, K\}. \end{aligned} \quad (4.8)$$

However, finding an analytical solution to $\hat{\boldsymbol{\Phi}}_K$ is intractable. Therefore, we utilize iterative optimization approaches such as the Expectation-Maximization (EM) algorithm [106, 107]. For ease of exposition, in the following paragraphs, we highlight the key components of the EM algorithm in the context of topology change detection. A detailed explanation is provided in Appendix C.2.

Recall that $\mathbf{Z}$ is a K -dimensional random vector of indicator variables taking values in $\{\mathbf{e}_1, \dots, \mathbf{e}_K\}$, where $\mathbf{e}_k$ are the standard basis vectors of $\mathbb{R}^K$. Here, we have, $f_{\mathbf{X}|\boldsymbol{\Phi}_K}(\mathbf{x}) =$

$\sum_{z \in \{e_1, \dots, e_K\}} f_{\mathbf{X}, \mathbf{Z} | \Phi_K}(\mathbf{x}, z)$, where $f_{\mathbf{X}, \mathbf{Z} | \Phi_K}(\mathbf{x}, z)$ denotes the joint density of $\mathbf{X}$ and $\mathbf{Z}$. EM algorithm is an iterative algorithm, where in iteration $l + 1$, the estimate of Φ_K , denoted by, $\Phi_K^{l+1} = (\pi_1^{l+1}, \dots, \pi_K^{l+1}, \phi_1^{l+1}, \dots, \phi_K^{l+1})$, is computed by solving the following maximization problem [104]:

$$\begin{aligned} \Phi_K^{l+1} &= \operatorname{argmax}_{\Phi_K} \sum_{n=1}^S \\ &[\mathbb{E}_{\mathbf{Z} \sim f_{\mathbf{Z} | \mathbf{X}, \Phi_K^l}} [\ln(f_{\mathbf{X}, \mathbf{Z} | \Phi_K}(\mathbf{x}^{(n)}, \mathbf{Z})) | \mathbf{X} = \mathbf{x}^{(n)}]] \\ &\text{such that } \sum_{k=1}^K \pi_k = 1, \pi_k \geq 0 \text{ for all } k \in \{1, \dots, K\}, \end{aligned} \quad (4.9)$$

where $f_{\mathbf{Z} | \mathbf{X}, \Phi_K^l}(\cdot)$ denotes the conditional density of $\mathbf{Z}$ given $\mathbf{X}$, parameterized by the estimate of Φ_K at the previous iteration l , denoted by Φ_K^l , and the objective function $\sum_{n=1}^S \mathbb{E}_{\mathbf{Z} \sim f_{\mathbf{Z} | \mathbf{X}, \Phi_K^l}} [\ln(f_{\mathbf{X}, \mathbf{Z} | \Phi_K}(\mathbf{x}^{(n)}, \mathbf{Z})) | \mathbf{X} = \mathbf{x}^{(n)}]$ is expressed analytically in terms of the parameters $\pi_1, \dots, \pi_K$ and $\phi_1, \dots, \phi_K$ as we describe in the next paragraphs. Equation (4.9) defines the EM algorithm. The algorithm works as follows:

Expectation step: The key component of the expectation step is to estimate the likelihood of the n -th sample belonging to the k -th topology, that is $\mathbb{P}[\mathbf{Z} = \mathbf{e}_k | \mathbf{X} = \mathbf{x}^{(n)}]$, for each $k \in \{1, \dots, K\}$. At iteration $l + 1$, the likelihood is estimated based on the parameter estimates in the previous iteration, Φ_K^l . Note that, the likelihood values indicate the classification of each sample $\mathbf{x}^{(n)}$ under one of the K topologies (higher the likelihood, more the probability that $\mathbf{x}^{(n)}$ belongs to that topology). For brevity of notation, let $h_k^{n,l} := \mathbb{P}[\mathbf{Z} = \mathbf{e}_k | \mathbf{X} = \mathbf{x}^{(n)}, \Phi_K^l]$ be the likelihood of $\mathbf{x}^{(n)}$ belonging to k -th topology, parameterized by Φ_K^l . Using Baye's rule [104], we can show that $h_k^{n,l}$ can be expressed as follows:

$$h_k^{n,l} = \frac{\pi_k^l |\Sigma_k^l|^{-\frac{1}{2}} e^{-\frac{1}{2}(\mathbf{x}^{(n)} - \boldsymbol{\mu}_k^l)^T (\Sigma_k^l)^{-1} (\mathbf{x}^{(n)} - \boldsymbol{\mu}_k^l)}}{\sum_{k=1}^K \pi_k^l |\Sigma_k^l|^{-\frac{1}{2}} e^{-\frac{1}{2}(\mathbf{x}^{(n)} - \boldsymbol{\mu}_k^l)^T (\Sigma_k^l)^{-1} (\mathbf{x}^{(n)} - \boldsymbol{\mu}_k^l)}}, \quad (4.10)$$

where $\pi_k^l, \boldsymbol{\mu}_k^l$, and Σ_k^l are the estimates of the corresponding quantities at iteration l . Once $h_k^{n,l}$ is computed, we can analytically express the objective function of (4.9), that is $\sum_{n=1}^S \mathbb{E}_{\mathbf{Z} \sim f_{\mathbf{Z} | \mathbf{X}, \Phi_K^l}} [\ln(f_{\mathbf{X}, \mathbf{Z} | \Phi_K}(\mathbf{x}^{(n)}, \mathbf{Z})) | \mathbf{X} = \mathbf{x}^{(n)}]$, as a function of Φ_K (which is to

be maximized in the maximization step) as follows:

$$\begin{aligned}
& \sum_{n=1}^S \mathbb{E}_{\mathbf{Z} \sim f_{\mathbf{Z}|\mathbf{X}, \Phi_K^l}} [\ln(f_{\mathbf{X}, \mathbf{Z}|\Phi_K}(\mathbf{x}^{(n)}, \mathbf{Z})) | \mathbf{X} = \mathbf{x}^{(n)}] \\
&= \sum_{n=1}^S \sum_{\mathbf{z} \in \{e_1, \dots, e_K\}} \ln(f_{\mathbf{X}, \mathbf{Z}|\Phi_K}(\mathbf{x}^{(n)}, \mathbf{z})) \mathbb{P}[\mathbf{Z} = \mathbf{z} | \mathbf{X} = \mathbf{x}^{(n)}, \Phi_K^l] \\
&= \sum_{n=1}^S \sum_{k=1}^K \ln(f_{\mathbf{X}, \mathbf{Z}|\Phi_K}(\mathbf{x}^{(n)}, e_k)) \mathbb{P}[\mathbf{Z} = e_k | \mathbf{X} = \mathbf{x}^{(n)}, \Phi_K^l]. \tag{4.11}
\end{aligned}$$

Now recall that given the event $\{\mathbf{Z} = e_k\}$, the conditional distribution of $\mathbf{X}$ is a Gaussian distribution parameterized by $\phi_k = (\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)$, and hence, $f_{\mathbf{X}, \mathbf{Z}|\Phi_K}(\mathbf{x}^{(n)}, e_k) = f_{\mathbf{X}|\mathbf{Z}, \phi_k}(\mathbf{x}^{(n)} | \mathbf{Z} = e_k) \mathbb{P}[\mathbf{Z} = e_k]$, where $f_{\mathbf{X}|\mathbf{Z}, \phi_k}(\cdot)$ is given in (4.5). Further, recall that $h_k^{n,l} = \mathbb{P}[\mathbf{Z} = e_k | \mathbf{X} = \mathbf{x}^{(n)}, \Phi_K^l]$, hence from (4.11) it follows that

$$\begin{aligned}
& \sum_{n=1}^S \mathbb{E}_{\mathbf{Z} \sim f_{\mathbf{Z}|\mathbf{X}, \Phi_K^l}} [\ln(f_{\mathbf{X}, \mathbf{Z}|\Phi_K}(\mathbf{x}^{(n)}, \mathbf{Z})) | \mathbf{X} = \mathbf{x}^{(n)}] \\
&= \sum_{n=1}^S \sum_{k=1}^K \ln(f_{\mathbf{X}|\mathbf{Z}, \phi_k}(\mathbf{x}^{(n)} | \mathbf{Z} = e_k) \mathbb{P}[\mathbf{Z} = e_k]) h_k^{n,l} \\
&= \sum_{n=1}^S \sum_{k=1}^K \ln(\pi_k f_{\mathbf{X}|\mathbf{Z}, \phi_k}(\mathbf{x}^{(n)} | \mathbf{Z} = e_k)) h_k^{n,l}, \tag{4.12}
\end{aligned}$$

where in the last equality, we use the definition $\pi_k = \mathbb{P}[\mathbf{Z} = e_k]$. Note that at iteration l , $h_k^{n,l}$ is computed for each $k \in \{1, \dots, K\}$ using (4.10), and $f_{\mathbf{X}|\mathbf{Z}, \phi_k}(\cdot)$ is given in (4.5). Therefore, the objective function of (4.9) is analytically expressed as a function of the parameters $\Phi_K = (\pi_1, \dots, \pi_K, \phi_1, \dots, \phi_K)$. In the maximization step, we maximize the aforementioned analytical function with respect to the parameters Φ_K .

Maximization step: In the M-step, we maximize the function (4.12) with respect to the parameters π_k , and $\phi_k, k \in \{1, \dots, K\}$ under the constraint given in (4.9), which produces the updated values of π_k^{l+1} , $\boldsymbol{\mu}_k^{l+1}$ and $\boldsymbol{\Sigma}_k^{l+1}$ as shown below [104]:

$$\begin{aligned}
\pi_k^{l+1} &= \frac{\sum_{n=1}^S h_k^{n,l}}{S}, \quad \boldsymbol{\mu}_k^{l+1} = \frac{\sum_{n=1}^S h_k^{n,l} \mathbf{x}^{(n)}}{\sum_{n=1}^S h_k^{n,l}} \\
\boldsymbol{\Sigma}_k^{l+1} &= \frac{\sum_{n=1}^S h_k^{n,l} (\mathbf{x}^{(n)} - \boldsymbol{\mu}_k^{l+1})(\mathbf{x}^{(n)} - \boldsymbol{\mu}_k^{l+1})^T}{\sum_{n=1}^S h_k^{n,l}}. \tag{4.13}
\end{aligned}$$

For each iteration $l+1$, as we obtain the estimates of the entire parameter set Φ_K^{l+1} , we compute the value of the log-likelihood $\mathcal{L}(\Phi_K^{l+1} | \mathcal{X})$ using (4.7) and (4.6) and check for convergence by evaluating if $|\mathcal{L}(\Phi_K^{l+1} | \mathcal{X}) - \mathcal{L}(\Phi_K^l | \mathcal{X})| \leq \epsilon$, for a given tolerance $\epsilon \geq 0$. In summary, upon convergence of the EM algorithm for a given K , we obtain an estimate

of the parameters Φ_K , denoted by $\hat{\Phi}_K$. Next, we describe how the optimal value of K is chosen.

4.4.2 Finding the Number of Different Topologies

We run the EM algorithm for each $K \in \{1, \dots, \bar{K}\}$, where $\bar{K}$ is a design parameter chosen by the control center. For each value of K , upon estimating the parameters of the Gaussian components of the mixture model (4.6) using EM algorithm, we compute the silhouette score [108] to assess the separation between the Gaussian components. The silhouette score combines the mean intra-class and inter-class separation in such a way that for both over-fitting (K value higher than actual number of topologies) and under-fitting (K value lower than actual) scenarios, the score is low. In the subsequent paragraphs, we briefly describe the silhouette score and how we utilize the same to estimate the optimal value of K .

Recall that, for each value of K , upon the convergence of the EM algorithm, we obtain the parameter estimate $\hat{\Phi}_K$. As explained in the E-step, we compute the likelihood of each sample $\mathbf{x}^{(n)}$, belonging to k -th topology, that is $\mathbb{P}[\mathbf{Z} = \mathbf{e}_k | \mathbf{X} = \mathbf{x}^{(n)}, \hat{\Phi}_K]$ for each $k \in \{1, \dots, K\}$ using (4.10), where $\pi_k^l, \mu_k^l, \Sigma_k^l$ are replaced with their corresponding estimates $\hat{\pi}_k, \hat{\mu}_k$, and $\hat{\Sigma}_k$ respectively. To classify each sample under one of the K topologies, we find k for which the likelihood value is the highest. In particular, for a given value of K , $\mathbf{x}^{(n)}$ is estimated to be generated by topology $k^{(n)} = \operatorname{argmax}_{k \in \{1, \dots, K\}} \mathbb{P}[\mathbf{Z} = \mathbf{e}_k | \mathbf{X} = \mathbf{x}^{(n)}, \hat{\Phi}_K]$. Let the set of samples, classified under k -th topology, be denoted by

$$\mathcal{X}_k := \{\mathbf{x} \in \mathcal{X} : k = \operatorname{argmax}_{i \in \{1, \dots, K\}} \mathbb{P}[\mathbf{Z} = \mathbf{e}_i | \mathbf{X} = \mathbf{x}, \hat{\Phi}_K]\},$$

for each $k \in \{1, \dots, K\}$. Upon classifying the samples, next, we describe how the silhouette score is computed based on the classification results.

Suppose a sample $\mathbf{x}^{(n)}$ is estimated to be generated by k -th topology, that is $\mathbf{x}^{(n)} \in \mathcal{X}_k$. The silhouette score of $\mathbf{x}^{(n)}$, denoted by $s(\mathbf{x}^{(n)})$, is defined as follows: If $|\mathcal{X}_k| = 1$, then $s(\mathbf{x}^{(n)})$ is zero. Otherwise, let $a(\mathbf{x}^{(n)}) = \frac{1}{|\mathcal{X}_k| - 1} \sum_{\mathbf{x}^{(m)} \in \mathcal{X}_k, m \neq n} d(\mathbf{x}^{(n)}, \mathbf{x}^{(m)})$ be the mean distance between the sample $\mathbf{x}^{(n)}$ and all other samples classified under the same topology k , where $d(\mathbf{x}^{(n)}, \mathbf{x}^{(m)})$ denote the distance between $\mathbf{x}^{(n)}, \mathbf{x}^{(m)}$. Further, let $b(\mathbf{x}^{(n)}) = \min_{j \in \{1, \dots, K\}, j \neq k} \frac{1}{|\mathcal{X}_j|} \sum_{\mathbf{x}^{(m)} \in \mathcal{X}_j} d(\mathbf{x}^{(n)}, \mathbf{x}^{(m)})$ be the minimum mean distance of $\mathbf{x}^{(n)}$ from all other samples classified under topology $j \neq k$. Then the silhouette score of $\mathbf{x}^{(n)}$ is defined as $s(\mathbf{x}^{(n)}) = \frac{b(\mathbf{x}^{(n)}) - a(\mathbf{x}^{(n)})}{\max(a(\mathbf{x}^{(n)}), b(\mathbf{x}^{(n)}))}$. Note that, $a(\mathbf{x}^{(n)})$ represents the

mean intra-class distance, and $b(\mathbf{x}^{(n)})$ represents the minimum value of mean inter-class distance between $\mathbf{x}^{(n)}$ and samples classified under topology $j \neq k, j \in \{1, \dots, K\}$. Note that, in the case of over-fitting, both intra-class and inter-class distances ($a(\mathbf{x}^{(n)})$ and $b(\mathbf{x}^{(n)})$) are low and on the other hand, in the case of under-fitting, both intra-class and inter-class distances are high. Hence, for both under-fitting and over-fitting scenarios, $s(\mathbf{x}^{(n)})$ is low. In the case of a good fit, inter-class distance is expected to be higher than the intra-class distance, that is $a(\mathbf{x}^{(n)}) < b(\mathbf{x}^{(n)})$, and hence a value of $s(\mathbf{x}^{(n)})$ closer to one is desirable. We compute the average silhouette score $s_K = \frac{1}{S} \sum_{\mathbf{x}^{(n)} \in \mathcal{X}} s(\mathbf{x}^{(n)})$ for each $K \in \{2, \dots, \bar{K}\}$. The optimal value of K is chosen as the one that results in the highest value of average silhouette score, that is $\hat{K} = \operatorname{argmax}_{K \in \{2, \dots, \bar{K}\}} s_K$. Note that, $K = 1$ indicates that there is no topology change during the sample collection window. Hence to consider such a scenario, we compare $s_{\hat{K}}$ to an empirically tuned threshold s_{th} , and if $s_{\hat{K}} \leq s_{th}$, we declare no change is observed, that is $\hat{K} = 1$. The entire topology change detection algorithm is summarized in Algorithm 4. In summary, upon running Algorithm 4, we obtain $\hat{K}$ and meet the objective of change detection. We further obtain $\hat{\Phi}_{\hat{K}}$, as required to meet the objectives of change localization. Apart from that, we obtain $\{k^{(n)}\}_{n=1}^S$ that classifies each phasor sample under $\hat{K}$ different topologies, and in the next section, we describe how we utilize the same towards meeting the objective of change instant detection.

4.4.3 Change instant detection

In this section, we discuss how the change instants are estimated once a change in topology is detected using the methodology described in the previous Sections 4.4.1 and 4.4.2. Recall that, upon running Algorithm 4, we obtain $\hat{K}$ and for each observed sample, $\mathbf{x}^{(n)} \in \mathcal{X}$, we obtain which of the $\hat{K}$ different topologies the sample is generated from, that is $k^{(n)}, n = 1, \dots, S$. Further, recall that timestamp $t^{(n)}$, associated with each sample $\mathbf{x}^{(n)}$ is also collected by the control center. Based on the classification of each sample $\mathbf{x}^{(n)}$, we classify the associated timestamp $t^{(n)}$ as well. In particular, denoting the set of timestamps by $\mathcal{T} = \{t^{(n)}\}_{n=1}^S$, we define $\mathcal{T}_k := \{t^{(n)} \in \mathcal{T} : k = k^{(n)}\}, k \in \{1, \dots, \hat{K}\}$ to denote the set of timestamps of the voltage and phase angle samples estimated to be generated by k -th topology. In summary, for each topology $k \in \{1, \dots, \hat{K}\}$, we obtain the set of voltage and phase angle samples generated by that topology, $\mathcal{X}_k$, and the set of timestamps $\mathcal{T}_k$, indicative of the time at which the topology was active during the sample collection window. Next, we describe how the sets $\mathcal{T}_k$ are

Algorithm 4: Topology change detection

```

Function detectChanges ( $\mathcal{X} = \{\mathbf{x}^{(n)}\}_{n=1}^S, \bar{K}, \bar{l}, s_{th}, \epsilon$ ):
  for  $K = 1$  to  $\bar{K}$  do
    for  $l = 0$  to  $\bar{l}$  do
      Initialize  $\Phi_K^0$ ;
       $\mathcal{L}(\Phi_K^0 | \mathcal{X}) \leftarrow \sum_{n=1}^S \ln [f_{\mathbf{X} | \Phi_K^0}(\mathbf{x}^{(n)})]$ ;
      for  $k = 1$  to  $K$  do
        /* E-step */
        for  $n = 1$  to  $S$  do
          | Compute  $h_k^{n,l}$  using (4.10);
        end
        /* M-step */
        Compute  $\pi_k^{l+1}, \mu_k^{l+1}$ , and  $\Sigma_k^{l+1}$  using (4.13);
      end
       $\mathcal{L}(\Phi_K^{l+1} | \mathcal{X}) \leftarrow \sum_{n=1}^S \ln [f_{\mathbf{X} | \Phi_K^{l+1}}(\mathbf{x}^{(n)})]$ ;
       $\hat{\Phi}_K \leftarrow \Phi_K^{l+1}; \hat{\mathcal{L}}_K(\hat{\Phi}_K | \mathcal{X}) \leftarrow \mathcal{L}(\Phi_K^{l+1} | \mathcal{X}); \hat{l} \leftarrow l$ ;
      if  $|\mathcal{L}(\Phi_K^{l+1} | \mathcal{X}) - \mathcal{L}(\Phi_K^l | \mathcal{X})| \leq \epsilon$  then
        | break;
      end
    end
    /* Computation of silhouette score */
    if  $K \neq 1$  then
      | Compute the average silhouette score  $s_K$ ;
    end
  end
   $\hat{K} \leftarrow \operatorname{argmax}_{K \in \{2, \dots, \bar{K}\}} s_K$ ;
  if  $s_{\hat{K}} \leq s_{th}$  then
    |  $\hat{K} = 1$ ;
  end
  for  $n = 1$  to  $S$  do
    |  $k^{(n)} \leftarrow \operatorname{argmax}_{k \in \{1, \dots, \hat{K}\}} h_k^{n, \hat{l}}$ ;
  end
  return  $\hat{K}, \hat{\Phi}_{\hat{K}}, \{k^{(n)}\}_{n=1}^S$ ;

```

utilized to find the number of changes and the change instants.

Suppose for each $k \in \{1, \dots, \hat{K}\}$, the histogram of the timestamps in $\mathcal{T}_k$ is plotted on the time axis. Then the overlapping regions between the histograms of $\mathcal{T}_i$ and $\mathcal{T}_j, i, j \in \{1, \dots, \hat{K}\}, i \neq j$, would indicate the time when the topology is changed from i to j . However, since the overlapping regions highly depend on the bin widths chosen for the histograms, obtaining a single change instant from each overlapping region is difficult to generalize. Therefore, we estimate the probability density function of the timestamps in $\mathcal{T}_k$, from its histogram to obtain the best estimates of such change instants. The density function is estimated using Kernel Density Estimation (KDE)

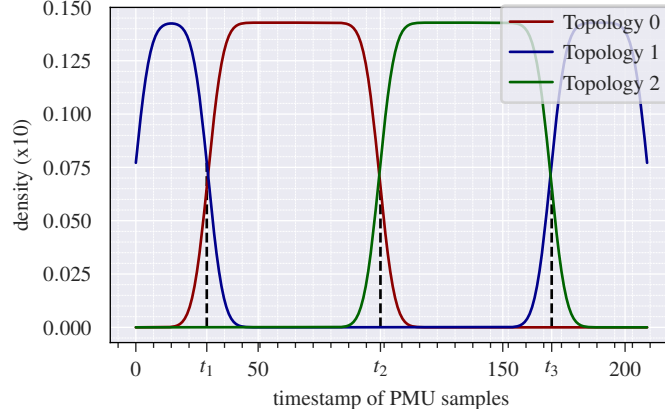


Figure 4.2: Illustration of the proposed change instant detection technique

[104] method. Let $\hat{f}_{T|k}(t)$ denote the estimated density function of the samples in $\mathcal{T}_k$. In KDE, the density is estimated as follows:

$$\hat{f}_{T|k}(t) = \frac{1}{|\mathcal{T}_k|h} \sum_{t^{(i)} \in \mathcal{T}_k} K_r\left(\frac{t-t^{(i)}}{h}\right),$$

where $K_r(\cdot)$ is a non-negative function known as Kernel function, and $h > 0$ is a design parameter termed as bandwidth [104]. Note that, the intersecting point between the density functions of the samples in $\mathcal{T}_i$, and $\mathcal{T}_j, i \neq j$ is indicative of the instant of change between topology i and j . The methodology is illustrated in Fig. 4.2 for an example with three different topologies. For ease of illustration, the timestamps are considered to be an array of integers between 0 and 209. Suppose, after running the EM algorithm, samples with timestamps $\{0, \dots, 29, 170, \dots, 209\}$ are found to be generated by topology 1, and samples with timestamps $\{30, \dots, 99\}, \{100, \dots, 169\}$ are found to be generated by topology 0 and 2 respectively. Using a Gaussian kernel with bandwidth $h = 5$, the corresponding density estimates are shown in Fig. 4.2. Observe that, the intersecting points at t_1, t_2 , and t_3 are found to be at 29, 100, and 170 which are the instants of changes from topology 1 to 0, 0 to 2 and 2 to 0 respectively. The entire algorithm is described in Algorithm 5, where $\bar{C} > 0$, and $\epsilon_n > 0$ are design parameters chosen by the control center. Therefore, upon running Algorithm 5, we obtain the set of change instants, $\mathcal{C} = \{c_1, \dots, c_C\}$, and the total number of changes, $C = |\mathcal{C}|$, meeting the change instant detection objective, described in Section 4.3. In the next subsection, we describe how we utilize the estimates of $\hat{K}, \hat{\Phi}_{\hat{K}}$ and the change instants in $\mathcal{C}$ as found from Algorithm 4 and 5, towards meeting the remaining objective of change localization.

Algorithm 5: Change instant detection

```

Function getChangeInstants ( $\mathcal{T} = \{t^{(n)}\}_{n=1}^S, \hat{K}, h, K_r(\cdot), \overline{C}, \epsilon_n$ ):
    /* Initialization
     $\mathcal{C} \leftarrow []$ ;  $j \leftarrow 0$ ;
    for  $k = 1$  to  $\hat{K}$  do
         $\mathcal{L}_k \leftarrow \text{zeroes}(S)$ ;
        Compute  $\mathcal{T}_k$ ;
        for  $n = 1$  to  $S$  do
             $\mathcal{L}_k[n] \leftarrow \ln[\hat{f}_{\mathcal{T}|k}(t^{(n)})]$ ;
        end
    end
    for  $k_1 = 1$  to  $\hat{K} - 1$  do
        for  $k_2 = k_1 + 1$  to  $\hat{K}$  do
             $\mathbf{d} \leftarrow |\mathcal{L}_{k_1} - \mathcal{L}_{k_2}|$ ;
             $\mathbf{d}_s \leftarrow \text{argsort } \mathbf{d} \text{ in ascending order;}$ 
            for  $i = 0$  to  $\overline{C}$  do
                if  $i == 0$  then
                    Append  $c_j = t^{(\mathbf{d}_s[i])}$  to  $\mathcal{C}$ ;  $j \leftarrow j + 1$ ;
                end
                else
                     $q \leftarrow |t^{(\mathbf{d}_s[i])} - t^{(\mathbf{d}_s[i-1])}|$ ;
                    if  $q \geq \epsilon_n$  then
                        Append  $c_j = t^{(\mathbf{d}_s[i])}$  to  $\mathcal{C}$ ;  $j \leftarrow j + 1$ ;
                    end
                end
            end
        end
    end
    return  $\mathcal{C}$ ;
    */

```

4.4.4 Topology Estimation and Change Localization

Note that, $\hat{\Sigma}_k$, learned from Algorithm 4, represents the covariance of the voltage magnitude and phase angles of the samples collected from topology $k \in \{1, \dots, \hat{K}\}$. We employ a learning methodology [88] that utilizes $\hat{\Sigma}_k$ to estimate the topology using an empirically tuned parameter $\tau \geq 0$. The algorithm is shown in Algorithm 6. Further, note that to enhance the reliability of the system it is necessary to find out the location of the changes and the type (outage/addition). For such purpose, we utilize the estimated change instants $\mathcal{C}$ from Algorithm 5 along with the estimated topologies $\{\mathcal{E}_k\}_{k=1}^{\hat{K}}$. In particular, first, we find the associated topology between two consecutive change instants. This can be accomplished by finding the topology to which the timestamps between two consecutive change instants belong (that is $\mathcal{T}_k$). Subsequently, we can compare the topologies before and after each change instant to find out the line

Algorithm 6: Learning Grid Topology [88]

```
Function getTopology ( $\{\hat{\Sigma}_k\}_{k=1}^{\hat{K}}, \tau \in \mathbb{R}^+, \mathcal{V}$ ):  
  for  $k = 1$  to  $\hat{K}$  do  
     $\mathcal{E}_k \leftarrow \phi$ ;  
     $\begin{bmatrix} J_{VV,k} & J_{V\theta,k} \\ J_{\theta V,k} & J_{\theta\theta,k} \end{bmatrix} \leftarrow \text{inverse}(\hat{\Sigma}_k)$ ;  
    foreach  $i, j \in \mathcal{V}$  do  
      if  $J_{VV,k}(i, j) + J_{\theta\theta,k}(i, j) < -\tau$  then  
         $\mathcal{E}_k \leftarrow \mathcal{E}_k \cup \{\{i, j\}, \{j, i\}\}$ ;  
      end  
    end  
  end  
return  $\{\mathcal{E}_k\}_{k=1}^{\hat{K}}$ ;
```

Algorithm 7: Change location detection

```
Function getChanges ( $\{\mathcal{E}_k\}_{k=1}^{\hat{K}}, \mathcal{C}, \{k^{(n)}\}_{n=1}^S, \mathcal{T} = \{t^{(n)}\}_{n=1}^S$ ):  
  for  $i = 1$  to  $|\mathcal{C}| - 1$  do  
     $E \leftarrow []$ ;  $Outages \leftarrow []$ ;  $Additions \leftarrow []$ ;  
     $A \leftarrow \{n : \mathcal{C}[i] \leq t^{(n)} \leq \mathcal{C}[i+1]\}$ ;  $B \leftarrow \{k^{(n)} : n \in A\}$ ;  
     $k \leftarrow k^{(n)}$  having maximum occurrence in  $B$ ;  
    Append  $\mathcal{E}_k$  to  $E$ ;  
    if  $i > 1$  then  
       $Outages[i] \leftarrow \{(i, j) : (i, j) \notin E[i], (i, j) \in E[i-1]\}$ ;  
       $Additions[i] \leftarrow \{(i, j) : (i, j) \in E[i], (i, j) \notin E[i-1]\}$ ;  
    end  
  end  
return  $Outages, Additions$ ;
```

outages/additions that occurred at that instant. The strategy is shown in Algorithm 7. The entire workflow of our proposed methodology is summarized in Fig. 4.3. In the next section, we demonstrate the real-time hardware experimental results of our proposed technique.

4.5 Hardware-in-the-loop Validation

4.5.1 Setup description

A hardware-in-the-loop-based demonstration of the proposed topology change identification and localization on various test systems is conducted using the test setup developed in our laboratory as shown in Fig. 4.4(a). There are two important components in the demonstration [109]:

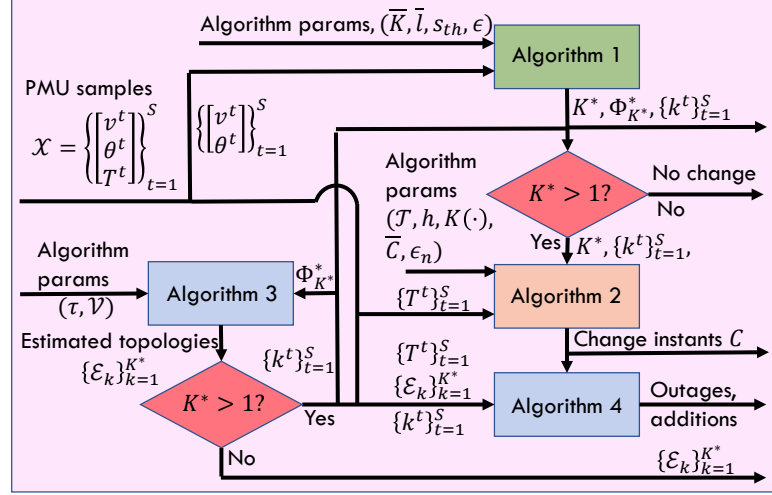
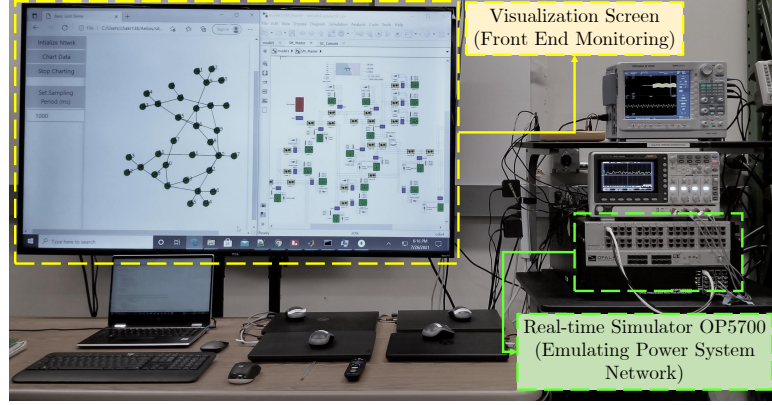
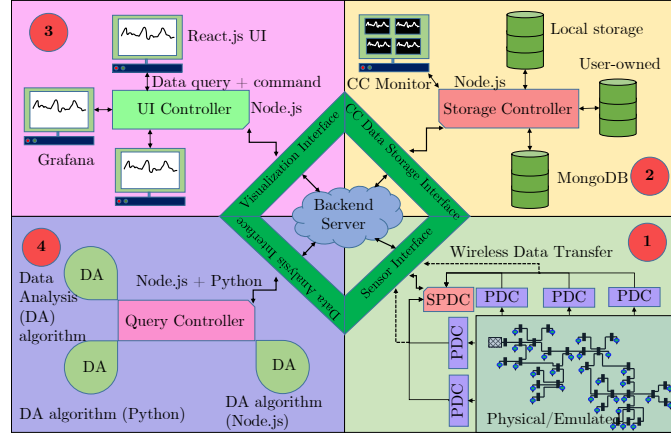


Figure 4.3: Change detection and topology estimation flowchart

1. The power distribution system networks under study are emulated using multi-timescale real-time simulation method, with time-step of $T_s = 50\mu s$ and $T_{ss} = 5ms$ via eMEGASIM and ePHASORSIM platform respectively inside the OP5700 RT-simulator (RTS) manufactured by OPAL-RT. The emulated power distribution systems contain multiple grid-following (GFL) inverter-based resources (IBRs) and 4 of these GFL-IBRs are controlled via controller-hardware-in-the-loop interfacing methods. Four low-cost Texas Instruments TMS320F28379D, 16/12-bit floating-point 200-MHz Delfino microcontroller boards are interfaced with the RTS to control the emulated power circuits of the GFL-IBRs.
2. Measurement vector (voltage magnitudes and phase angles of the buses obtained from the Phasor Measurement Units (PMUs) emulated inside RTS) is created by accumulating the measurements and the packaged data is sent via communication channels to the back-end server, as shown in Fig. 4.4(b). Phasor measurements are sent using user datagram protocol (UDP) to the back-end server via Socket.IO at a rate of 0.1 and 1 sec respectively. Data aggregation algorithms are currently developed using Node.js v14.16 with Python 3.8 support. The server workstation contains MongoDB v4.4 based data storage with data structure schema enforced by Mongoose v5.11. For the user interfaces (UIs), React.js v17.0 is used to develop our web application, and D3 library v6.2 to render the grid network. All



(a)



(b)

Figure 4.4: Figure of (a) the laboratory-based controller-hardware-in-the-loop test system with OP5700 RT-simulator (RTS), (b) the end-to-end cyber-physical framework for grid monitoring.

communications between each component of our architecture are implemented using Socket.IO v3.0 server-client protocol. The topology change identification and topology estimation algorithms are developed using Python 3.8.

Real-time load profiles (both active and reactive power) for all the loads in the test system networks are fed into the simulator box as input. The bus voltage phasor magnitude, phase angle, active and reactive power consumed by each load are computed from the three-phase instantaneous bus voltages, and load currents. The structural data flow involving the components described in this section is as follows: (i) The framework

is initialized by a user command, sent from the UI, which tethers the emulated PMUs and MongoDB to the back-end. (ii) The phasor measurements are continuously received, restructured, and aggregated by the back-end and are stored in MongoDB. (iii) The user initiates the topology change identification algorithm from the UI and the back-end starts feeding the data to the algorithm using standard MongoDB queries utilizing the storage controller. (iii) Upon receiving the required amount of phasor data, the algorithm runs in parallel and returns the estimated changes, change instants, and the topologies to the back-end once completed. (iv) Back-end stores the topology information to MongoDB and relays the same to the UI controller which processes the information and shows the relevant information on the UI. The following test system networks are used in this study:

IEEE-33 Bus Network

We take a modified IEEE 33-bus distribution network, with a total active and reactive demand of 3.715 MW and 2.3 MVar, respectively [110]. The test system is a 12.66 kV system with one feeder substation, feeding a network with 33 buses and 3 looping branches. For topology change identification, those 3 looping branches (switchable tie lines) are used in this study.

226 Bus Suburb Network

The 226-bus loopy network is based on the 3- ϕ , 13.8 kV, University of Minnesota campus network, that is power by the university's energy management system, and the 3- ϕ , 0.48 kV surrounding suburb Minneapolis distribution network, that is powered by Xcel Energy. This test system has 1 commercial-scale medical center of 700 kVA, 6 community buildings, 1 mobile auxiliary generation unit, 1 cold load station. Each building contains multiple DERs such as 100 kVA PV GFL inverters, 100 kVA Battery GFL inverters, 50 kVA diesel generation sets etc as well as 50-300 kVA controllable and 50-100 kVA non-controllable loads. The suburb power network contains 5 residential apartments, each contain 2-7 kVA roof-top PV systems, 1 kWh in-house battery systems, 2-7 kVA in-house loads. The total load in the network is 5.12 MW. More details can be found in [111].

1262 bus network

The 1262-bus network is based on the ‘ckt7’ of the large representative feeder, developed by the Electric Power Research Institute (EPRI) [112]. The network voltage is 12.5 kV, fed by a sub-transmission system of 115 kV via 14 feeders. The primary length of the network is 12.9 km, feeding 5694 customer loads. The original network is modified by adding few buses via switchable tie-lines to make the network loopy.

4.5.2 Topology change identification performance

In this section, we demonstrate the performance of our proposed topology change detection algorithm and discuss the results obtained by integrating the same into our topology estimation algorithm to localize the line outages/additions. The algorithm parameters are taken as follows: $\bar{K} = 10$, $\bar{l} = 200$, $s_{th} = 0.45$, $\epsilon = 0.001$, $\bar{C} = 8$, $\epsilon_n = 50$. Further, we consider a phasor measurement noise of $\pm 1\%$ of the full-scale measurement. For ease of exposition, the timestamps of the phasor measurement samples are normalized such that the timestamp of the initial sample in the sample collection window is 0, and the timestamps of the subsequent samples are incremented by 1.

In the case of IEEE 33-bus network, during the sample collection window, we open the circuit breaker on the branch connecting bus 25 and 6 to emulate a line outage, upon collecting the sample with timestamp 3160. For reference, in Fig. 4.5c, we show the actual time instant of the change where we designate the topology 0 and 1 as the corresponding topology before and after the line outage respectively. Note that, the number of different topologies encountered during the sample collection window is not known a priori. Therefore, we run Algorithm 4 with the collected phasor samples for each possible number of topologies ranging from 2 to $\bar{K}$, and compare the average silhouette scores which are shown in Fig. 4.5a. Note that our algorithm correctly estimates the optimal number of topologies as 2 producing the highest average silhouette score of 0.72. Subsequently, we run Algorithm 5 to estimate the time instant of the change. We estimate the probability distribution of the timestamps of the phasor samples, classified under topology 0 and 1 by Algorithm 4, using a Gaussian kernel with bandwidth $h = 5$. Fig. 4.5b shows the obtained distributions and in zoomed inset, we highlight the intersection point of the two distributions. Notice that, our proposed methodology correctly estimates the topology change instant as indicated by the intersection point. Once the topology change is detected and the instant is identified, we identify the change location using the topology estimation algorithm. We feed the estimated covariance

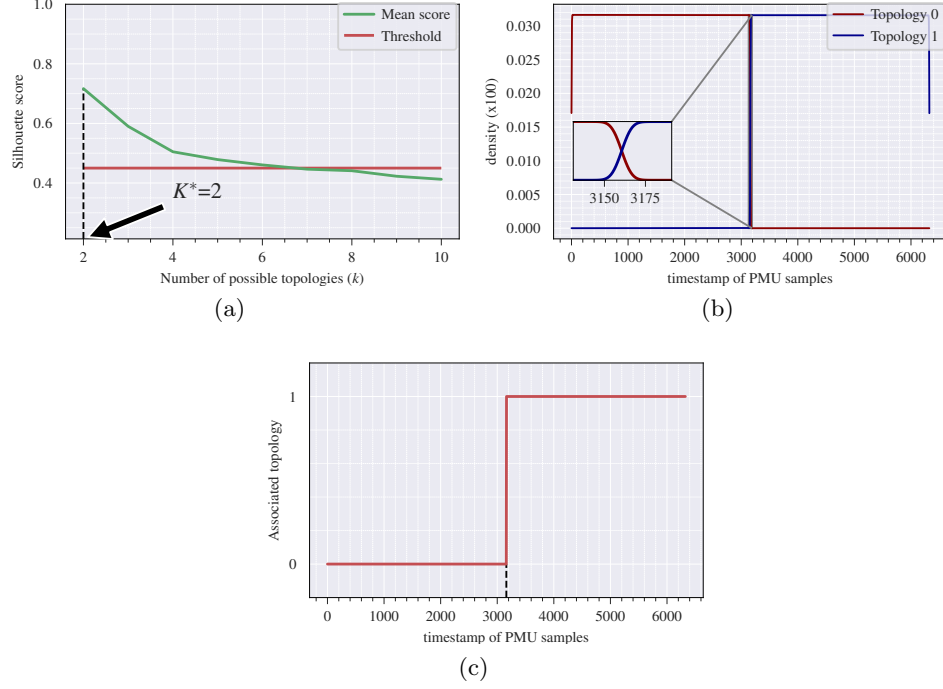


Figure 4.5: IEEE 33-bus: (a) Mean silhouette score of the phasor samples obtained by running Algorithm 4 for each $K \in \{2, \dots, \bar{K} = 10\}$. The highest value of the average silhouette score is 0.72 with $K = 2$, indicating two different topologies are observed during the sample collection window, (b) Estimated density functions of the timestamps of the phasor samples classified under topology 0 and 1. The density functions intersect at the timestamp value of 3160 indicating the change instant, (c) Topology is actually changed upon collecting the phasor sample with timestamp 3160, indicating the actual time instant. The value of the timestamp at the intersection of the density functions in (b) matches with the actual instant.

matrices of the phasor samples, obtained from Algorithm 4, categorized under topology 0 and 1, to our topology learning algorithm 6. Here we learn each of the topologies, $\mathcal{E}_0, \mathcal{E}_1$. Subsequently, we run Algorithm 7 to compare $\mathcal{E}_0$ and $\mathcal{E}_1$ for change localization. Here, we define the accuracy of detection as the ratio of the number of correctly detected outages and additions to the actual number of outages and additions. We also define false localization as the ratio of the total number of incorrectly estimated outages and additions to the total number of branches. As shown in Table 4.1, our proposed strategy correctly identifies the locations of all the changes. Using our CPS situational awareness and visualization framework, shown in Fig. 4.4, we further visualize the locations of the

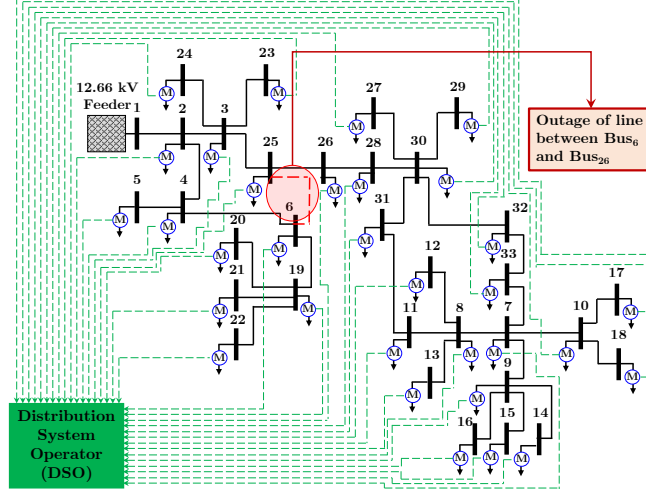


Figure 4.6: Line outage detected by the proposed topology change identification and proposed topology estimation technique.

changes. An illustration of the same is shown in Fig. 4.6. Note that, the false localization depends on the topology learning accuracy of Algorithm 6. We remark that the false localization percentage of 3% is due to 2% topology estimation error. We emphasize that a low false localization rate indicates that our proposed methodology provides a smaller subset of the total branches to inspect for possible outages, including the actual one. Such information can help the system operator to identify the outage in a short amount of time, thus improving reliability.

Table 4.1: IEEE 33-bus: Change localization performance

Network	Change Instants	Accuracy (%)	False localization (%)
33	3160	100	3
226	9115	100	1.3
	18232	100	1.3
1262	2735	100	1.9

On the 226-bus network, we emulate both line outage and addition upon acquiring phasor samples with timestamps 9115 and 18232 respectively. The line outage is performed between buses 104 and 204, and the addition is done between buses 106 and 206

in real-time on RTS. We designate the actual topology by topology-0, and the topologies after the first and second change are designated by 1, and 2 respectively. Fig. 4.7c shows the actual change instants and the associated topologies. Similar to the IEEE 33-bus

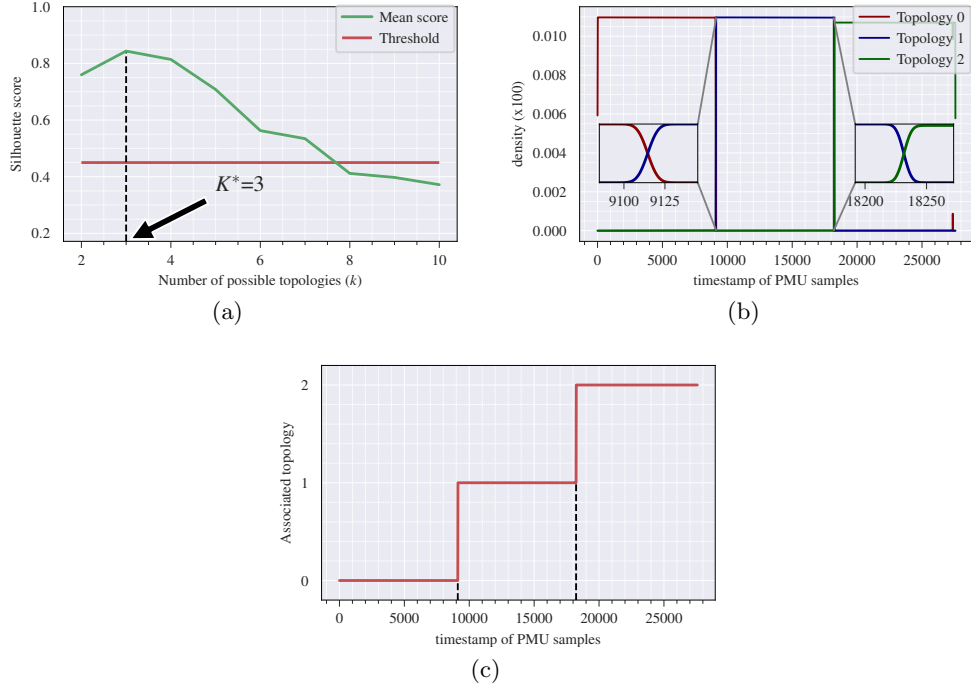


Figure 4.7: 226-bus: (a) Mean silhouette score of the phasor samples obtained by running Algorithm 4 for each $K \in \{2, \dots, \bar{K} = 10\}$. The highest value of the average silhouette score is 0.84 with $K = 3$, indicating three different topologies are observed during the sample collection window, (b) Estimated density functions of the timestamps of the phasor samples classified under topology 0, 1, 2. The density functions intersect at the timestamp value of 9115, and 18232, indicating the change instants, (c) Topology is actually changed upon collecting the phasor sample with timestamp 9115 and 18232, indicating the actual time instant. The values of the timestamps at the intersections of the density functions in (b) match with the actual instants.

case, here our proposed Algorithm 4 indicates that the optimal number of topologies is 3 with the highest silhouette score of 0.84, as illustrated in Fig. 4.7a. Further, we plot the estimated densities of the sample timestamps categorized under topology 0, 1 and 2 in Fig. 4.7b. Note that, we correctly estimate the topology change instants as indicated by the intersection points of the densities. Subsequently, to localize the changes and identify the types (outage/addition), we run Algorithm 6 and obtain an accuracy of 97%

in topology estimation. We detect the order and type of changes using Algorithm 7, and the localization result is tabulated in Table 4.1. Similar to the IEEE 33-bus network, here we obtain an accuracy of 100% and a false localization of 1.3%. Thus, our proposed methodology provides a very small subset of branches, for possible outages and additions for each change. Note that, the subsets include the actual changes as indicated by the accuracy, and thus can aid the system operator in identifying the changes easily without the requirement of inspecting every branch in the network.

To showcase the scalability of our proposed methodology, we perform a real-time simulation on the 1262-bus network where we emulate multiple line outages and additions at the same instant. Here in Fig. 4.8c, we show the instant of the changes. Note that, our proposed methodology correctly identifies the optimal number of topologies as 2 and the change instant as shown in Fig. 4.8a and Fig. 4.8b respectively. Upon running Algorithm 6, we obtain a topology estimation accuracy of 98.3% for each of the two topologies. In Table 4.1, we show the change localization performance which corroborates the efficacy of our proposed methodology even in a very large network.

Next, we demonstrate the statistical stability of our proposed change detection algorithm. We perform a real-time simulation on the 226-bus network, where we emulate 350 random changes, including both line outages and additions, over a span of one hour. In Table 4.2, we show the performance metrics. Out of 350 changes, we correctly

Table 4.2: Change estimation performance

	Count	Accuracy (%)	False positive (%)	False negative (%)
Total changes	350	-	-	-
Correct detection	338	96.57	-	-
False detection	4	-	0.014	-
Missed detection	12	-	-	0.034

identify 328 resulting in an accuracy of 96.57%. Further, only 4 false detections are observed and only 12 changes are undetected, leading to a false positive and false negative percentage of 0.014 and 0.034 respectively. Such low false positives and false negatives corroborate the efficacy of our proposed methodology in detecting topology changes.

Remember that, the accuracy of our proposed change localization Algorithm 7, depends on the topology estimation accuracy. Hence, to complete our analysis, we focus on the topology estimation performance in the next section.

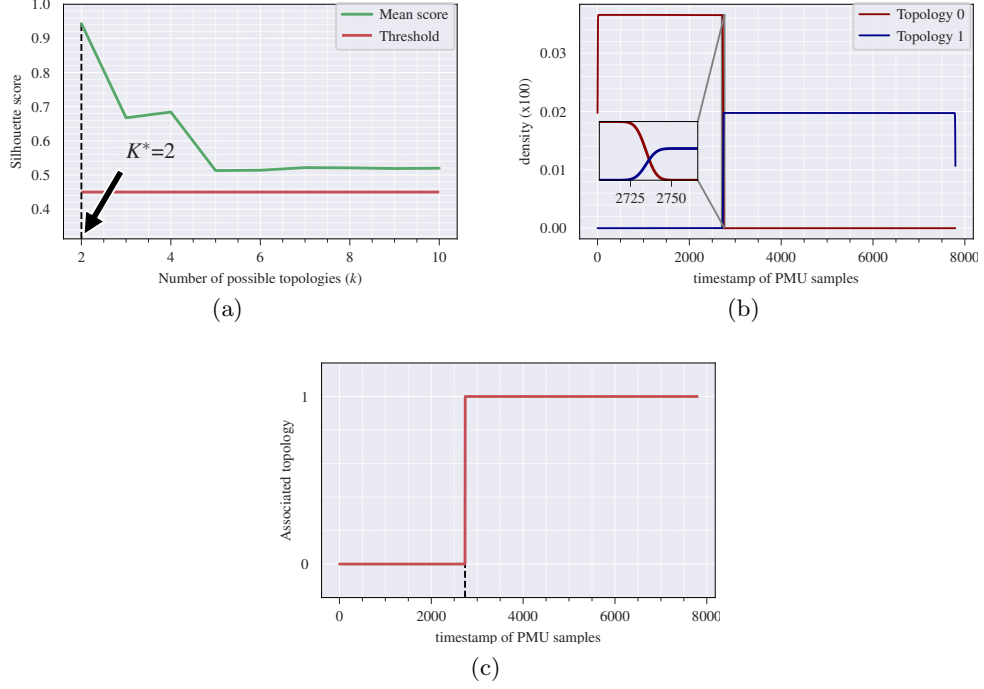


Figure 4.8: 1262-bus: (a) Mean silhouette score of the phasor samples obtained by running Algorithm 4 for each $K \in \{2, \dots, \bar{K} = 10\}$. The highest value of the average silhouette score is 0.96 with $K = 2$, indicating two different topologies are observed during the sample collection window, (b) Estimated density functions of the timestamps of the phasor samples classified under topology 0 and 1. The density functions intersect at the timestamp value of 2736 indicating the change instant, (c) Topology is actually changed upon collecting the phasor sample with timestamp 2736, indicating the actual time instant. The value of the timestamp at the intersection of the density functions in (b) matches with the actual instant.

4.5.3 Topology estimation performance

In this section, we study the performance of the topology learning algorithm 6. Here, we collect various numbers of phasor samples from the actual topology for all three distribution networks, IEEE 33-bus, 226-bus, and 1262-bus, and estimate the corresponding covariance matrices. We subsequently run Algorithm 6 to learn the topologies. As an efficiency metric, we define the estimation error as the ratio of the sum of the number of false lines identified and true lines missed, to the total number of lines in the network. We also separately analyze the false positives (number of false lines identified over the

total number of branches) and false negatives (number of lines missed over the total number of branches) that contribute to the total error. In Fig. 4.9, 4.10, and 4.11, we illustrate the topology learning performance under varying numbers of samples for IEEE 33-bus, 226-bus, and 1262-bus network respectively. Further, for demonstration and comparison purposes, we show the estimation performance considering no measurement noise as well. For all the plots, the dashed lines indicate the mean value of the corresponding metrics averaged over 1000 runs.

From Fig 4.9a, 4.9b, and 4.9c, it is observed that the measurement noise degrades the estimation accuracy of IEEE 33-bus, however, with an increased number of samples, it is possible to estimate the topology with absolute accuracy. Further, notice that the false negative rate is always lower than the false positives. Even with only 2000 phasor samples, we achieve a false negative rate of 0%, which corroborates the effectiveness of the topology learning algorithm in identifying critical line outages in near real-time. Apart from line outage, another critical component in situational awareness is to derive islanding information from the line outages. An erroneous detection of islands can lead to inefficient grid operation and economic losses. Here, in Fig. 4.9d, we plot the number of islands erroneously detected by our topology learning algorithm. Notice that our algorithm does not detect any islands, thus establishing confidence in the outcome.

Similar plots are obtained in the 226-bus network and are shown in Fig. 4.10. Here, we obtain mean error rates of 5.2% and 5.9%, without and with measurement noise respectively for 5500 samples. Further, the low variance of the error rate for 1000 runs, as shown in the zoomed inset in Fig. 4.10a, indicates high confidence in the outcome of the learning algorithm. In this case, we observe a slightly higher false negative rate than false positive as shown in Fig. 4.10b and 4.10c. It is attributed to the fact that the 226-bus network represents a part of the University of Minnesota campus and surrounding suburbs, and thus it is extremely loopy along with multiple PVs distributed across the network. Further, the average false negative rate of 3% with 5500 samples translates to incorrect detection of 5 islands in the topology as presented in Fig. 4.10d.

In Fig. 4.11, we demonstrate the topology estimation performance for the 1262-bus network. Here with 2500 samples, we achieve a mean error of 0.2% and 3% without and with measurement noise respectively as indicated in Fig. 4.11a. Similar to the IEEE 33-bus case, here we achieve lower false negatives compared to false positives. In the 1262-bus network we have numerous leaf nodes, and therefore, even with a small error percentage, we observe a higher number of erroneously identified islands as shown in Fig. 4.11d. However, in the case of clean phasor measurements, we do not identify any

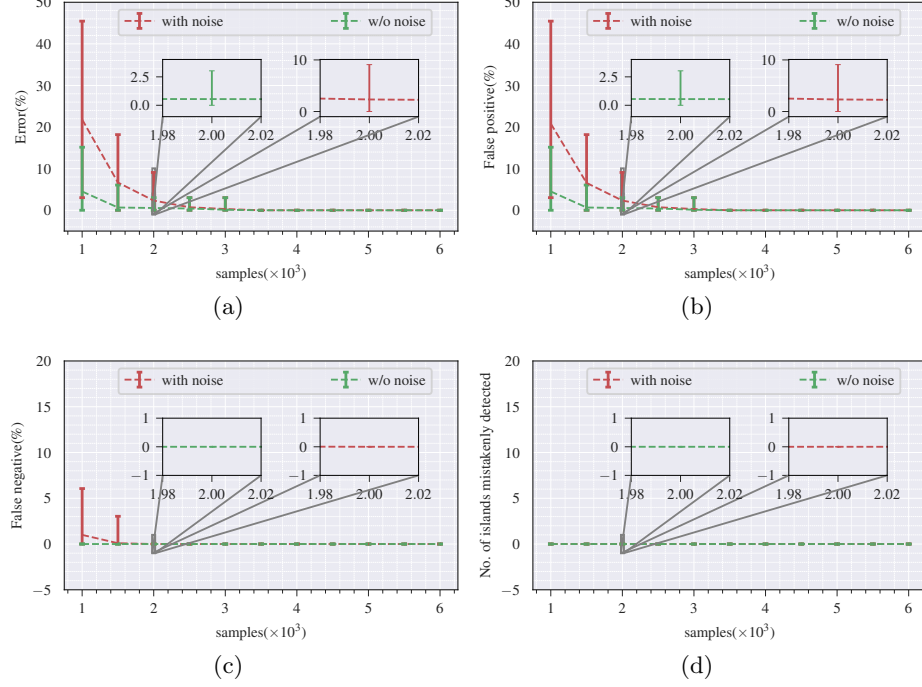


Figure 4.9: IEEE 33-bus: (a) Topology estimation error, (b) False positive percentage, (c) False negative percentage, (d) Number of islands mistakenly detected, for 1000 runs with various numbers of phasor samples. Dashed lines indicate the mean values of the corresponding metrics.

islands.

To summarize, we observe that, a higher number of loops in the network degrades the performance of the topology learning algorithm as indicated by our analysis of the 226-bus network. Here, the IEEE-33 bus and 1262-bus network have less number of loops compared to the total number of branches in the networks, leading to a much lower estimation error. In such networks, we observe a lower false negative percentage compared to the false positives. Further, in a large system with many leaf nodes, as in the case of the 1262-bus network, the estimation error translates to a higher number of erroneously found islands. To this end, We remark that our topology estimation performance analysis provides a comprehensive guideline to the system operators.

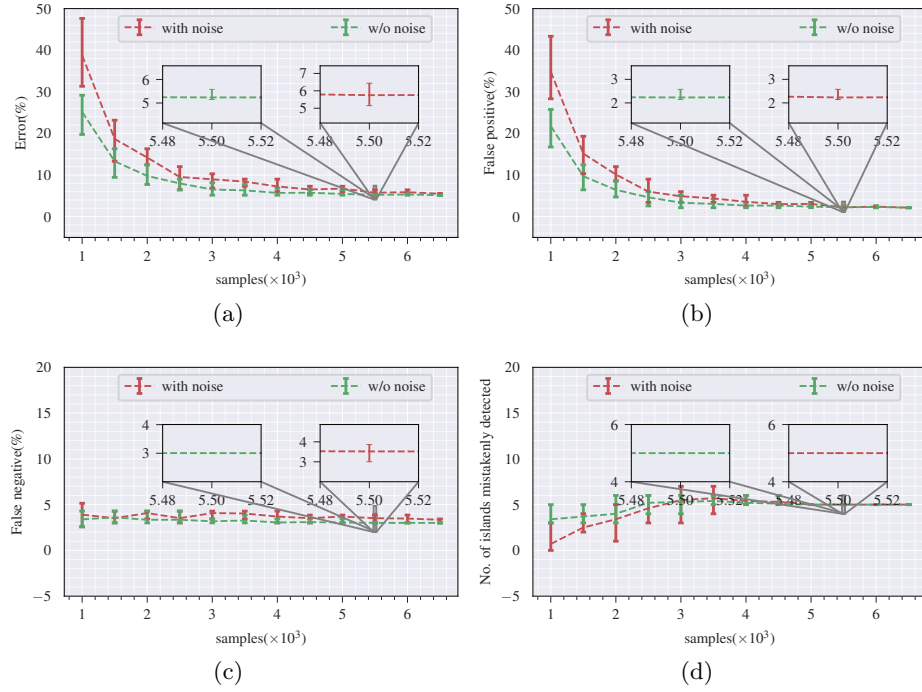


Figure 4.10: 226-bus: (a) Topology estimation error, (b) False positive percentage, (c) False negative percentage, (d) Number of islands mistakenly detected, for 1000 runs with various numbers of phasor samples. Dashed lines indicate the mean values of the corresponding metrics. The extremely loopy structure of the network contributes to a lower estimation accuracy that translates to a higher number of erroneously identified islands.

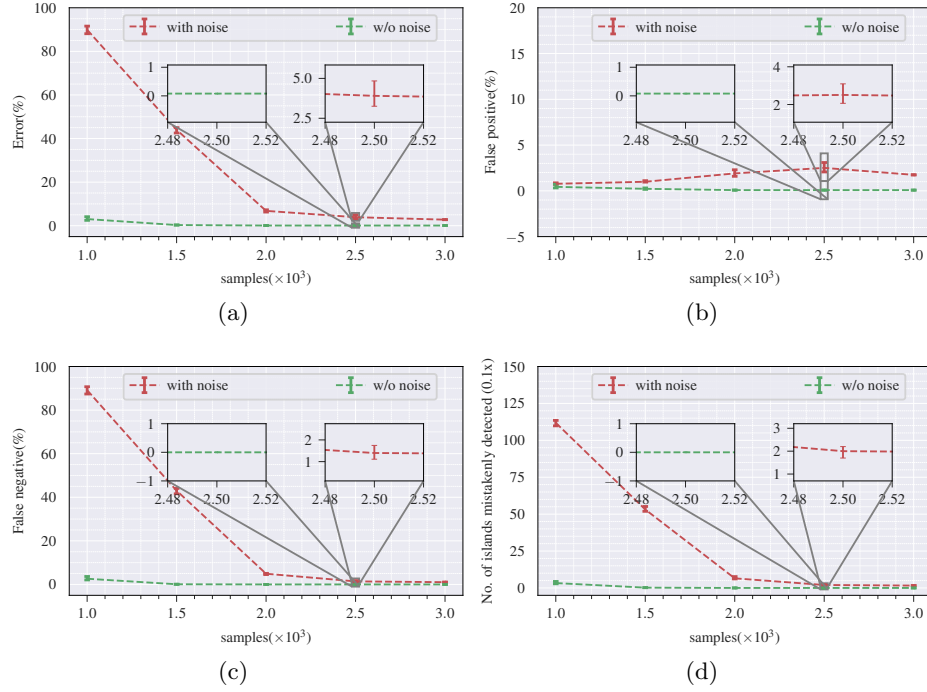


Figure 4.11: 1262-bus: (a) Topology estimation error, (b) False positive percentage, (c) False negative percentage, (d) Number of islands mistakenly detected, for 1000 runs with various numbers of phasor samples. Dashed lines indicate the mean values of the corresponding metrics. A very large network with numerous leaf nodes can have a higher number of erroneously detected islands even with a lower topology estimation error when phasor measurement noise is present.

Chapter 5

Resilient System Operation: A Cyber-Security Analysis

In this chapter, we analyze the resiliency of power distribution systems under cyber-attack vulnerabilities. We discuss two types of cyber-attacks in this chapter: (i) cyber-topology attack, and (ii) false data injection (FDI) attack on state-feedback load frequency control architecture for microgrids. Under a cyber-topology attack, the attacker manipulates the breaker status measurements to falsify the status of transmission lines, making the attacked branches appear offline while they are operational, to disrupt the electricity market operation. Here the objective is to disrupt the electricity market operations since the decision of electricity prices at different buses in a network depends on the knowledge of the topology. In the first part of the chapter, we analyze such vulnerabilities and propose a methodology to identify the critical branches in a network that must be protected to avoid electricity market disruption. In the second part of this chapter, we consider FDI attacks on the measurements and control commands in a load frequency controller deployed in a microgrid. We establish the impact of such attacks on the stability of the system and propose an attack mitigation strategy.

5.1 Cyber-topology Attack to Disrupt Electricity Market Operation

We propose a reinforcement learning (RL)-based methodology to analyze vulnerability of real-time (RT) electricity market under grid topology attack. In RT electricity market, the electricity prices at different buses in the grid network are decided based on the locational marginal price (LMP). LMP is derived from the solution to DC optimal power flow (DCOPF) which depends on the grid topology, generation cost, and real-time demand. Hence, an attacker can manipulate the topology information to alter the solution to DCOPF leading to alteration of LMPs, thus harnessing monetary profit. Our analysis entails realistic cyber-topology attack, that is, the attacker can only manipulate the breaker status, but not the physical topology, and it has no knowledge of the topology prior to attack. Under such cyber-topology attack, our proposed RL-based methodology identifies the critical breakers in the power network that, if attacked, can lead to large deviation in LMP from the actual value and disrupt the electricity market. We instantiate our proposed technique in IEEE-39 and 300 bus network and establish that the critical branches, identified by our algorithm, are crucial in terms of maintaining the stability of RT market, hence must be protected by the grid operator.

5.1.1 State-of-the-art

Increased penetration of distributed energy resources as well as infusion of advanced metering infrastructure to modern power grids have caused a paradigm shift in electricity market operation. Further, the impetus to create carbon-free power grids has propelled the interest towards deregulation of electricity markets [113]. Generally, deregulated markets are driven by Regional Transmission Organizations (RTO) such as Pennsylvania-New Jersey-Maryland, Independent System Operators New England etc. In real-time (RT) market operation, RTOs clear the market according to locational marginal price (LMP) [114] which is computed based on generation cost and real-time demand. LMP is defined as the marginal cost of supplying one unit increment of active power demand at a specific bus in the power network. To obtain LMP, DC optimal power flow (DCOPF) [114] is utilized. The solution to DCOPF depends on the generation cost of the generators in the network, power demand, physical network constraint and grid topology. Therefore, by launching a surreptitious topology attack [75], that manipulates the topology structure information, DCOPF solution can

be altered, leading to erroneous LMP values. Such vulnerability can be exploited by attackers to alter LMPs at certain locations in the network to harness financial profits. Therefore, it is extremely important for the grid operators to protect the market from such cyber-attacks.

Prior Works

Usually, the grid operators emulate potential behavior of attackers to perform network vulnerability analysis and devise protection mechanisms. Successful attack emulation exposes the vulnerable points in the system that are chosen to be protected by the grid operators. Hence, discovering and analysing various types of cyber-attacks are of utmost importance to the operators. In existing literature, profit-oriented cyber attacks, to manipulate LMPs, are analysed in the context of false data injection (FDI) attacks, altering the power injection and flow measurements [115]. Various FDI-based cyber attack strategies, altering the state estimation results are analysed in [116, 117, 118, 119, 115]. However, such FDI attacks assume that only the power measurements are attacked but the underlying grid topology is not manipulated. In fact, topology error also contributes to significant deviation in LMP [120]. Despite the importance, topology attacks in the context of electricity market operation, are not explored much. In [121, 122, 123], disruption of electricity markets due to topology attacks are analysed. However, in the existing works, it is assumed that either the attacker can physically trip a line or, in case of cyber-topology attack (modifies only the circuit breaker status measurements), have knowledge of the topology prior to crafting an attack. In practical scenario, gaining such capabilities is extremely difficult for attackers. Further, most of the existing works consider one-shot attacks, where multiple breakers are manipulated simultaneously instead of less detectable sequential attacks. Hence, there is a need to analyze profit-oriented sequential attack strategies considering practical attack constraints. In summary, there is very limited literature on system vulnerability analysis tailored to cyber-topology attack on electricity markets considering practical constraints of attackers. To address this research gaps, in this work we propose a RT market vulnerability analysis, under cyber-topology attack, to enhance the resiliency and reliability of electricity market operation.

Key Contributions

The main contributions are:

1. We propose a reinforcement learning (RL) based sequential topology attack mechanism to expose the vulnerability of electricity market against monetary profit-oriented sequential cyber-attacks. Our proposed method incorporates all practical constraints of an attacker, such as the attacker can only modify the measurements and cannot alter the topology physically. and it does not have any information of the grid topology structure.

2. Grid operators can utilize our methodology to identify how many and which measurement devices are to be protected to shield the electricity market from LMP manipulation. We establish that securing the measurement devices, identified by our proposed technique, safeguards the electricity markets from LMP deviation due to falsified topology information.

5.1.2 Attack Model and Problem Description

Real-time electricity market and LMP calculation

Let the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ represent the power network, where $\mathcal{V}$ is the set of N buses and $\mathcal{E}$ is the set of undirected branches. Let $\mathcal{V}_g \subseteq \mathcal{V}$ represent the set of generator buses. Further, let P_{gi} , D_i , and θ_i denote real power injection, load, and voltage phase angle at bus i respectively, and x_{ij} be the line reactance of branch (ij) . In DC power flow model (DCPF), the power flow equation can be written as $\mathbf{P}_g - \mathbf{D} = \mathbf{H}\boldsymbol{\theta}$, where $\mathbf{P}_g := [P_{g1}, \dots, P_{gN}]^T$, $\mathbf{D} := [D_1, \dots, D_N]^T$, $\boldsymbol{\theta} := [\theta_1, \dots, \theta_N]^T$, and $\mathbf{H} \in \mathbb{R}^{N \times N}$ embeds the topology structure as follows: $H(i, j) = \sum_{k: (ik) \in \mathcal{E}} \frac{1}{x_{ik}}$, if $i = j$, $H(i, j) = -\frac{1}{x_{ij}}$, if $i \neq j, (ij) \in \mathcal{E}$, and zero otherwise. In RT markets, LMP is computed from DCOPF [114] where the objective is to compute generation set points by solving the following optimization problem: Given, $\mathbf{D}, \mathbf{H}$ and generation cost functions $C_i(P_{gi}), i \in \mathcal{V}_g$,

$$\underset{P_{gi}, i \in \mathcal{V}_g, \theta_j, j \in \mathcal{V}}{\text{minimize}} \sum_{i \in \mathcal{V}_g} C_i(P_{gi}) \quad (5.1)$$

$$\text{s.t. } \mathbf{P}_g - \mathbf{D} = \mathbf{H}\boldsymbol{\theta}, \quad (5.2)$$

$$P_{ij}^{\min} \leq P_{ij} \leq P_{ij}^{\max}, \quad (5.3)$$

$$P_{gi}^{\min} \leq P_{gi} \leq P_{gi}^{\max}, i \in \mathcal{V}_g, \quad (5.4)$$

where $P_{ij}, P_{ij}^{\min}, P_{ij}^{\max}$ are power flow, and minimum and maximum limits of the same, through branch (ij) , and $P_{gi}^{\min}, P_{gi}^{\max}, i \in \mathcal{V}_g$ are the minimum and maximum generation output limits of the generator connected to bus i . The operator solves (5.1) at a certain

interval to optimally set generator outputs to minimize total generation cost. By definition, the Lagrangian multiplier, $\lambda^* = [\lambda_1^*, \dots, \lambda_N^*]$, associated with the constraint (5.2), evaluated at the optimal solution of (5.1) are the LMPs of the corresponding buses [114]. In the next section, we discuss how the dependency of LMP on the grid topology can be exploited by the attackers to manipulate the RT market.

Attacker model

Note that, by altering the grid topology $\mathbf{H}$, the constraint (5.2) can be altered and in turn the LMPs of the buses can be changed. Let the target of the attacker be to alter the LMP of a target bus $k \in \mathcal{V}$ by topology manipulation. Essentially the attacker tries to spoof the line breaker status of the active lines in such a way that the LMP of the target bus, λ_k^a , deviates from the actual value λ_k^* by a certain amount δ_k^a chosen by the attacker. We consider a line removing topology attack, where the attacker intends to falsify tripping of a line (ij) so that in the manipulated breaker status measurement, status of the line (ij) is made 0 while the actual line is in service. However, manipulation of only breaker status measurements can be easily detected from the power injection measurements at bus i, j and flow measurements from the line (ij) as those would not conform to power balance at buses i and j . Let $P_i = P_{gi} - D_i$ denote the net actual power injection measurement at bus $i \in \mathcal{V}$ before the attack. To maintain power balance, the attacker needs to modify the injection and flow measurements in the following way: (1) flow measurement at line (ij) is made 0, (2) injection measurement at bus i is modified from P_i to $P_i + P_{ij}$, (3) injection measurement at bus j is modified from P_j to $P_j - P_{ij}$. Such attack strategy is illustrated in Fig. 5.1 and can create a cyber-tripped line by fully masking the actual breaker status [75]. Under the line removing attack,

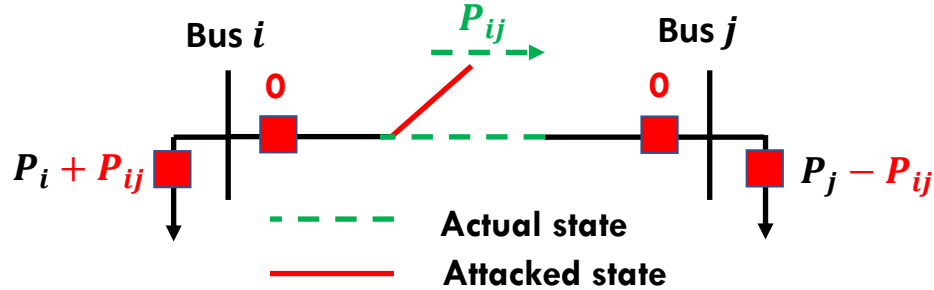


Figure 5.1: Line removing attack

the grid operator modifies $\mathbf{H}$ according to the falsified breaker status measurements and calculates LMP based on the erroneous topology. The objective of the attacker is to carefully choose a set of lines $\{(ij)\} \subseteq \mathcal{E}$ that are to be attacked so that the LMP deviation at the target bus, $\lambda_k^a - \lambda_k^* \geq \delta_k$.

Attack constraints

In a power network, the lines connected to the generators are crucial to maintain seamless power flow in the entire system and thus are protected from any cyber-attack. In other words, attacking such branches can easily alarm the grid operator. We denote the set of such protected branches by $\mathcal{E}_p$. We assume that the attacker does not attack breaker status measurement at a branch $(ij) \in \mathcal{E}_p$. Further, the resources of the attacker is constrained in terms of total number of breakers that can be attacked, otherwise, a large change in the network can alarm the grid operator. Moreover, the attack must be sequential rather than one-shot to avoid alarming the operator of any unusual behavior (multiple line outages simultaneously). In summary, we assume the following constraints of the attacker:

1. The attacker cannot attack any protected branch $(ij) \in \mathcal{E}_p$.
2. It cannot physically trip a transmission line.
3. It cannot attack more than $N_a \leq |\mathcal{E}|$ number of lines.
4. The attacker cannot attack multiple lines simultaneously.

Problem formulation

The goal of the attacker is to modify the LMP of the target bus k by a certain amount δ_k , chosen by the attacker, by means of breaker spoofing attack as described in Section 5.1.2. Thus it aims to find the set of lines $\{(ij)\} \subseteq \mathcal{E}$ that are to be attacked and the attack sequence, to achieve the goal $\lambda_k^a - \lambda_k^* \geq \delta_k$, under the constraints described in Section 5.1.2. Note, after each attack, $\mathbf{H}$ and $\mathbf{D}$ are modified, leading to different solutions to DCOPF, LMP, and power flows, which in turn affects the choice of next attack action. Thus, the attacker needs to find a proper attack sequence to reach the objective. Further, the methodology can be utilized by the grid operator to assess criticality of the branches in a network. The outcome of our proposed framework identifies

the branches that are to be protected by the control center to prevent undesirable LMP deviation. Next, we describe our proposed RL-based approach.

5.1.3 Proposed Reinforcement Learning Solution

Power grid environment in RL framework

The principal structure of RL relies upon a Markov Decision Process (MDP) [124]. An MDP is described by a tuple $(\mathbf{S}, \mathbf{A}(\mathbf{s}), \{P_{\mathbf{s}'|\mathbf{s}}^{\mathbf{a}}\}, \gamma, R)$ in discrete time $t \in \mathbb{N}$, where:

1. $\mathbf{S}$ is the state space of the power grid, and is represented by a set of breaker status of all the branches present in the system. The state of the grid, $\mathbf{s} \in \mathbf{S}$, is defined by $\mathbf{s} := \{s_{ij} : (ij) \in \mathcal{E}\}$, where, $s_{ij} = 1$ if branch (ij) is not cyber-tripped, and zero otherwise. We denote the state of the grid at time t by $\mathbf{s}^t$.
2. $\mathbf{A}(\mathbf{s})$ is the set of all possible actions of the attacker at any state $\mathbf{s} \in \mathbf{S}$. We denote the action performed by the attacker at a state $\mathbf{s}$ by $\mathbf{a} \in \mathbf{A}(\mathbf{s})$, where $\mathbf{a} := \{(b_{ij}, P_{ij}) : s_{ij}=1, (ij) \in \mathcal{E}\}$, and b_{ij} is -1 if branch (ij) is to be attacked and 0 otherwise, and P_{ij} is the power flow in branch (ij) prior to attack. Attack action involves modification of breaker status by the operation $s_{ij}+b_{ij}$ and injection measurement modification as described in Section 5.1.2. We denote the set of all possible actions by $\mathbf{A} = \cup_{\mathbf{s} \in \mathbf{S}} \mathbf{A}(\mathbf{s})$, and the action taken at time t by $\mathbf{a}^t$.
3. $P_{\mathbf{s}'|\mathbf{s}}^{\mathbf{a}} := \mathbb{P}(\mathbf{s}^{t+1} = \mathbf{s}' | \mathbf{s}^t = \mathbf{s}, \mathbf{a}^t = \mathbf{a})$ denotes the transition probability of reaching $\mathbf{s}'$ at $t+1$ by taking action $\mathbf{a}$ in $\mathbf{s}$ at t .
4. $\gamma \in [0, 1]$ is a constant discount factor.
5. $R : \mathbf{S} \times \mathbf{A} \times \mathbf{S} \rightarrow \mathbb{R}$ is the reward function where $R(\mathbf{s}^{t+1}, \mathbf{a}^t, \mathbf{s}^t)$ generates the reward for taking action $\mathbf{a}^t \in \mathbf{A}(\mathbf{s}^t)$ in state $\mathbf{s}^t$ and thus reaching the state $\mathbf{s}^{t+1}$.

Due to intractability of computing system dynamics under sequential topology attack, which involves computation of DCOPF, we employ Q-learning [125] to learn the optimal attack actions that maximizes expected sum of discounted reward. At each time t , the attacker chooses $\mathbf{a}^t$ at a state $\mathbf{s}^t$. The action is chosen based on the following ϵ -greedy policy: given any $\epsilon \in [0, 1]$, with probability $(1 - \epsilon)$, $\mathbf{a}^t = \arg\max_{\mathbf{a}' \in \mathbf{A}(\mathbf{s}^t)} Q^t(\mathbf{s}^t, \mathbf{a}')$ is chosen, where $Q^t : \mathbf{S} \times \mathbf{A} \rightarrow \mathbb{R}$ is the Q-value [124] of state-action pairs, and with probability ϵ , a random action is chosen. Upon performing the action $\mathbf{a}^t$, the grid operator receives falsified breaker status, $\mathbf{s}^{t+1}$, and load redistribution as described in Section 5.1.2. Hence, it constructs a new topology $\mathbf{H}^{t+1}$, and load vector, $\mathbf{D}^{t+1}$, and dispatches new LMPs by solving (5.1). The attacker observes the changes in LMP at the target bus k and receives a reward $R(\mathbf{s}^{t+1}, \mathbf{a}^t, \mathbf{s}^t)$. Then it updates the Q-value of

$(\mathbf{s}^t, \mathbf{a}^t)$ as follows:

$$Q^{t+1}(\mathbf{s}^t, \mathbf{a}^t) = Q^t(\mathbf{s}^t, \mathbf{a}^t) + \alpha[R(\mathbf{s}^{t+1}, \mathbf{a}^t, \mathbf{s}^t) + \gamma \max_{\mathbf{a}'} Q^t(\mathbf{s}^{t+1}, \mathbf{a}') - Q^t(\mathbf{s}^t, \mathbf{a}^t)], \quad (5.5)$$

until the attack objective $\lambda_k^a - \lambda_k^* \geq \delta_k$ is satisfied.

Design of reward function

We design the reward function considering all possible consequences of breaker spoofing. Let $N_\theta(\mathbf{s})$ denote the total number of cyber-tripped lines at any state $\mathbf{s} \in \mathbf{S}$. As described in Section 5.1.2, the attacker must achieve the attack objective by cyber-tripping no more than N_a number of breakers. Thus, the goal state for the attacker is a state $\mathbf{s}$ for which $\lambda_k^a - \lambda_k^* \geq \delta_k$ and $N_\theta(\mathbf{s}) \leq N_a$. The attacker is rewarded for reaching such a goal state. However, the grid operator will get cautious if it tries to attack multiple branches in sequence to reach the goal state. Therefore, the attacker accumulates small penalties for each attack action to ensure the goal state is reached by minimal number of attacks. Further, if a cyber-tripped line breaks the grid into islands, the attacker is hugely penalized as such an attack is easily detectable. Moreover, as cyber-tripping of lines modifies grid topology and thus changes the constraint (5.2) of DCOPF problem (5.1), any attack that results into a topology for which (5.1) has no solution is hugely penalized. Further, once generation set-points are computed from (5.1), and dispatched, the system runs according to actual topology and loads, and not the falsified one. Hence, the attacker is penalized if the generation set-points cause any branch overflow in the actual network. Therefore, for our proposed model, the reward function is designed as follows:

$$R(\mathbf{s}', \mathbf{a}, \mathbf{s}) = \begin{cases} +C_1, & \text{if } \lambda_k^a - \lambda_k^* \geq \delta_k \text{ and } N_\theta(\mathbf{s}') \leq N_a, \\ -C_2, & \text{if } \mathbf{s}' \text{ corresponds to an isolated system,} \\ -C_3, & \text{if } \mathbf{s}' \text{ corresponds to a grid topology } \mathbf{H} \\ & \text{for which (5.1) has no solution,} \\ -C_4, & \text{if solution to (5.1) with falsified topology} \\ & \text{leads to branch overflows} \\ & \text{in actual system,} \\ -C_5, & \text{otherwise,} \end{cases}$$

where $C_1, \dots, C_5$ are positive constants chosen by the designer based on the severity of each scenario and operational consideration. Further, as described in Section 5.1.2, we consider that the attacker cannot attack the any protected branch, thus $P_{s's}^a = 0$, if $(ij) \in \mathcal{E}_p$ for an $\mathbf{a} = \{b_{ij}, P_{ij}\}$, and one otherwise. Our proposed RL-based cyber-topology attack algorithm is presented in Algorithm 8.

Algorithm 8: RT-market vulnerability analysis

```

Function findVulner ( $nRun, nEpisode, \mathcal{E}, \mathcal{E}^p, \mathbf{H}, \mathbf{P}, \mathbf{D}$ ):
    Initialize  $\mathbf{S}, \mathbf{A}$  from  $\mathcal{E}$ ; Initialize  $Q(\mathbf{s}, \mathbf{a})$  for all  $\mathbf{s} \in \mathbf{S}, \mathbf{a} \in \mathbf{A}$ ;
    Solve problem (5.1); Calculate  $\lambda_k^*$  and  $P_{ij}^0$  for all  $(ij) \in \mathcal{E}$ ;
    AttackSeq  $\leftarrow []$ ;
    for  $i = 1$  to  $nRun$  do
        for  $t = 1$  to  $nEpisode$  do
             $\mathcal{E}^t \leftarrow \mathcal{E}; \mathbf{H}^t \leftarrow \mathbf{H}; \mathbf{D}^t \leftarrow \mathbf{D}; \mathbf{P}^t \leftarrow \mathbf{P}$ ;
            while True do
                if  $(ij)^t \notin \mathcal{E}_p$  then
                     $H^t(i, j) \leftarrow 0; H^t(j, i) \leftarrow 0; D^t(i) \leftarrow D^t(i) + P_{ij};$ 
                     $D^t(j) \leftarrow D^t(j) - P_{ij}; \mathcal{E}^t \leftarrow \mathcal{E}^t \setminus \{(ij)\}$ ;
                end
                Append  $(ij)$  to AttackSeq;
                Update  $\mathbf{s}^{t+1}$  from  $\mathcal{E}^t$ ; Update  $N_\theta(\mathbf{s}^{t+1})$ ;
                Solve problem (5.1); Update  $\mathbf{P}^t, \lambda_k^a, P_{ij}^t \forall (ij) \in \mathcal{E}^t$ ;
                Solve DCPF with  $\mathcal{E}, \mathbf{H}, \mathbf{P}^t, \mathbf{D}$ ;
                Calculate and update reward using  $R(\cdot)$ ;
                if  $\lambda_k^a - \lambda_k^* \geq \delta_k$  then break ;
                Update Q-value using (5.5);
                 $(ij)^{t+1} \leftarrow \begin{cases} \operatorname{argmax}_{\mathbf{a} \in \mathbf{A}(\mathbf{s})} Q^t(\mathbf{s}, \mathbf{a}), \text{ with } \Pr(1-\epsilon) \\ \text{any } \mathbf{a} \in \mathbf{A}(\mathbf{s}) \text{ with } \Pr(\epsilon) \end{cases}$  ;
                 $\mathbf{a}^{t+1} \leftarrow \{b_{ij} = -1, P_{ij}^t\}$ ;
            end
            Update  $\epsilon$ ; Reset  $N_\theta(\mathbf{s}), \forall \mathbf{s} \in \mathbf{S}$ ; Reset total reward;
        end
        Reset  $Q(\mathbf{s}, \mathbf{a}), N_\theta(\mathbf{s}), \forall \mathbf{s} \in \mathbf{S}, \mathbf{a} \in \mathbf{A}$ ; Reset total reward;
    end
return AttackSeq;

```

5.1.4 Simulation Results

We deploy our proposed RL-based topology attack-defense mechanism in two benchmark systems: (1) IEEE-39 bus [126] and IEEE-300 bus system. IEEE-39 bus system has 10 generators connected to buses $\mathcal{V}_g = \{30, \dots, 39\}$. We consider cost functions of the form $C_i(P_{gi}) = c_{i2}P_{gi}^2 + c_{i1}P_{gi} + c_{i0}, i \in \mathcal{V}_g$, for the generators. The parameters are: $c_{i2} = 0.02$ \$/MW²h for all $i \in \mathcal{V}_g, c_{i1} = 0.1, 0.5$ \$/MWh for $i = 37, 38$, and 0.3 \$/MWh for the rest,

$c_{i0}=0.4, 0.5, 0.5$ \$/h for $i=30, 31, 32$ and 0.2 \$/h for rest. The system and simulation parameters are shown in Table 5.1. Algorithm 8 is implemented in Python 3.8. DCOPF, LMPs are calculated using PyPower tool. We execute our algorithm for 1000 runs with 4000 episodes each. For each run, we start with an exploration probability (ϵ) of 0.8, which is reduced at an interval of 100 episodes to reach the final value, 0.001, at the end of the run. Fig. 5.2 illustrates attacker performance, averaged over 1000 runs when only

Table 5.1: IEEE-39 bus system parameters and Q-learning parameters

Total number of branches, N	46
Total generation & load	7367 & 6630 MW
Attacker target bus, k	8
Attack objective, δ_k	18% LMP deviation
Attacker constraint, N_a	10% of total branches
Initial branch protection set, $\mathcal{E}_p$	$\{1, 4, 13, 16, 19,$ $32, 33, 36, 38, 40, 45\}$
Discount factor, γ	0.9
Reward values, C_1, C_2, C_3, C_4, C_5	50, 50, 50, 50, 10

the breakers at $\mathcal{E}_p$ are secured. Fig. 5.2a shows that, upon convergence of our algorithm, the attacker is able to modify the LMP at the target bus by 19.37% of the actual value. Fig. 5.2b presents the convergence curve of the number of sequential attacks required to achieve the target. It indicates that the attacker needs to modify only two breakers to reach the goal state. Further, the frequency of attacks at any particular branch is indicative of criticality of that branch, in other words, the attacker learns to attack the branches that cause maximum impact on LMP. Securing such branches are critical for the grid operator. In Fig. 5.4, we plot the density of number of times the attacker attacks each branch to reach the goal state upon convergence, for 1000 runs. It is found that manipulating the breaker status of branch 7 and 12 in sequence enables the attacker to achieve the target. Next, we present how the information about critical branches can be beneficial for a grid operator to enhance RT market resiliency. Fig. 5.3 shows the attacker performance, averaged over 1000 runs, when the breakers installed at branches 7 and 12 are protected in addition to $\mathcal{E}_p$. Fig. 5.3a establishes that the attacker can modify the LMP at target bus by 14.2%, thus not meeting the target. Further, Fig. 5.3b shows that the number of attack converges to two, to achieve the final LMP deviation value of 14.2%. In Fig. 5.4, we plot the density of number of times the attacker attacks

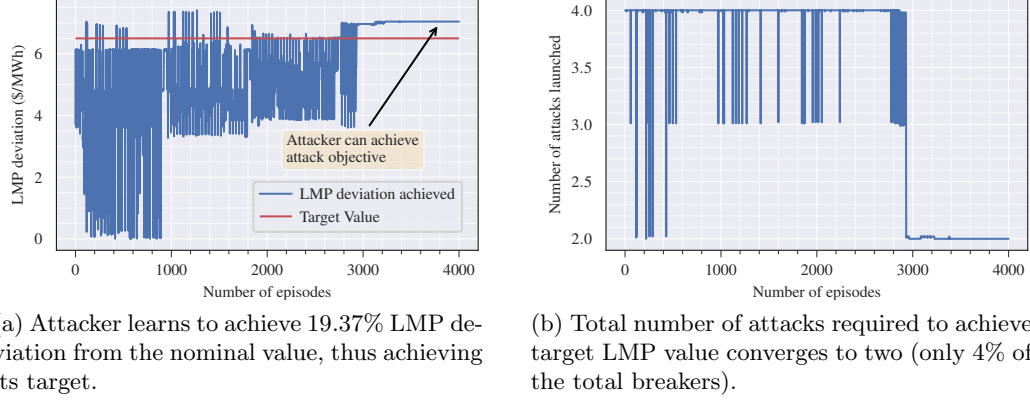


Figure 5.2: Attacker performance in IEEE-39 bus system, averaged over 1000 runs, when only the breakers in $\mathcal{E}_p$, are secured.

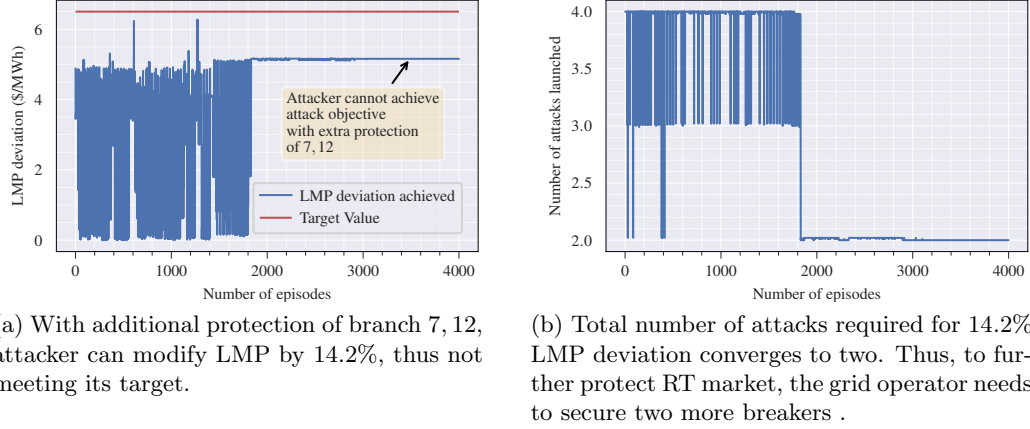


Figure 5.3: IEEE-39 bus: Attacker performance, averaged over 1000 runs, when grid operator protects breakers installed at branch 7, 12 in addition to $\mathcal{E}_p$.

each branch, for 1000 runs, to reach the final state. It is found that, upon convergence, the attacker attacks branches 15 and 30 in sequence to reach the final LMP deviation value. Hence, to further restrict the attacker's capability, the grid operator needs to protect branch 15 and 30 in addition to branch 7 and 12, and all the initially protected branches. The attacker performance is summarized in Table 5.2.

To establish the scalability of our proposed algorithm, we deploy the same on IEEE-300 bus system having 411 branches and a total of 32.7 GW generation and 29.4 GW load. Attacker target bus, k , is chosen to be 245 and the attack objective is 10% LMP deviation. We execute our algorithm for 100 runs with 5000 episodes in each. Initial

branch protection set is composed of all branches connected to the generator buses and Q-learning parameters are kept same as previous. Further, we analyze the attack performance with two different N_a values of 3% and 1%. For $N_a = 3\%$, we observe that the attacker can achieve 11.9% LMP deviation as seen from Fig. 5.5a. Further, we find that it needs to attack only 5 breakers (1% of the total breakers). The results are summarized in Table 5.3. We omit the attack convergence and branch criticality plots to conserve space. When $\mathcal{E}_p$ is augmented with the previously found critical branches, we observe that the attacker is able to achieve only 5.9% LMP deviation, thus cannot meet the objective, as illustrated in Fig. 5.5b. When we further restrict the attacker by choosing $N_a = 1\%$, we observe that it cannot manipulate the LMP significantly, as shown in Fig. 5.5c, and thus, cannot disrupt RT market operation. In

Table 5.2: IEEE 39 bus system vulnerability analysis results

Parameters	Protection: $\mathcal{E}_p$	Protection: $\mathcal{E}_p \cup \{7, 12\}$
LMP deviation achievable	19.37%	14.2%
Number of attacks required	2	2
Attack action sequence	Branch 7 $\rightarrow$ 12	Branch 15 $\rightarrow$ 30

Table 5.3: IEEE 300 bus system vulnerability analysis results

Parameters	$\mathcal{E}_p$	$\mathcal{E}_p \cup \{1, 2, 204, 326, 348\}$
LMP deviation	11.9%	5.9%
Number of attacks	5	7
Attack sequence	1, 2, 204, 326, 348	216, 320 – 325

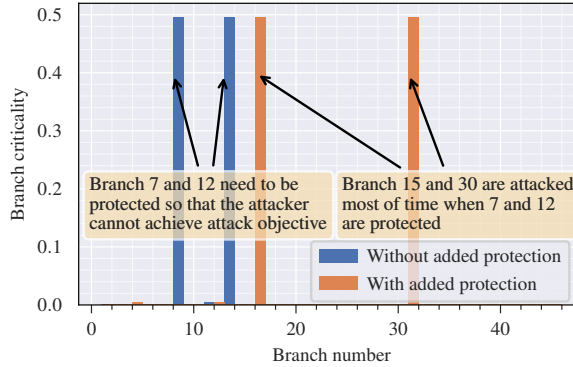
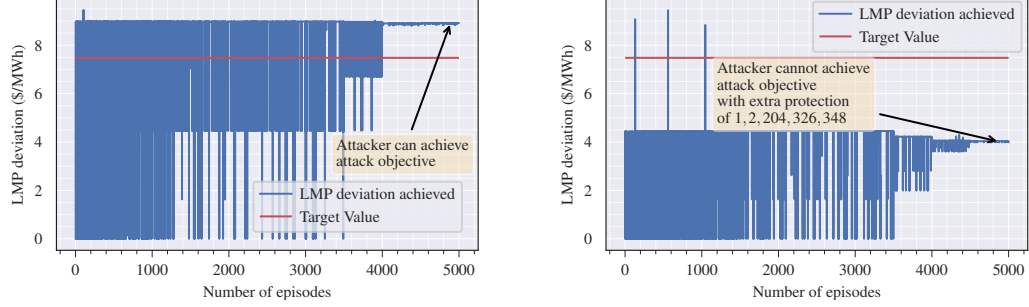


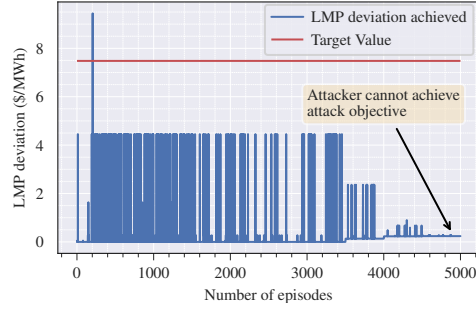
Figure 5.4: IEEE-39 bus: Histogram of attacked branches, upon convergence of our proposed algorithm for 1000 runs.

summary, the results corroborate the utility of our proposed method in identifying the



(a) IEEE-300 bus, $N_a=3\%$: Attacker learns to achieve 11.9% LMP deviation, thus achieving its target.

(b) IEEE-300 bus, $N_a=3\%$: With additional protection, attacker cannot achieve its target, as it can modify LMP only by 5.9% of the nominal value.



(c) With $N_a = 1\%$, and additional protection of branches 1, 2, 204, 326, 348, attacker cannot disrupt RT market.

Figure 5.5: Attacker performance in IEEE-300 bus, averaged over 100 runs.

critical branches of power distribution networks, that are to be protected, to mitigate the impact of cyber-topology attacks on electricity market operations. In the next part of the chapter, we discuss another type of cyber-attack: FDI attacks on microgrid tertiary control infrastructure.

5.2 False Data Injection Attack on Microgrid Secondary Control

In this part of the chapter, we propose a cyber-defense mechanism for observer-based state-feedback controllers for load frequency control of microgrids to enhance resiliency

against cyber-attacks. Load frequency control is required to compensate frequency deviation of a microgrid caused by inherently uncertain renewable energy sources. However, load frequency control becomes vulnerable to cyber attacks like false data injection due to the integration of communication infrastructures. These attacks can destabilize the system by surreptitiously modifying the system parameters along with the measurement and control commands. By designing a unique defense mechanism-based observer that can, not only estimate the system states but also detect unknown and undesired disturbances such as cyber-attacks, this work proposes a cyber-attack resilient state feedback controller-based load frequency control for compensating the frequency deviation of microgrids. A controller hardware-in-the-loop-based experimental demonstration of the proposed method is conducted by emulating a microgrid test system inside the OP5700 RT-simulator (RTS) and by realizing the proposed strategy, developed using Python 3.7.1, on a server workstation. The experimental results corroborate the efficacy of the proposed method in maintaining the stability of the system and in enhancing the resiliency of the operation against false data injection.

5.2.1 State-of-the-art

The uncertain nature of RESs introduces difficulties in balancing generation and demand thus causing system frequency deviation. Further, scenarios like electrification of remote areas and recovery from natural disasters require MGs to be operated in islanded mode, where main grid support to mitigate frequency/voltage fluctuation is not present. Thus, the uncertainty of RESs in MGs weakens system stability and resiliency [127].

Prior Works

To deal with such problems, normally, two control responses are incorporated, (i) primary control loop (droop) which adjusts the governor-based systems to address the frequency deviation (ii) secondary control loop containing load frequency controller (LFC), which supports the primary control loop and fine tunes the frequency [128]. Secondary control incorporates distribution management systems, installed at the control center to monitor the states of the MG and dispatch control commands to maintain stable frequency via communication infrastructure. However, integration of communication layers makes the secondary control loop vulnerable to cyber-attacks like Denial-of-Service (DoS) [129], Time Delay Switch (TDS) or Signal Delay Attack [130], and False Data Injection (FDI) [131]. These unpredictable attacks can hamper the control performance

of the secondary load frequency control and as a result destabilize the entire microgrid system [132].

In the current state-of-the-art, there are research works on the load frequency control for MG under cyber-attacks [133, 134, 135, 136, 137]. Most of the research works have focused on the detection and estimation of attacks in the MG. However, there are limited techniques in the literature to detect and compensate the attacks automatically in real time. Moreover, there is a common drawback for many strategies to assume that adversarial scenarios are built upon a single snapshot of the operating moment. This assumption might not be realistic for today's MG with persistent varying dynamics and intermittency of renewable energy sources. Few strategies are present in order to mitigate the above-mentioned practical challenges. However, existing detection and mitigation methods suffer from high computation and slow detection rate. In order to fill this gap, this chapter proposes a simple and effective defense mechanism against FDI for enhancing resiliency of MG operation.

Key Contributions

The contributions of this work are as follows:

1. It proposes an unique defense mechanism-based observer, that can be employed with the LFC unit, to mitigate the cyber disruptions in the LFC operation.
2. The proposed observer can, not only estimate the MG states, but also detect the unknown and undesired disturbances e.g. FDI-based cyber attacks.
3. The proposed defense mechanism-based observer, when employed with LFC unit, is capable of compensating the frequency deviation, caused by primary layer, while the FDI-based cyber attack is present both in measurements as well as in control signals.

5.2.2 Attack Model and Problem Description

A three-phase MG test system under study contains an uncontrolled RESs i.e. wind generation and solar inverter system, two electric vehicle systems (EVSs) and a critical infrastructure unit (e.g. medical hospital). Moreover, it contains a diesel generation unit with P - f -droop law based governor control, also called primary control layer of MG control hierarchy and a grid-forming inverter system, responsible for emulating virtual

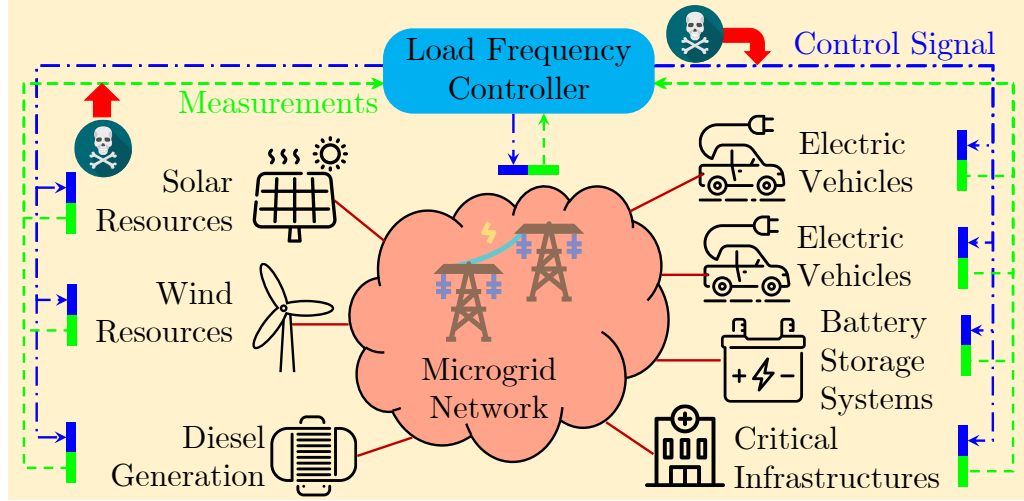


Figure 5.6: Schematic of a MG with renewable and diesel generation based resources supplying load of a critical infrastructures along with load frequency control and communication medium prone to possible cyber-attacks.

inertia and maintaining a stable voltage and frequency of the MG. It is known that in MG control hierarchy, primary control layer is responsible for stabilizing the voltage and frequency of the system by maintaining the generation demand balance. However, the primary control layer often leads to a frequency deviation from the desired nominal value even in steady state and is unable to provide the power to compensate for the deviation. A secondary control layer for load frequency control is employed to restore the frequency to the desired nominal value. The focus of this work lies in the secondary layer of LFC. In this work, a full-state feedback control for LFC is considered which is supposed to ensure a stable and closer to rated system frequency by compensating the deviation caused by primary layer. In the scenario considered here, the control efforts of the LFC unit includes the active power set-points for EVS and diesel generation unit. Moreover, the control operation requires measurements that need to be transmitted to the LFC from the system and control efforts that need to be transmitted back to the system via communication medium. As a result, the communication medium is affected by cyber-attacks such as FDI before the signal is received by the actual system or by the LFC unit. Here, FDI is classified under two categories: i) FDI in measurements, and ii) FDI in control signals. A generic cyber-physical system with the targeted application is shown in Fig. 5.6. Literature suggests that such FDI attacks can potentially degrade the MG operation and even lead to total system collapse by making it unstable in worst

cases [133, 134, 135, 136, 137]. In light of this motivation, the main problem statement of this work is to design a defense mechanism that can be employed with the conventional LFC unit in order to enhance resiliency against both types of FDI cyber attacks.

System Model

To synthesize the proposed control operation, the system, defined in Section 5.2.2, is modeled where the system is linearized at the operating point for ease of analysis. The linearized state-space model of the system is described by

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) + D_w d_w(t), \\ y(t) &= Cx(t), \end{cases}$$

$$A = \begin{bmatrix} 0 & \frac{1}{2H_t} & 0 & \frac{1}{2H_t} & \frac{1}{2H_t} \\ 0 & -\frac{1}{T_d} & \frac{1}{T_d} & 0 & 0 \\ -\frac{1}{RT_g} & 0 & -\frac{1}{T_g} & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{T_{e1}} & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{T_{e2}} \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{T_g} & 0 & 0 \\ 0 & \frac{1}{T_{e1}} & 0 \\ 0 & 0 & \frac{1}{T_{e2}} \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad (5.6)$$

$$D_w = \begin{bmatrix} -\frac{1}{2H_t} & 0 & 0 & 0 & 0 \end{bmatrix}^T,$$

$$x(t) = [\Delta f(t) \ \Delta P_{DG}(t) \ \Delta X_G(t) \ \Delta P_{EVS1}(t) \ \Delta P_{EVS2}(t)]^T$$

$$u(t) = [\Delta P_{DG}^*(t) \ \Delta P_{EVS1}^*(t) \ \Delta P_{EVS2}^*(t)]^T$$

where, $x(t) \in \mathbb{R}^n$ denotes the states, $u(t) \in \mathbb{R}^p$ the input, $y(t) \in \mathbb{R}^q$ the measurements, $d_w(t)$ the combined load and RES output power fluctuation, Δf the frequency deviation, $\Delta P_{DG}(t)$ the diesel generation output power variation, $\Delta X_G(t)$ the governor position variation, and $\Delta P_{EVS1}(t)$ and $\Delta P_{EVS2}(t)$ are EVS1 and EVS2 output power variations, $\Delta P_{DG}^*(t)$ the DG governor control signal, $\Delta P_{EVS1}^*(t)$ and $\Delta P_{EVS2}^*(t)$ the EVS1 and EVS2 control signals. In the subsequent analysis, we omit the time variable t for brevity of notation.

Attack Model

As stated in Section 5.2.2, we consider FDI in both actuator signals and measurements. We assume the actuator attack vector in control signal, $f_a \in \mathbb{R}^r$, is stealthily added to the actuator input u . Without loss of generality, we assume that only EVS-1 actuator is attacked. Further, we assume that the dynamics of the measurement attack vector $f_m \in \mathbb{R}^q$ is modeled by an linear time-invariant system [138] given by:

$$\begin{cases} \dot{f}_m &= A_m f_m + B_m u, \\ y_m &= C_m f_m, \end{cases} \quad (5.7)$$

where $A_m \in \mathbb{R}^{q \times q}$, $B_m \in \mathbb{R}^{q \times p}$ and $C_m \in \mathbb{R}^{q \times q}$ are chosen by the attacker and $y_m \in \mathbb{R}^q$ is injected to the actual measurement y to maliciously alter the measurements. Thus the system's state-space model under FDI attacks can be written as follows:

$$\begin{aligned} \dot{x}_a &= Ax_a + Bu + D_w d_w + D_a f_a \\ y_a &= Cx_a + C_m f_m, \end{aligned} \quad (5.8)$$

where D_a is constructed by extracting the second column of B . Next we analyse the impact of such attacks on the closed-loop system states.

Observer-based State-feedback Controller for Conventional Load Frequency Control

In this work, we focus on observer-based state feedback control because all the states of the system are not directly measurable, as can be seen from (5.6). In the case of a conventional Luenberger observer, designed for the system (5.6), the estimated system states are given by,

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}), \quad (5.9)$$

where L is the observer gain. L is designed such a way that $(A - LC)$ has eigen values with negative real parts. Once the states are observed they are fed back to the system using the negative feedback law $u = -K\hat{x}$, where K is the feedback gain matrix. K is designed such that $(A - BK)$ has eigen values with negative real parts along with considering control objective and actuator saturation limits. In a conventional state-feedback

load frequency control, the presence of FDI attacks in actuator signals and/or measurements are not considered, making the system vulnerable to cyber-attacks. Next we show how an attacker can manipulate the system states using carefully crafted actuator and measurement FDI attacks.

Impact of FDI Attacks on Conventional Load Frequency Controller

The closed loop system equation with observer-based state feedback control of (5.8) under FDI attacks can be written as:

$$\begin{aligned}\dot{x}_a &= Ax_a - BK\hat{x}_a + D_w d_w + D_a f_a \\ &= (A - BK)x_a + BK e_{x_a} + D_w d_w + D_a f_a,\end{aligned}\tag{5.10}$$

where $e_{x_a} := x_a - \hat{x}_a$ is the state estimation error. Thus, from (5.8) and (5.10), the following can be derived:

$$\begin{aligned}\dot{e}_{x_a} &= \dot{x}_a - \dot{\hat{x}}_a = Ax_a - BK\hat{x}_a + D_w d_w + D_a f_a \\ &\quad - A\hat{x}_a + BK\hat{x}_a - L(Cx_a + C_m f_m - C\hat{x}_{cla}) \\ &= (A - LC)e_{x_a} + D_w d_w + D_a f_a - LC_m f_m.\end{aligned}\tag{5.11}$$

Note that, the error can be directly controlled by the attacker using attack inputs f_a and f_m . The transfer function from f_a to e_{x_a} can be written as $T_{efa} = (sI - A + LC)^{-1}D_a$. Similarly, the transfer function from f_m to e_{x_a} can be written as $T_{efm} = -(sI - A + LC)^{-1}LC_m$. Further, the transfer function from f_a to x_a can be written as $(sI - A + BK)^{-1}D_a$. Thus, by carefully crafting D_a and C_m the system operation can be hampered by the attacker or the system states can be driven to attacker's desired values. In order to enhance the resiliency of the MG operation and control against such FDI attacks, a modification of the conventional observer is required which we discuss next.

Required Modification of Conventional Architecture

In this section we analyze how the design of conventional observers (5.9), and feedback control law must be modified to mitigate the impacts of FDI attacks. We state the following theorem to illustrate the design and utility of the same.

Theorem 8. Consider a state observer of the following form,

$$\dot{\hat{x}}_a = A\hat{x}_a + Bu + K_1\hat{f}_a - K_2\hat{f}_m + L(y_a - C\hat{x}_a), \quad (5.12)$$

and a state feedback control law as follows,

$$u = -K\hat{x}_a - K_3\hat{f}_a, \quad (5.13)$$

where $\hat{x}_a, \hat{f}_a, \hat{f}_m$ are the estimated closed-loop state of the system (5.8), estimate of f_a , estimate of f_m respectively, and K_1, K_2, K_3 are matrices of appropriate dimensions. Assume that $\hat{f}_a \rightarrow f_a$ and $\hat{f}_m \rightarrow f_m$ as $t \rightarrow \infty$. Then x_a , and the state estimation error, denoted by $e_a := x_a - \hat{x}_a$, do not depend on f_a and f_m as $t \rightarrow \infty$ if $BK_3 = K_1 = D_a$, and $K_2 = LC_m$.

Proof. If K_1, K_2, K_3 are chosen such that $BK_3 = K_1 = D_a$, and $K_2 = LC_m$, then from (5.12) and (5.13) we obtain,

$$\begin{aligned} \dot{x}_a &= (A - BK)x_a + BK e_{x_a} + BK_3 e_{f_a} + D_w d_w, \\ \text{where, } \dot{e}_{x_a} &= \dot{x}_a - \dot{\hat{x}}_a = (A - LC)e_{x_a} + D_w d_w + K_1 e_{f_a} - K_2 e_{f_m}, \\ e_{f_a} &:= \hat{f}_a - f_a, \quad e_{f_m} := \hat{f}_m - f_m. \end{aligned}$$

Under the assumption that $\hat{f}_a \rightarrow f_a$ and $\hat{f}_m \rightarrow f_m$ as $t \rightarrow \infty$, we have $e_{f_a} \rightarrow 0$, $e_{f_m} \rightarrow 0$ as $t \rightarrow \infty$. Hence, x_a and e_{x_a} do not depend on f_a and f_m as $t \rightarrow \infty$. $\square$

Remark 1. Note that in order to deploy the modified observer and control law as given in (5.12) and (5.13), accurate estimates of f_a and f_m are required, which necessitates additional machinery thus increasing complexity. Moreover, without further modification of the observer, it is exigent to get estimates of the FDI attacks. We alleviate the requirement of additional methodologies to estimate the system states, and FDI attacks separately by our proposed architecture as described in the next section. We emphasize that in our proposed mechanism, impact of FDI attacks are mitigated while estimating the system states and FDI attacks in tandem.

5.2.3 Proposed Solution

In this section, we discuss our proposed solution to mitigate the impacts of FDI attacks on the closed-loop system states. Our solution is built upon the estimates of the FDI

attacks and unaltered closed-loop states simultaneously. The proposed methodology is detailed in the subsequent paragraphs.

First we create an augmented state space model as follows:

$$\begin{aligned}
\dot{x}_{aug} &= A_{aug}x_{aug} + B_{aug}u + D_{aug}d, \quad y_{aug} = C_{aug}x_{aug}, \\
x_{aug} &= [x \ f_m]^T \in \mathbb{R}^{n+q}, d = [d_w \ f_a]^T \in \mathbb{R}^{r+1}, y_{aug} = y \in \mathbb{R}^q \\
A_{aug} &= \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & A_m \end{bmatrix}, B_{aug} = [B \ B_m]^T \\
D_{aug} &= [D_w \ D_a], C_{aug} = [C \ C_m]^T.
\end{aligned} \tag{5.14}$$

Note that the vector d is unknown to the MG operator. In order to utilize the control law (5.13), d must be estimated from the input u and measument y_{aug} which is also manipulated by the attacker. We assume that $\text{rank}(C_{aug}D_{aug}) = \text{rank}(D_{aug}) = r+1 \leq q$ and $\text{rank}(C_{aug}) = q$. We denote an identity matrix and null matrix in $\mathbb{R}^{k \times k}$ by I_k and 0_k respectively. Then the following theorem establishes how d, f_a and f_m can be accurately estimated from (5.14) and the impact of FDIs on closed-loop system states can be attenuated at the same time.

Theorem 9. *Consider a state transformation $x_{aug} = T[z_1 \ z_2]^T$, where $T = [N \ D_{aug}]$, $N \in \text{nullspace}(D_{aug}^T)$. Further consider a non-singular matrix $U = [C_{aug}D_{aug} \ Q]$, $Q \in \text{nullspace}((C_{aug}D_{aug})^T)$. Denote the estimates of x_{aug} and d by $\hat{x}_{aug}$ and $\hat{d}$ respectively. Then $\hat{x}_{aug}$ and $\hat{d}$ can be estimated from u and y_{aug} , and $\hat{x}_a = [I_n \ 0_q]\hat{x}_{aug}$, $\hat{f}_a = [0_1 \ I_r]\hat{d}$, and $\hat{f}_m = [0_n \ I_q]\hat{x}_{aug}$ can be made asymptotically convergent to x_a, f_a and f_m respectively.*

Proof. Let the transformed state space model be $[\dot{z}_1 \ \dot{z}_2]^T = \bar{A}[z_1 \ z_2]^T + \bar{B}u + \bar{D}d$, and $y_{aug} = \bar{C}[z_1 \ z_2]^T$, where,

$$\begin{aligned}
\bar{A} &= T^{-1}A_{aug}T = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}, \bar{B} = T^{-1}B_{aug} = \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix}, \\
\bar{D} &= T^{-1}D_{aug} = \begin{bmatrix} 0 \\ I \end{bmatrix}, \bar{C} = C_{aug}T = [C_{aug}N \ C_{aug}D_{aug}].
\end{aligned} \tag{5.15}$$

Let, $U^{-1} = [U_1 \ U_2]^T$ with $U_1 \in \mathbb{R}^{(r+1) \times q}$ and $U_2 \in \mathbb{R}^{(q-r-1) \times q}$. From (5.14) and (5.15), we

get,

$$\begin{aligned}\dot{z}_1 &= (\bar{A}_{11} - \bar{A}_{12}U_1C_{aug}N)z_1 + \bar{B}_1u + \bar{A}_{12}U_1y_{aug} \\ U_2y_{aug} &= U_2C_{aug}Nz_1,\end{aligned}\tag{5.16}$$

and $z_2 = U_1y_{aug} - U_1C_{aug}Nz_1$. Observe that the state-space model (5.16) with states z_1 and output U_2y_{aug} does not depend on unknown input d . Hence, a conventional Luenberger observer with gain matrix L_{z_1} can be designed to obtain an estimate of z_1 , denoted by $\hat{z}_1$, where,

$$\begin{aligned}\dot{\hat{z}}_1 &= (\bar{A}_{11} - \bar{A}_{12}U_1C_{aug}N)\hat{z}_1 + \bar{B}_1u + \bar{A}_{12}U_1y_{aug} \\ &+ L_{z_1}(U_2C_{aug}Nz_1 - U_2C_{aug}N\hat{z}_1).\end{aligned}\tag{5.17}$$

Denote the estimation error of z_1 by e_{z_1} . Then, $\dot{e}_{z_1} = \dot{z}_1 - \dot{\hat{z}}_1 = (\bar{A}_{11} - \bar{A}_{12}U_1C_{aug}N - L_{z_1}U_2C_{aug}N)e_{z_1}$. Therefore, by proper choice of observer gain L_{z_1} , the eigen values of $(\bar{A}_{11} - \bar{A}_{12}U_1C_{aug}N - L_{z_1}U_2C_{aug}N)$ can be designed to have positive real parts so that as $t \rightarrow \infty$, e_{z_1} goes to zero. Further, the estimation error of z_2 can also be made zero asymptotically as $e_{z_2} = z_2 - \hat{z}_2 = U_1C_{aug}Ne_{z_1}$. Note that the estimated states of (5.14) can be written as

$$\hat{x}_{aug} = T \begin{bmatrix} \hat{z}_1 & U_1y_{aug} - U_1C_{aug}N\hat{z}_1 \end{bmatrix}^T.\tag{5.18}$$

Therefore, as $t \rightarrow \infty$, $\hat{z}_1 \rightarrow z_1$ and $\hat{z}_2 \rightarrow z_2$ and hence, $\hat{x}_{aug} \rightarrow x_{aug}$ which implies $\hat{x}_a \rightarrow x_a$. Let $e_{fm} := f_m - \hat{f}_m$ be the estimation error of the measurement attack vector. Therefore, $e_{fm} = [0_n \ I_q](x_{aug} - \hat{x}_{aug})$ asymptotically converges to zero, leading to $\hat{f}_m \rightarrow f_m$. Next we show how $\hat{d}$ can be estimated from $\hat{z}_1, u$ and y_{aug} and made asymptotically convergent to d .

From (5.15), $\hat{d}$ can be estimated [139] as follows,

$$\hat{d} = U_1\dot{y}_{aug} + G_1\hat{z}_1 + G_2y_{aug} + G_3u,\tag{5.19}$$

$$\begin{aligned}\text{where, } G_1 &= U_1C_{aug}NL_{z_1}U_2C_{aug}N + U_1C_{aug}N\bar{A}_{12}U_1C_{aug}N \\ &\quad - U_1C_{aug}N\bar{A}_{11} - \bar{A}_{21} + \bar{A}_{22}U_1C_{aug}N, \\ G_2 &= -U_1C_{aug}NL_{z_1}U_2 - U_1C_{aug}N\bar{A}_{12}U_1 - \bar{A}_{22}U_1, \\ G_3 &= -U_1C_{aug}N\bar{B}_1 - \bar{B}_2.\end{aligned}$$

Thus,

$$\begin{aligned}
\dot{d} &= U_1 \dot{y}_{aug} + G_1 \hat{z}_1 + G_2 y_{aug} + G_3 u \\
&= U_1 \dot{y}_{aug} + (U_1 C_{aug} N)(L_{z_1} U_2 C_{aug} N \hat{z}_1 + \bar{A}_{12} U_1 C_{aug} N \hat{z}_1 \\
&\quad - \bar{A}_{11} \hat{z}_1) - \bar{A}_{21} \hat{z}_1 + \bar{A}_{22} U_1 C_{aug} N \hat{z}_1 - U_1 C_{aug} N L_{z_1} U_2 y_{aug} \\
&\quad - U_1 C_{aug} N \bar{A}_{12} U_1 y_{aug} - \bar{A}_{22} U_1 y_{aug} - U_1 C_{aug} N \bar{B}_1 u - \bar{B}_2 u \\
&= U_1 \dot{y}_{aug} - U_1 C_{aug} N \dot{\hat{z}}_1 - \bar{A}_{21} \hat{z}_1 + \bar{A}_{22} U_1 C_{aug} N \hat{z}_1 \\
&\quad - \bar{A}_{22} U_1 y_{aug} - \bar{B}_2 u.
\end{aligned}$$

Note that, $\hat{z}_2 = U_1 y_{aug} - U_1 C_{aug} N \hat{z}_1$. Further, from (5.15), we have $\dot{z}_2 = \bar{A}_{21} z_1 + \bar{A}_{22} z_2 + \bar{B}_2 u + d$. Therefore, $\hat{d} = U_1 \dot{y}_{aug} + \dot{\hat{z}}_2 - U_1 \dot{y}_{aug} - \bar{A}_{21} \hat{z}_1 - \bar{A}_{22} \hat{z}_2 - \bar{B}_2 u = d + \bar{A}_{21}(z_1 - \hat{z}_1) + \bar{A}_{22}(z_2 - \hat{z}_2) - (\dot{z}_2 - \dot{\hat{z}}_2) = d + \bar{A}_{21} e_{z_1} + \bar{A}_{22} e_{z_2} - \dot{e}_{z_2}$. As e_{z_1} and e_{z_2} asymptotically converge to zero, $\hat{d}$ converges to d . Let $e_{f_a} := f_a - \hat{f}_a$ be the estimation error of the actuator attack input f_a . Therefore, $e_{f_a} = [0_1 \ I_r](d - \hat{d})$ asymptotically converges to zero, leading to $\hat{f}_a \rightarrow f_a$. $\square$

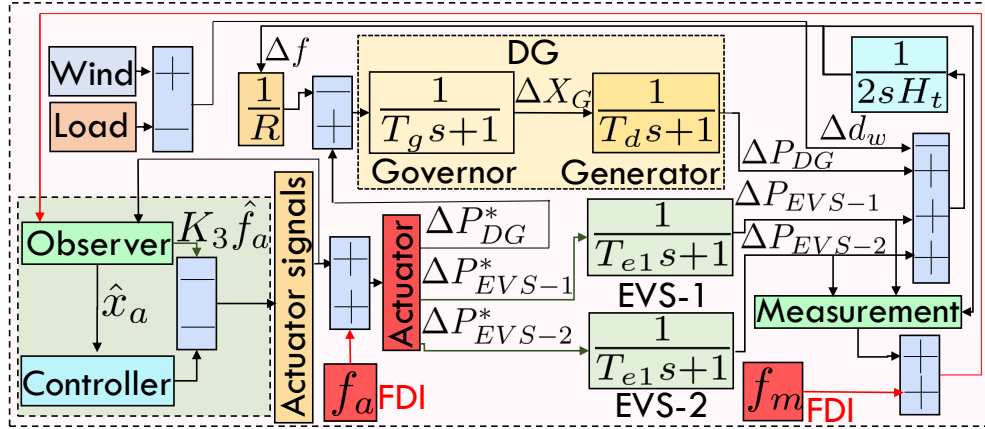


Figure 5.7: Block diagram of the proposed cyber-secured MG control architecture

We propose to utilize the control input law (5.13) along with estimate equations (5.18) and (5.19), detailed in Theorem 9, to mitigate the impact of FDI attacks on MG system operation. The block diagram of our proposed architecture is shown in Fig. 5.7. We summarize the design steps for our proposed attack-resilient MG operation architecture in Algorithm 9. In the next section we present the utility of the same on a real-time

Algorithm 9: Attack-resilient MG design architecture

Function getEstimates ($A_{aug}, B_{aug}, C_{aug}, D_{aug}, u, y_{aug}$):

STEP 1: Transformation matrices
 $N = \text{nullspace}(D_{aug}^T); T = [N \ D_{aug}];$
 $Q = \text{nullspace}((C_{aug}D_{aug})^T); U = [C_{aug}D_{aug} \ Q];$

STEP 2: State transform
 $\bar{A} = T^{-1}A_{aug}T = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}; \bar{B} = T^{-1}B_{aug} = \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix};$
 $\bar{C} = C_{aug}T = [C_{aug}N \ C_{aug}D_{aug}]; U^{-1} = [U_1 \ U_2]^T;$
 Extract B from B_{aug} , D_a from D_{aug} ;

STEP 3: Observer and controller gain selection
 Design L_{z_1} s.t. $(\bar{A}_{11} - \bar{A}_{12}U_1C_{aug}N - L_{z_1}U_2C_{aug}N)$ has eigen values with negative real parts;
 Design K s.t. $(A - BK)$ has eigen values with negative real parts;
 Design K_3 such that $BK_3 = D_a$;

STEP 4: Design matrices
 $G_1 = U_1C_{aug}NL_{z_1}U_2C_{aug}N + U_1C_{aug}N\bar{A}_{12}U_1C_{aug}N - U_1C_{aug}N\bar{A}_{11} - \bar{A}_{21} + \bar{A}_{22}U_1C_{aug}N;$
 $G_2 = -U_1C_{aug}NL_{z_1}U_2 - U_1C_{aug}N\bar{A}_{12}U_1 - \bar{A}_{22}U_1;$
 $G_3 = -U_1C_{aug}N\bar{B}_1 - \bar{B}_2;$

STEP 5: Estimations and updates
 $\hat{z}_1 = (\bar{A}_{11} - \bar{A}_{12}U_1C_{aug}N)\hat{z}_1 + \bar{B}_1u + \bar{A}_{12}U_1y_{aug} + L_{z_1}(U_2C_{aug}Nz_1 - U_2C_{aug}N\hat{z}_1);$
 $\hat{d} = U_1\dot{y}_{aug} + G_1\hat{z}_1 + G_2y_{aug} + G_3u;$
 $\hat{x}_{aug} = T[\hat{z}_1 \ (U_1y_{aug} - U_1C_{aug}N\hat{z}_1)]^T;$
 $\hat{x}_a = [I_n \ 0_q]\hat{x}_{aug}; \hat{f}_a = [0_1 \ I_r]d; \hat{f}_m = [0_n \ I_q]\hat{x}_{aug};$
 $u = -K\hat{x}_a - K_3\hat{f}_a;$

return $\hat{x}_a, \hat{f}_a, \hat{f}_m, u;$

OPAL-RT based emulation platform along with a controller hardware-in-loop, emulating our proposed observer. We establish the efficacy of our methodology by a detailed comparison between the following scenarios:

(1) Method A: Here we assume that the MG operator does not consider the presence of the measurement attack. Method A relates to conventional LFC control where the measurements are assumed to be unaltered by cyber-attacks. In our hardware experiments, we show that operation of such a system can be greatly hampered by measurement attacks.

(2) Method B: Here we design our controller and observer as proposed in Theorem 9. We establish that our proposed methodology performs better compared to method A and the system operation is resilient to FDI attacks.

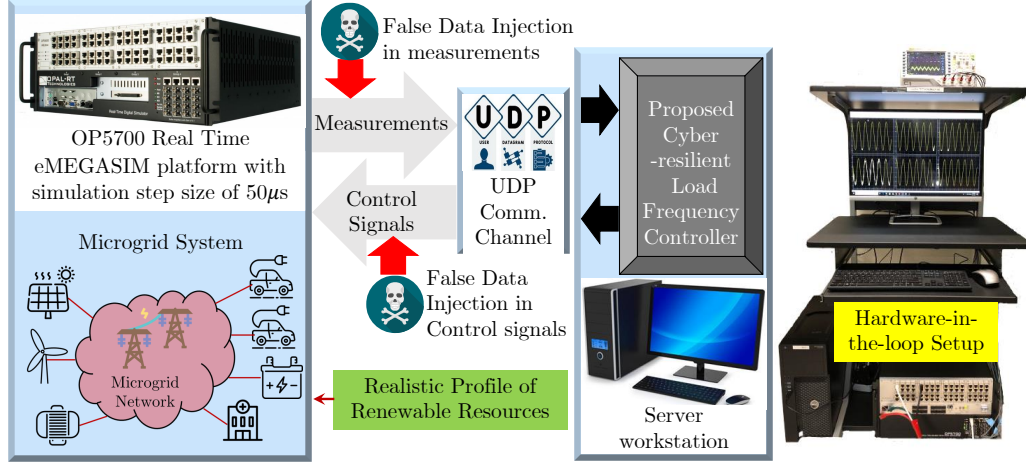


Figure 5.8: Schematic of the CHIL-based demonstration of the load frequency control with the proposed defence mechanism against cyber-attack.

5.2.4 Controller Hardware-in-the-loop based Validation

Control Hardware-in-the-loop Configuration

Fig. 5.8 shows the controller hardware-in-the-loop setup developed for the validation of the LFC of a MG with the proposed defense mechanism. There are three important components in the demonstration: (i) the MG, emulated using the eMEGASIM platform with a simulation step size of $50\mu\text{s}$ inside the OP5700 RT-simulator (RTS), manufactured by OPAL-RT; (ii) the LFC unit with the proposed defence mechanism, running on a server workstation, developed using Python 3.7.1, to enhance the resiliency against cyber attacks; and (iii) a device control gateway, realized using standard User Datagram Protocol (UDP), that interfaces the MG and the LFC unit and emulates the communication layer that continuously listens to the measurements sent from the MG and dispatch control signals to the MG. Both the false data injection (i.e. in measurements as well as the control signals) are also emulated inside the RT-simulator. A representative of attacks in output power measurements of EVS-1 and EVS-2 are shown in Fig. 5.9. Realistic profiles of renewable units are fed into the real-time simulator[140, 141]. The $3-\phi$, 60 Hz, 480 V MG test system under study contains 1 renewable energy resource i.e. wind generation and solar inverter system of 50 kW, 1 diesel generation unit of 70 kW with P - f -droop law with co-efficient $R=2.5$ Hz/p.u. and governor control with time constant, $T_g=0.1$ s, 1 grid-forming inverter system, responsible for virtually emulating inertia of $H_t=7.11$ s in the MG and for maintaining a stable voltage and frequency of

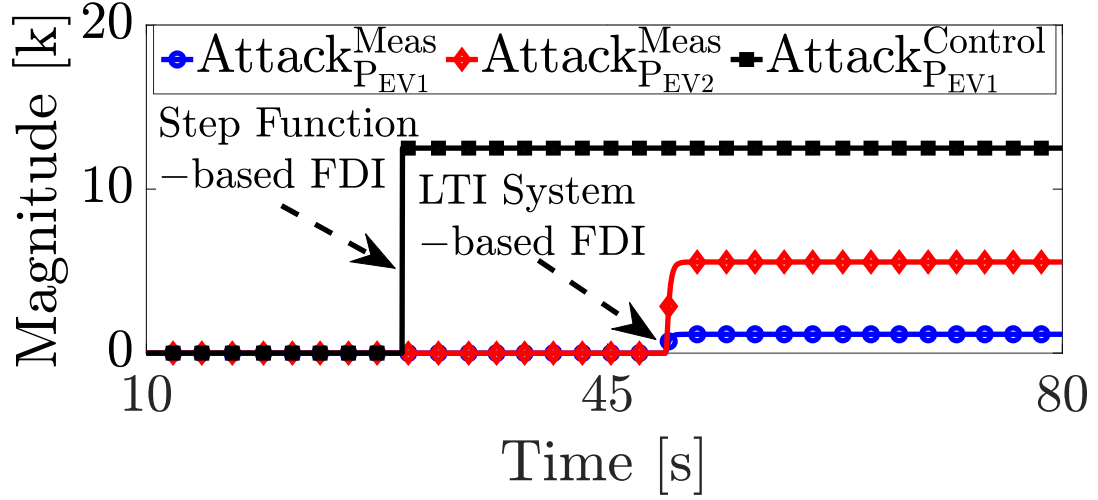


Figure 5.9: Actual false data injection in active power reference of EVS-1 and output active power measurements of EVS-1 and EVS-2.

the MG, 2 electric vehicle system of 50 kW each with time constants $T_{e1}=1.1s$, $T_{e2}=0.9s$ and 1 critical infrastructure unit (e.g. medical hospital) of 200 kW.

Demonstrations and Results

A test case is constructed in order to validate the performance of the proposed defence mechanism of the LFC unit. The total demonstration time is of 70s where

- from $t = 10s$ to $t = 30s$, the MG is operating with no attacks in measurement as well as control signals.
- at $t = 30s$, a false data is injected in the control signal, set points of EVS-1, P_{EVS1}^* , (plotted signal in Fig. 5.9 in black and with square marker).
- at $t = 50s$, a false data is injected in the measurements, P_{EVS1}^{Meas} and P_{EVS2}^{Meas} , (plotted signal in Fig. 5.9 in red and blue with circle and diamond marker respectively).

The performance of the estimation of the proposed architecture in this application is shown in Fig. 5.10. Performance of the estimation of one internal state (e.g. active power output of diesel generation, P_{Diesel}) and estimation of unknown input to the system (e.g. the FDI in control unit, $Attack_{P_{EV1}}^{Control}$) are shown in Fig. 5.10 (a) and Fig. 5.10 (b) respectively. As observed from the result, the performance of the estimator

is fairly accurate. For P_{Diesel} , the estimations have mean error of -6.4% where error reduces asymptotically to $\approx -7\%$. Whereas for $\text{Attack}_{P_{EV1}}^{\text{Control}}$, the estimations have mean error of -14.48% where error reduces asymptotically to $\approx -2.5\%$. It is important to note here that the estimation performance is resilient of cyber-attack through out the demonstration time.

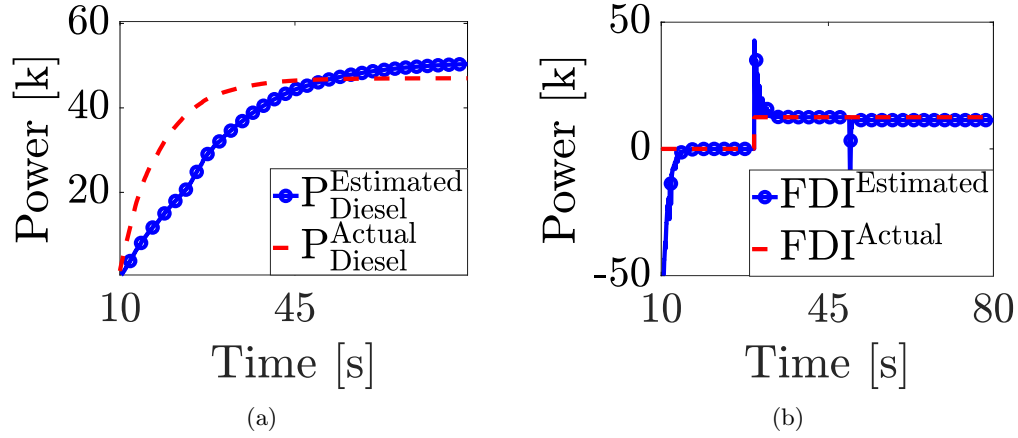


Figure 5.10: Performance of the proposed estimator, (a) internal state (e.g. active power output of diesel generation, P_{Diesel}) estimation, (b) Estimate of FDI in control unit, f_a) estimation.

The performance of the proposed defense mechanism for the load frequency control of the MG will be discussed briefly. To showcase the advantage of the proposed defense mechanism, the performance of the load frequency control with “Method A” and with “Method B” are analysed. Fig. 5.11 shows the frequency of the MG where the load frequency control is associated with the proposed defense mechanism based observer. It is observed that in “Method A”, the system becomes unstable after the attack is initialized in the measurement as the frequency increases continuously. Whereas, the load frequency control with the “Method B”-based defence mechanism of observer makes sure that the system is always stable irrespective of the unwanted cyber-attack both in measurements and control signals. Fig. 5.12 and Fig. 5.13 show the output active power of the EVS 1 and 2 as a result of the load frequency control with defense mechanism-based observer of “Method A” and “Method B” respectively. It is observed that the system is becoming unstable and the LFC is not able to make corrective control action via generating stable and bounded power set-points when “Method A”-based defensive

mechanism is employed for both the EVS after the cyber-attack in measurement initiates. However, the same system is stable through out the demonstration and the load frequency control is able to make corrective control action via generating stable and bounded power set-points when “Method B”-based defensive mechanism is employed for both the EVS after the cyber-attack in measurement initiates.

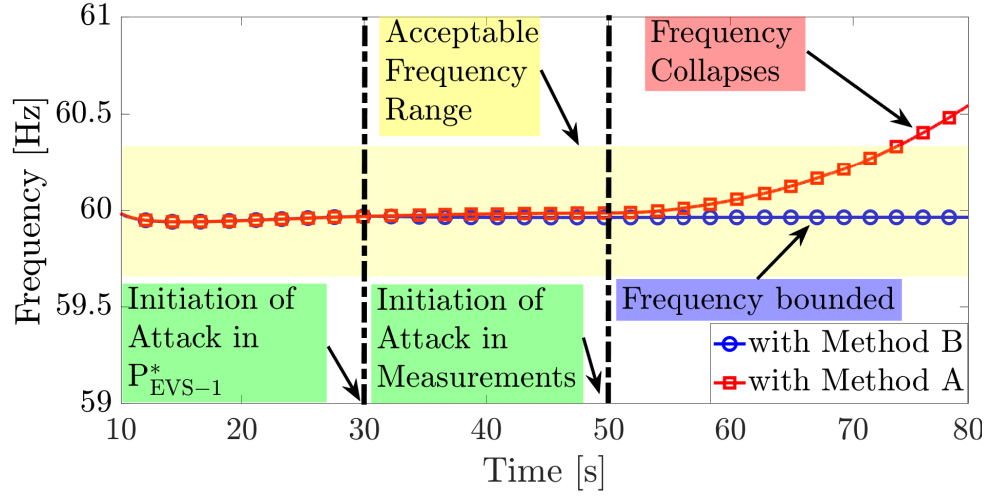


Figure 5.11: Frequency of the MG as a result of the the load frequency control with proposed defense mechanism-based observer under cyber-attacks both in measurements and control signals.

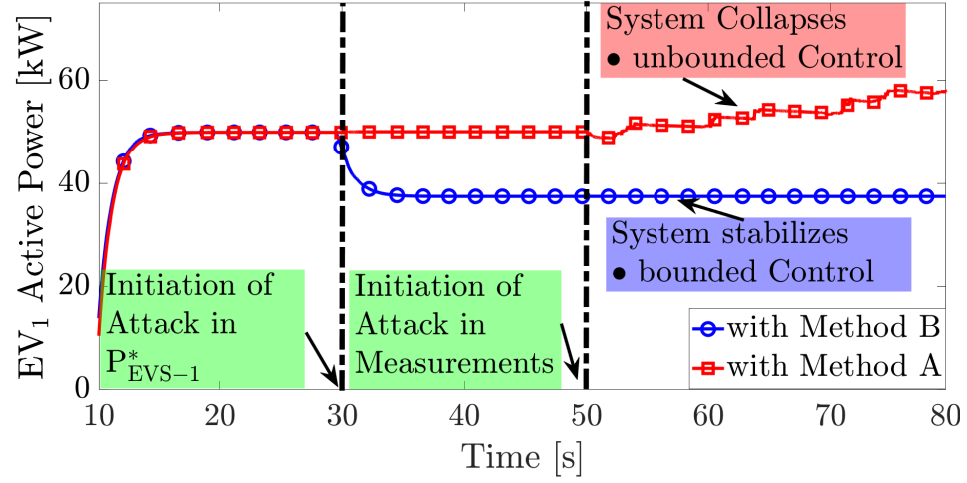


Figure 5.12: Output active power of the EVS-1 as a result of the the load frequency control with proposed defense mechanism-based observer under cyber-attacks both in measurements and control signals.

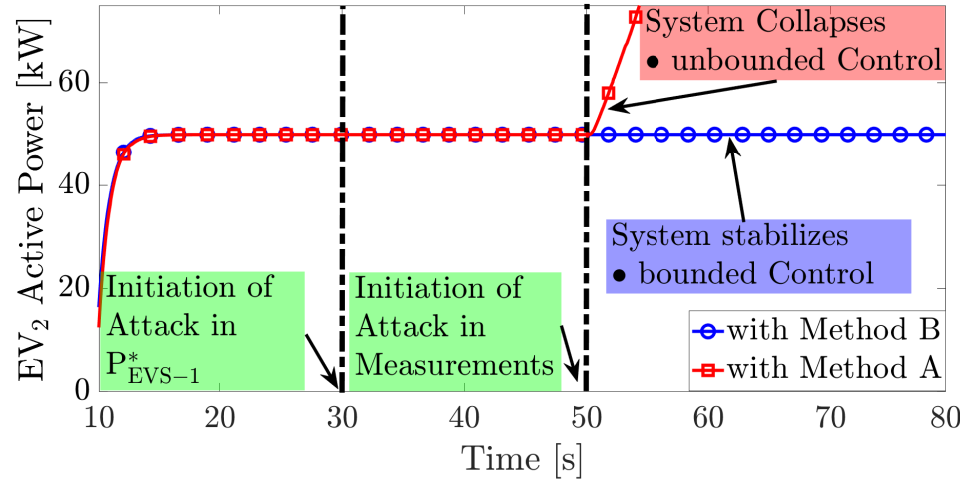


Figure 5.13: Output active power of the EVS-2 as a result of the the load frequency control with proposed defense mechanism-based observer under cyber-attacks both in measurements and control signals.

Chapter 6

An End-to-End Cyber-Physical Infrastructure for Smart Grid Control and Monitoring

6.1 Gist of the Chapter

In this chapter, we propose a generic cyber-physical framework, developed in our laboratory, for smart grid control and monitoring in real-time. Our framework is composed of four key elements: (1) *system layer* which embeds a physical or emulated power system network, (2) *data analysis layer* to execute real-time data-driven grid analysis algorithms, (3) *backend layer* with a generic data storage framework which supports multiple databases with functionally different architectures, and (4) *visualization layer* where multiple customized or commercially available user interfaces can be deployed concurrently for grid control and monitoring. These four layers are interlinked via bi-directional communication channels. Such a flexible and scalable framework provides a cohesive environment to enhance smart grid situational awareness. We demonstrate the utility of our proposed architecture with several case studies where we estimate a modified IEEE-33 bus distribution network topology entirely from synchrophasor measurements, without any prior knowledge of the grid network, and render the same on visualization platform. Three demonstrations are included with single and multiple system operators having complete and partial measurements.

6.2 State-of-the-art: Cyber-Physical Laboratory Testbeds

The increased penetration of distributed energy resources to the traditional distribution grid and a complex nexus of Information and Communications Technology (ICT) infrastructure, built on top of it, represents a rapid paradigm shift in the way the grid is controlled and supervised. With the manifestation of ICT, two-way communication between the utilities and the customers is becoming more apparent. However, before deploying the entire ICT network to physical grids, it is required to define its communication framework and assess the performance in terms of flexibility, scalability, and security [142]. This necessitates the design of a cyber-physical (CP) infrastructure that replicates the real system through simulation and modeling. In addition to the communication, control and monitoring framework, it is required to integrate data-driven intelligent algorithms to support critical operations that require real-time status of the entire grid.

6.2.1 Prior Works and Key Contributions of the Chapter

There are numerous examples of CP experimental environments, each developed with distinct utility and application, such as Phasor Measurement Units (PMUs) and Phasor Data Concentrator (PDC) framework testing [143], supervisory control and data acquisition [144], wide-area monitoring and control [145, 146], cyber-security [147, 148] etc. Further, the frameworks differ in terms of core components. For example, real-time simulator [149], power hardware-in-loop [150] are used to emulate physical grids, and network simulator [151], generalized radio based wireless communication [152], publish-subscribe [153] protocol are used to emulate ICT. Moreover, some experimental environments are tailored to specific applications performed by distribution system operator (DSO), such as state-estimation [154], energy management [155], demand-response [156] etc. Although these diversities are beneficial to address specific challenges associated with smart grids, a unified framework is more suitable in experimental level to provide flexibility and scalability along with fidelity. Apart from unification, another important feature of CP framework is the capability to provide situational awareness of the grid. Modern distribution grids have meshy network structure compared to traditional radial networks. The complexity of such network is further exacerbated by geographical span of the system which may be divided into several sub-regions supervised by independent

authorities. This poses a challenge in acquiring precise information about the grid network which is essential to ensure situational awareness. However, to the best of the authors knowledge, the state-of-the-art lacks an unified CP experiment platform which encapsulates the capability to provide real-time situational awareness. This chapter presents a flexible and scalable CP framework for smart grid control and monitoring manifesting the following major contributions:

- (i) design of a *generic* CP framework that is *flexible* to fit physical, emulated or custom smart grid components.
- (ii) implementation of a *scalable* framework capable of adapting to existing, customized and future ICT infrastructure.
- (iii) development of an architecture that is suitable for both (a) research environments where a support to customized grid analysis algorithms is the prime objective, and (b) enterprise-level where the fidelity to real-time complete situational awareness of the same is of utmost importance.

In this chapter, we utilize grid topology estimation technique [157] for situational awareness using our proposed framework. However, any generic smart grid control and monitoring functionalities can be tethered to our architecture.

6.3 Overview of the Architecture

Our smart grid control and monitoring infrastructure is composed of four distinct layers that can communicate to each other, to provide complete configurability in terms of system control, monitoring and data analytics. Fig. 6.1(a) illustrates the architectural overview of the communication network between these layers. At the core of our proposed framework, there is a backend server that interconnects (1) *System layer*, constituting a physical (or emulated) grid and associated measurement devices, (2) *Backend layer*, involving data storage, local monitoring, (3) *Visualization layer*, where customized user-interfaces (UIs) can be deployed, and (4) *Data analysis layer*, where analytics algorithms can be executed by querying the stored data from the backend layer. Subsequent sections describe each layer with its specific functionalities.

6.3.1 System Layer

The system layer embeds a physical or emulated grid with several measurement devices deployed at different places in the network. Usually smart grids are equipped with

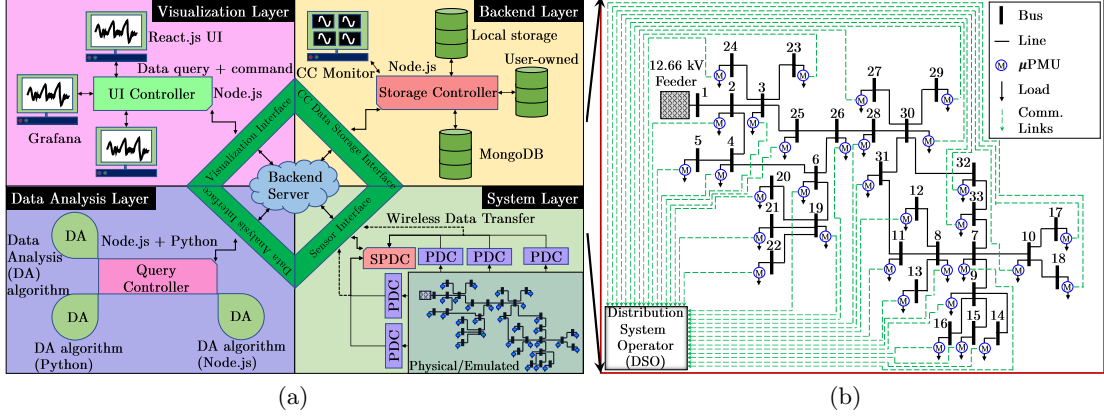


Figure 6.1: Overview of the Proposed End-to-End Cyber-Physical Framework for Smart Grid Control and Monitoring, (a) generic architecture showing interconnected System, Backend, Data Analysis and Visualization layers, (b) deployment of the proposed framework on IEEE-33 bus distribution network

PMUs, power flow meters, and various other sensors with specific functionalities such as breaker status measurement etc. While the traditional power flow measurement can be directly relayed to the control center (CC), PMUs require data aggregation at different points to efficiently manage the volume of data. PMUs utilize user datagram protocol (UDP) datagrams that are transmitted to PDCs. We develop multiple virtual PDCs, each aggregating distinct set of PMU measurements from the grid via Ethernet. Each PDC decodes the UDP datagrams from the PMUs under its supervision, and takes care of time alignment and missing or lagging measurements. Our data collection algorithm employs circular-buffers to store aggregated PMU data according to their timestamps. Transmission delays and missing measurements are taken care of using buffer and timer combinations. All the PDCs send their respective measurements, via wireless communication, to a virtual super PDC (SPDC) which aggregates them and sends to the CC via backend server. We assign unique identification numbers (UIDs) to all the PDCs and SPDC for authentication purpose. If backend server receives any data packet with unknown UIDs, the packet is dropped. A *sensor interface*, for bi-directional communication between SPDC and backend server, enables CC to remotely control the system, as shown in Fig. 6.1. Note that our framework is agnostic to the type of measurement devices as long as the data is received with valid UIDs. Hence, along with PMU measurements from SPDC, distributed power measurements and breaker

status, from respective sensors, can also be relayed to the backend server.

6.3.2 Backend and Data Analysis Layer

The backend server, maintained by the CC, is responsible for integrating the system with other components of our framework. The received measurements from SPDC are stored in a database. The *CC data storage interface* is an application programming interface (API), with abstraction layer, to support different database (DB) architectures. Multiple different DBs can be accessed using common function call from CC data storage interface, while DB specific functionalities and queries are implemented in *Storage Controller* block. Such a generic framework provides scalability in terms of supported DB architectures. Stored data can be asynchronously fetched for real-time monitoring and further data-driven analysis of the grid. However, resource intensive data analytics and learning algorithms can throttle the backend server if run directly on the server workstation. Therefore, we implement a separate *Data Analysis (DA) Layer* with a query based *DA interface* which routes the queries to the respective DB via CC data storage interface. Note that, several independent DA algorithm can be run in parallel with our architecture. If the queries from multiple DA algorithms are different in terms of structure and the data they request, proper routing of the same is handled in *Query Controller* block. The results of these analyzes are published on user web applications utilizing a visualization interface as shown in Fig. 6.1. Upon receiving the data transfer request from the DA layer, for result publication, backend server routes it to a UI controller in the visualization layer. For continuous monitoring of the system, backend can be instructed to automatically send data from the storage to the UIs at certain intervals configured by the user. We encapsulate all data and commands in JSON format appended with unique event names denoting the types of requests being made. Asynchronous event-based Socket.IO protocol is used to decode the event names and execute the intended operations. We use certain pre-registered event names for common functionalities like *start* and *stop* to control data flow, however, customized events are also supported to enhance scalability. For example, an user or CC can send custom commands to SPDC by encapsulating it in a *route* event with SPDC's UID, so that backend routes it to SPDC without trying to decode the customized command which can only be decoded by SPDC. Our backend server is fully scalable and can be used to establish communication from any point to another, thus providing a distributed yet cohesive framework for smart grid monitoring.

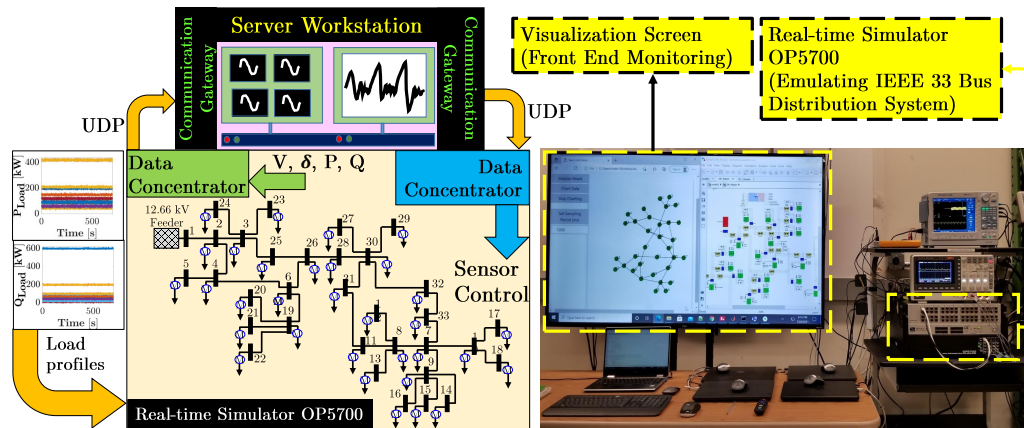


Figure 6.2: Lab prototype showing the software-hardware co-simulation workflow

6.3.3 Visualization Layer

The visualization layer is composed of UIs attached to the rest of the layers via a *visualization interface*. Similar to CC data storage interface, it is an API which maintains bi-directional communication with the backend server. Incoming and outgoing data are processed in an UI controller responsible for restructuring the data based on the UI architecture. For example, time series data visualization on Grafana requires the data to be structured in a certain format. Similarly, customized UIs may require their respective data structures. All of these tasks are handled by the UI controller. Users can send control commands to modify functionalities of other components, for example, stopping data transfer from a certain PDC, if any intruder is suspected. Apart from time series data, we support real-time graphical model rendering. The information about the nodes and the edges of a graph, along with other necessary details, are encapsulated in JSON format which is decoded by the UI controller and rendered on the web application. Therefore, the real-time grid network can be visualized on the UI enhancing the situational awareness of the entire grid. In the subsequent sections, we describe several case studies for a DSO utilizing our architecture to estimate grid topologies from PMU data and render the power networks on UI.

6.4 Case-study Demonstration

6.4.1 Architecture Specifications

Our demonstration entails topology estimation of modified IEEE-33 bus distribution network, shown in Fig. 6.1(b). The power network is emulated inside OP5700 real-time simulator box, manufactured by OPAL-RT. The real-time simulation is conducted using eMEGASIM-based EMT-simulation platform of RTLAB with simulation step size of $50\mu s$. Real-time load profiles (both active and reactive power) for 32 loads in the network are fed into the simulator box as input to 32 three-phase dynamic loads. PMUs are also emulated that measure the three-phase instantaneous bus voltages, load currents and compute the bus voltage phasor magnitude, phase angle, active and reactive power consumed by each load. Measurement vector is created by accumulating the measurements and the packaged data is sent via communication channels to backend server, serving as SPDC, realized in a dedicated workstation. PMU data are sent using UDP to SPDC and it relays the same to backend server via Socket.IO at a rate of 0.1 and 1 sec respectively. Data aggregation algorithm of SPDC are currently developed using Node.js v14.16 with Python 3.8 support. The server workstation contains MongoDB v4.4 based data storage with data structure schema enforced by Mongoose v5.11. For the UIs, we use React.js v17.0 to develop our web application, and D3 library v6.2 to render grid network. All communications between each component of our architecture are implemented using Socket.IO v3.0 server-client protocol. We implement network reconstruction algorithm from PMU data, described in the next section, using Python 3.8. The laboratory prototype of the demonstration is shown in Fig. 6.2. The structural data-flow involving the components described in Section 6.3 is as follows: (i) The framework is initialized by a user command, sent from the UI, which tethers the emulated PMUs and MongoDB to the backend. (ii) The PMU data are continuously received, restructured, and aggregated by the backend and are stored in MongoDB. (iii) The user initiates topology estimation algorithm from the UI and the backend starts feeding the data to the algorithm using standard MongoDB queries utilizing the storage controller. (iii) Upon receiving required amount of PMU data, the algorithm runs in parallel and returns the estimated topology to the backend once completed. (iv) Backend stores the topology information to MongoDB and relays the same to the UI controller which processes the information and shows the topology on the UI.

Algorithm 10: Learning Grid Topology [157]

```
Function getTopology ( $[\mathbf{V} \ \boldsymbol{\theta}]^\top \in \mathbb{R}^{2N \times S}, \tau \in \mathbb{R}^+, \mathcal{V}$ ):  
   $\mathcal{E} \leftarrow \emptyset$ ;  $\boldsymbol{\Sigma}_{(\mathcal{V}, \boldsymbol{\theta})} \in \mathbb{R}^{2N \times 2N} \leftarrow \text{covariance}([\mathbf{V} \ \boldsymbol{\theta}]^\top)$ ;  
   $\begin{bmatrix} \mathbf{J}_{\mathbf{V}\mathbf{V}} & \mathbf{J}_{\mathbf{V}\boldsymbol{\theta}} \\ \mathbf{J}_{\boldsymbol{\theta}\mathbf{V}} & \mathbf{J}_{\boldsymbol{\theta}\boldsymbol{\theta}} \end{bmatrix} \leftarrow \text{inverse}(\boldsymbol{\Sigma}_{(\mathcal{V}, \boldsymbol{\theta})})$ ;  
  foreach  $i, j \in \mathcal{V}$  do  
    if  $\mathbf{J}_{\mathbf{V}\mathbf{V}}(i, j) + \mathbf{J}_{\boldsymbol{\theta}\boldsymbol{\theta}}(i, j) < -\tau$  then  
       $\mathcal{E} \leftarrow \mathcal{E} \cup \{\{i, j\}, \{j, i\}\}$ ;  
    end  
  end  
  return  $\mathcal{E}$ ;
```

6.4.2 Topology Estimation of modified IEEE-33 bus system

The topology estimation algorithm [157] utilizes voltage magnitude and phase angle measurements, received from the PMUs installed at the buses in the network. The topology is estimated with no a priori information on the branches of the network. Let the grid structure be represented by an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the set of buses excluding the slack bus (Bus 1) and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of operational branches in the network. This network is to be determined from S number of time-series samples from PMUs installed at the buses in $\mathcal{V}$. Let $|\mathcal{V}| = N$ and $[p_i, q_i]^\top \in \mathbb{R}^2$ denote the fluctuation in non-reference nodal active and reactive power injections around the mean values. We assume that $[p_i, q_i]^\top$ is uncorrelated from $[p_j, q_j]^\top$ if $j \neq i$ and $[p_i, q_i]^\top \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_i)$ for all $i, j \in \mathcal{V}$, where $\boldsymbol{\Sigma}_i$ is non-zero covariance matrix. We take uncorrelated p_i, q_i with variance of 10^{-4} . Algorithm 10 is used to estimate $\mathcal{G}$ with an empirically tuned non-negative constant τ . [157] is referred for specific details and the proofs of the algorithm which we omit here to conserve space.

6.4.3 Demonstration-1

In this demonstration, a DSO (shown in Fig. 6.3) has the access of all 32 PMUs in the IEEE-33 bus network. The proposed CP architecture is utilized to find the efficacy of the topology estimation algorithm. It uses various numbers of time-series samples of measured voltage phasor magnitude and angle from the complete sets of PMU (i.e. 32 PMUs). Measurement noise (within $\pm 1\%$ of full scale measurement) of sensors are also included to showcase the accuracy of the estimation. As a metric for efficiency, error in estimation is defined as the ratio of the sum of number of false lines identified and true lines missed, to the total number of lines in the network. Fig. 6.4 shows the error plot

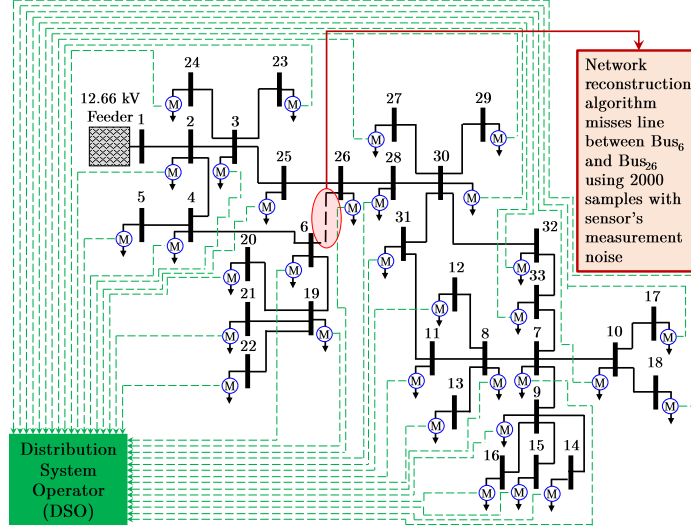


Figure 6.3: Demonstration-1: DSO reconstructs grid topology, in real time, using comprehensive set of PMU data utilizing proposed cyber-physical framework

of the topology estimation algorithm used in the architecture w.r.t various number of samples with and without sensor noises. It is observed that measurement noise degrades the performance of the algorithm. As a part of demonstration, two experimental results are showcased here. As shown in Fig. 6.3, with comprehensive sets of 2000 samples of phasor measurements without noise from 32 PMUs, the DSO estimates the topology with absolute accuracy using the estimation algorithm. However, when measurements are corrupted with sensor noises, the DSO misses line connecting bus 6 and 26 as shown in Fig. 6.3. The efficacy of the algorithm in this case-study with 2000 samples are also shown in Fig. 6.4.

6.4.4 Demonstration-2

In this demonstration, a DSO (shown in Fig. 6.5) has partial access to PMUs in the IEEE-33 bus network. Due to malfunction, PMUs at bus $\{4, 5, 6\}$ and $\{19, \dots, 22\}$ are not sending measurements to the DSO. The proposed CP architecture is utilized by the DSO to estimate the topology using the topology estimation algorithm with 2000 time-series samples of measured voltage phasor magnitude and angle from the partial sets of PMU. The estimated topology is shown in Fig. 6.5. It can be observed that the algorithm finds the network with all visible PMUs. However, as PMUs at buses $\{4, 5, 6\}$

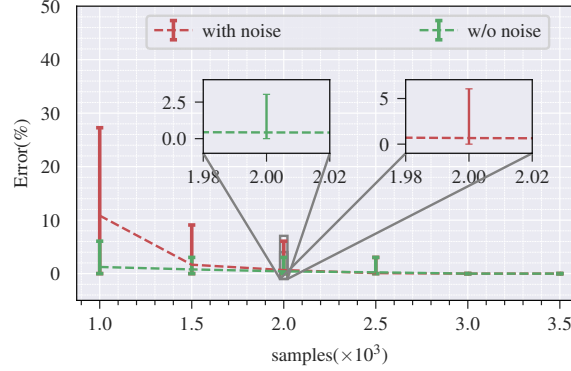


Figure 6.4: Topology estimation error plot for 1000 runs with various number of samples of phasor measurements from 32 PMUs in IEEE 33 bus network used for Demonstration-1. Dashed line indicates the mean error rate.

and $\{19, \dots, 22\}$ are not visible, the algorithm finds a spurious link between bus 2 and 26. This spurious link suggests the DSO that there is possibly a loopy network between bus 2 and 26. In fact, network connecting buses $\{4, 5, 6\}$ and $\{19, \dots, 22\}$ is the actual hidden loopy network in the original system.

6.4.5 Demonstration-3

In this demonstration, two DSOs (as shown in Fig. 6.6) have partial access to PMUs in the IEEE-33 bus network. Due to locational advantage, DSO-1 has access to PMUs at buses $\{2, \dots, 6\}$ and $\{19, \dots, 26\}$. DSO-2 has access to PMUs at buses $\{7, \dots, 18\}$ and $\{27, \dots, 33\}$. The tie-line between bus 26 and 28 is an additional information that both DSOs know. The proposed architecture is utilized by both the DSOs to find the efficacy of the algorithm using various numbers of time-series samples of measured voltage phasor magnitude and angle, to estimate respective topologies (networks inside blue box for DSO-1 and green box for DSO-2). Sensor noises are also included to showcase the accuracy of the estimation. Fig. 6.7 shows the error plot of the same w.r.t various number of samples for this particular case. One experimental result is showcased in Fig. 6.6, with partial sets of 2000 samples from PMUs at buses $\{2, \dots, 6\}$ and $\{19, \dots, 26\}$ for DSO-1 and PMU measurements at buses $\{7, \dots, 18\}$ and $\{27, \dots, 33\}$ for DSO-2. Both estimate their respective topologies with absolute accuracy using the estimation algorithm.

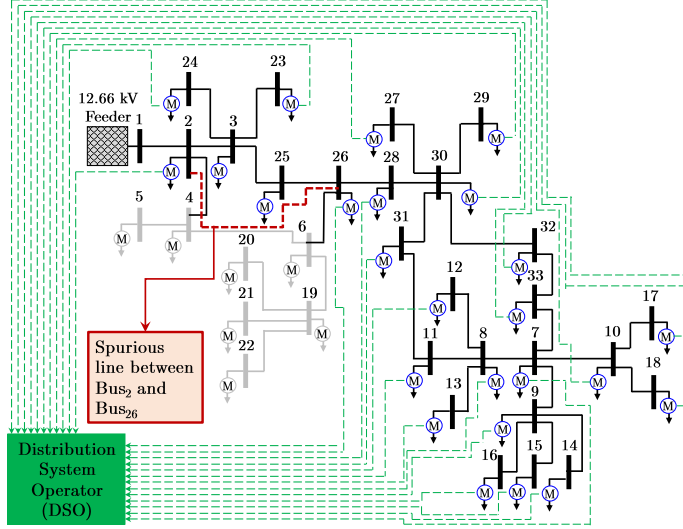


Figure 6.5: Demonstration-2: PMUs at buses 4 – 6 and 19 – 22 are malfunctioning. DSO reconstructs the grid topology using rest of the available PMU data.

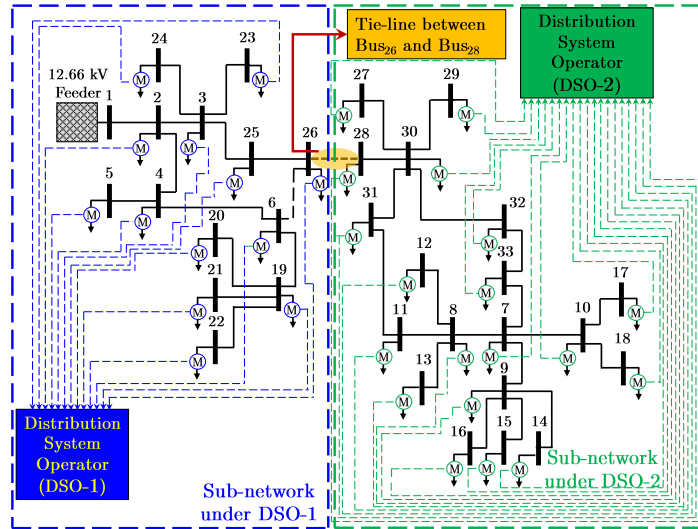


Figure 6.6: Demonstration-3: DSO-1 and 2 utilize PMU data from Sub-network-1 and 2 respectively and reconstruct the corresponding topologies, in real time.

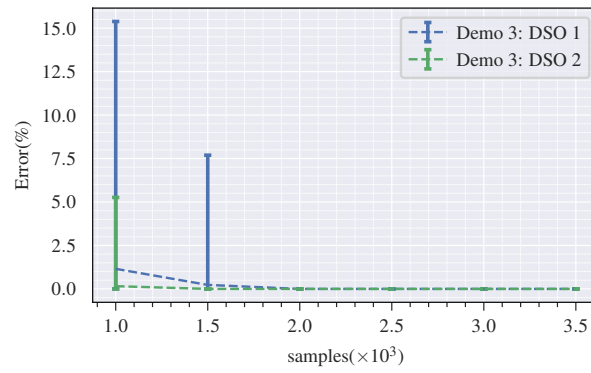


Figure 6.7: Topology estimation error plot for 1000 runs with various number of phasor measurement samples from PMUs deployed in their respective sub-networks used for Demonstration-3. Dashed line indicates the mean error rate.

Chapter 7

Conclusions and Future Research

7.1 Summary of Completed Works

This thesis makes several contributions to developing stochastic control frameworks to mitigate renewable production uncertainty, and thereby improving the reliability of the power distribution systems. Further, the thesis delves into situational awareness and cyber-security of distribution systems to improve resiliency.

In Chapter 2 a novel electricity market mechanism for an IMMIG containing multiple MGs, equipped with DER and critical infrastructure loads, and a DNO with centralized battery storage, is developed with a focus on controlling battery power output, via contracts made in the market in real-time, to meet the load demands of the MGs at the requested time with *almost sure guarantee*. The proposed algorithms mitigate uncertainties of meeting the load demands of the MGs, due to intermittent RESs generations, at the requested time, with *almost sure guarantee* which is crucial for critical infrastructures. We derive battery power capability control policy for IDMMG with no power sharing among the facilities and compare the results with IMMIG. The simulation study shows that IMMIG enables DNO to reduce the power provided by the battery than IDMMG, while still maintaining the guarantee of meeting the load demands of each MG.

Chapter 3 develops a battery energy capacity design methodology providing probabilistic guarantees of minimizing generation and load curtailment under stochastic RESs. In particular, the proposed BESs capacity design strategy ensures that the BESs do not reach their maximum capacity or get completely depleted throughout a chosen time horizon, with a pre-specified high probability. Such an operation guarantees that

if RES generation is more than the demand, the BES can be charged and when demand exceeds RES generation, the BESs can be utilized to supply loads, thus reducing both generation and load curtailments. We further extend the methodology to a two MG system, where along with the required BES capacity, we provide a power transfer policy between the MGs that reduces the BESs requirement while ensuring the BESs remain within their operating range with high probability. We analytically show that BES requirement is minimized if at any time, the MG having higher available energy in its BESs, transfers maximum power that can flow via the power line, to other. Further, as the power line capacity is increased the required BES capacity can be decreased up to a certain factor. We further provide a theoretical analysis of how the flow capacity of the transmission line, connecting the MGs, affect the required amount of BESs. We analytically establish that for a large operation horizon it is possible to reduce the transmission line capacities while maintaining the least amount of BESs, thus reducing the investment and operation cost of both the transmission lines and BESs in unison.

In Chapter 4, we propose a methodology to detect power grid topology changes, along with change time instant estimation and localization. We utilize the voltage magnitude and angle measurements to detect changes in the grid topology. Our proposed strategy is built upon the expectation-maximization algorithm and does not require any historical change data or prior network information to detect changes. We further propose a kernel density estimator-based algorithm to estimate the change instant from the timestamps of the phasor measurements. To localize the detected changes, we utilize the covariance matrix of the phasor samples, estimated from our change detection algorithm, and integrate the same with the estimated change instants to learn the underlying grid topologies corresponding to each change. We instantiate our proposed strategy on a modified IEEE 33-bus, 226-bus, and 1262-bus network and perform real-time hardware-in-the-loop experiments on RTS. Our comprehensive experimental validation demonstrates the efficacy and scalability of our proposed methodology.

Chapter 5 develops cyber-defense strategies toward making power distribution networks resilient to cyber-attacks. In the first part of the chapter, a reinforcement learning-based solution to assess vulnerability of RT electricity market under cyber-topology attack is proposed. We show that an attacker can surreptitiously modify line breaker status measurements to manipulate the LMP at different buses in the network. Our proposed algorithm identifies the critical breakers those, if attacked, can have a huge impact on real-time market operation, enabling the attacker to harness unethical monetary profits. We instantiate our proposed methodology in IEEE-39 bus system to identify

vulnerable breakers in the system, and show that protecting those breakers can minimize the impact of cyber-topology attack on RT market operation. In the second part, we propose a cyber-attack resilient state feedback controller based load frequency control for compensating the frequency deviation of MG. In particular, we design an unique defense mechanism-based observer that can, not only estimate the MG states but also detect the unknown and undesired disturbances such as cyber attacks. A CHIL-based demonstration of the proposed method is conducted using a RT-simulator and a server workstation. It is observed here that the proposed defense mechanism-based observer design with conventional state-feedback control of load frequency control for MG system is resilient against cyber-attacks both in measurements as well as the control signals.

In Chapter 6, we develop a cyber-physical testbed infrastructure, for smart grid control and monitoring in real-time. The framework is developed in laboratory setup. It consists of system, data analysis, backend and visualization layer. For demonstration, a real-time simulation of modified IEEE-33 bus distribution network is done with the control and monitoring layer, realized in server workstation. For validation, a topology estimation algorithm is executed in the workstation communicating with the test system and three case-studies are showcased. These validations form the basis for the authors to further extend and include various other control and monitoring functionalities in the framework. For future work, such functionalities will be included in order to shape the framework to be more generic and useful for real operation under smart grid paradigm.

7.2 Future Works

After delineating the contributions in developing a resilient and situationally aware multi-microgrid framework, this thesis also highlights a few eminent open research questions that must be addressed to enable the widespread adoption of stochastic control and optimization strategies for microgrids in future electric grid. A summary of open research topics that warrant further investigation is provided below:

- A relevant direction of future study is to develop a distributed control architecture for multi-microgrid systems to achieve probabilistic guarantees of demand-supply balance, minimizing over and under-production. A distributed stochastic control framework enjoys the advantages of improved reliability and security due to less dependency on a single-point-of-contact. Furthermore, distributed nature of the

framework has the potential to reduce the energy storage requirement by utilizing the diversity of RESs generation. Therefore, it is crucial to explore such methodologies to further improve the system reliability.

- To improve the economic efficiency of a multi-microgrid system, it is essential to develop an optimization framework to minimize operational cost of the system considering the physical constraints of the battery resources (such as capacity and ramp rates constraints etc.), along with power transmission cost, cost of power transaction from the grid, battery resource, and RESs operational cost. Such analysis reveals the economic utility of the stochastic control frameworks. In this context, the trade-offs between the technical and operational advantages and economic advantages must be assessed for a comprehensive system design guideline. We plan to devise such techno-economic analysis strategy in our future research works.
- In the context of battery capacity design, an important future direction is to extend the methodology to multi-microgrids connected with each other. Here, the dependency of the underlying topology on the required BESs capacity and resource pooling decisions are to be analyzed. We plan to devise a stochastic control policy for sharing power, in a system with multiple MGs connected to each other, with an objective of minimizing the BESs capacity as part of our future work. We further plan to provide a topology design guideline to minimize the resource requirement while providing probabilistic guarantees of demand-supply balance under renewable uncertainties.
- For situational awareness of power distribution networks, a relevant future direction is to explore the effect of various types of cyber-attacks on both technical and economical operation of the system. Although FDI attacks are most common in this regime, in recent years, there has been an influx of other types of attacks such as denial-of-service, jamming, and replay attacks etc. These attacks make a part of the network unobservable to the system operator. Hence, conventional measurement-based cyber-defense mechanisms must be augmented with strategies capable of detecting the attacks with partial measurements.

In conclusion, there are many challenging open questions regarding the design and implementation of resilient multi-microgrids frameworks bolstering the situational awareness of the same. The author hopes that the work conducted in this thesis may help to

answer some of these questions and serve as a base for a large variety of challenging future research directions.

References

- [1] Climate Central. Weather-related power outages rising. <https://www.climatecentral.org/climate-matters/weather-related-power-outages-rising>. [Online; accessed 15-June-2024].
- [2] NOAA. 2023: A historic year of u.s. billion-dollar weather and climate disasters. <https://www.climate.gov/news-features/blogs/beyond-data/2023-historic-year-us-billion-dollar-weather-and-climate-disasters>. [Online; accessed 15-June-2024].
- [3] EIA. Today in energy. <https://www.eia.gov/todayinenergy/detail.php?id=60822>. [Online; accessed 15-June-2024].
- [4] DoE. Electric emergency incidents and disturbance report. <https://www.oe.net1.doe.gov/oe417.aspx>. [Online; accessed 15-June-2024].
- [5] Nate Blair, Chad Augustine, Wesley Cole, Paul Denholm, Will Frazier, Madeline Geocaris, Jennie Jorgenson, Kevin McCabe, Kara Podkaminer, Ashreeta Prasanna, et al. Storage futures study: Key learnings for the coming decades. Technical report, National Renewable Energy Lab.(NREL), Golden, CO (United States), 2022.
- [6] Dharmesh Faquir, Nestoras Chouliaras, Vlachou Sofia, Kalopoulou Olga, and Leandros Maglaras. Cybersecurity in smart grids, challenges and solutions. *AIMS Electronics and Electrical Engineering*, 5(1):24–37, 2021.
- [7] IRENA. Global energy transformation: A roadmap to 2050, 2019. [Accessed 31 August 2020].

- [8] Yuqing Yang, Stephen Bremner, Chris Menictas, and Merlinde Kay. Battery energy storage system size determination in renewable energy systems: A review. *Renewable and Sustainable Energy Reviews*, 91:109–125, 2018.
- [9] Georgios Tsaousoglou, Juan S Giraldo, and Nikolaos G Paterakis. Market mechanisms for local electricity markets: A review of models, solution concepts and algorithmic techniques. *Renewable and Sustainable Energy Reviews*, 156:111890, 2022.
- [10] Jing Hu, Robert Harmsen, Wina Crijns-Graus, Ernst Worrell, and Machteld van den Broek. Identifying barriers to large-scale integration of variable renewable electricity into the electricity market: A literature review of market design. *Renewable and sustainable energy reviews*, 81:2181–2195, 2018.
- [11] Erik Ela, Michael Milligan, Aaron Bloom, Audun Botterud, Aaron Townsend, and Todd Levin. Evolution of wholesale electricity market design with increasing levels of renewable generation. *NREL Report*, 2014.
- [12] Hualei Zou, Shiwen Mao, Yu Wang, Fanghua Zhang, Xin Chen, and Long Cheng. A survey of energy management in interconnected multi-microgrids. *IEEE Access*, 7:72158–72169, 2019.
- [13] Yongxi Zhang, Zhao Yang Dong, Fengji Luo, Yu Zheng, Ke Meng, and Kit Po Wong. Optimal allocation of battery energy storage systems in distribution networks with high wind power penetration. *IET Renewable Power Generation*, 10(8):1105–1113, 2016.
- [14] Daniel Bienstock, Michael Chertkov, and Sean Harnett. Chance-constrained optimal power flow: Risk-aware network control under uncertainty. *Siam Review*, 56(3):461–495, 2014.
- [15] Lun Yang, Yinliang Xu, Hongbin Sun, and Wenchuan Wu. Tractable convex approximations for distributionally robust joint chance-constrained optimal power flow under uncertainty. *IEEE Transactions on Power Systems*, 37(3):1927–1941, 2021.
- [16] Yue Yang, Wenchuan Wu, Bin Wang, and Mingjie Li. Chance-constrained economic dispatch considering curtailment strategy of renewable energy. *IEEE Transactions on Power Systems*, 36(6):5792–5802, 2021.

- [17] Bala Kameshwar Poolla, Ashish R Hota, Saverio Bolognani, Duncan S Callaway, and Ashish Cherukuri. Wasserstein distributionally robust look-ahead economic dispatch. *IEEE Transactions on Power Systems*, 36(3):2010–2022, 2020.
- [18] Hang Shuai, Jiakun Fang, Xiaomeng Ai, Yufei Tang, Jinyu Wen, and Haibo He. Stochastic optimization of economic dispatch for microgrid based on approximate dynamic programming. *IEEE Transactions on Smart Grid*, 10(3):2440–2452, 2018.
- [19] Runzhao Lu, Tao Ding, Boyu Qin, Jin Ma, Xin Fang, and Zhaoyang Dong. Multi-stage stochastic programming to joint economic dispatch for energy and reserve with uncertain renewable energy. *IEEE Transactions on Sustainable Energy*, 11(3):1140–1151, 2019.
- [20] Hossein Farzin, Mahmud Fotuhi-Firuzabad, and Moein Moeini-Agtaie. Enhancing power system resilience through hierarchical outage management in multi-microgrids. *IEEE Transactions on Smart Grid*, 7(6):2869–2879, 2016.
- [21] Seyed Ali Arefifar, Martin Ordonez, and Yasser Abdel-Rady I Mohamed. Energy management in multi-microgrid systems—development and assessment. *IEEE Transactions on Power Systems*, 32(2):910–922, 2016.
- [22] Yun Liu, Hoay Beng Gooi, Yuanzheng Li, Huanhai Xin, and Jian Ye. A secure distributed transactive energy management scheme for multiple interconnected microgrids considering misbehaviors. *IEEE Transactions on Smart Grid*, 10(6):5975–5986, 2019.
- [23] Tianyi Li and Min Dong. Residential energy storage management with bidirectional energy control. *IEEE Transactions on Smart Grid*, 10(4):3596–3611, 2018.
- [24] Juan S Giraldo, Mohammed Ahsan Adib Murad, Taulant Kërçi, and Federico Milano. Impact of decentralized microgrids optimal energy management on power system dynamics. *Electric Power Systems Research*, 212:108337, 2022.
- [25] Hossein Khorramdel, Jamshid Aghaei, Benyamin Khorramdel, and Pierluigi Siano. Optimal battery sizing in microgrids using probabilistic unit commitment. *IEEE Transactions on Industrial Informatics*, 12(2):834–843, 2015.

- [26] Zhengmao Li, Yan Xu, Xue Feng, and Qiuwei Wu. Optimal stochastic deployment of heterogeneous energy storage in a residential multienergy microgrid with demand-side management. *IEEE Transactions on Industrial Informatics*, 17(2):991–1004, 2020.
- [27] Yue Yang, Wenchuan Wu, Bin Wang, and Mingjie Li. Analytical reformulation for stochastic unit commitment considering wind power uncertainty with gaussian mixture model. *IEEE Transactions on Power Systems*, 35(4):2769–2782, 2019.
- [28] Seyed Mehdi Hakimi, Arezoo Hasankhani, Miadreza Shafie-khah, and João PS Catalão. Stochastic planning of a multi-microgrid considering integration of renewable energy resources and real-time electricity market. *Applied Energy*, 298:117215, 2021.
- [29] Mohammadreza Daneshvar, Behnam Mohammadi-Ivatloo, Kazem Zare, and Sormayeh Asadi. Two-stage robust stochastic model scheduling for transactive energy based renewable microgrids. *IEEE Transactions on Industrial Informatics*, 16(11):6857–6867, 2020.
- [30] Zhao Yang, Junjie Hu, Xin Ai, Jiechen Wu, and Guangya Yang. Transactive energy supported economic operation for multi-energy complementary microgrids. *IEEE Transactions on Smart Grid*, 12(1):4–17, 2020.
- [31] Han-I Su and Abbas El Gamal. Modeling and analysis of the role of energy storage for renewable integration: Power balancing. *IEEE Transactions on Power Systems*, 28(4):4109–4117, 2013.
- [32] Bo Zhao, Xiangjin Wang, Da Lin, Madison M Calvin, Julia C Morgan, Ruwen Qin, and Caisheng Wang. Energy management of multiple microgrids based on a system of systems architecture. *IEEE Transactions on Power Systems*, 33(6):6410–6421, 2018.
- [33] Yinliang Xu, Hongbin Sun, and Wei Gu. A novel discounted min-consensus algorithm for optimal electrical power trading in grid-connected dc microgrids. *IEEE Transactions on Industrial Electronics*, 66(11):8474–8484, 2019.

- [34] Liang Che, Xiaping Zhang, Mohammad Shahidehpour, Ahmed Alabdulwahab, and Abdullah Abusorrah. Optimal interconnection planning of community microgrids with renewable energy sources. *IEEE Transactions on Smart Grid*, 8(3):1054–1063, 2015.
- [35] Wei Liu, Wei Gu, Jianhui Wang, Wenwu Yu, and Xinze Xi. Game theoretic non-cooperative distributed coordination control for multi-microgrids. *IEEE Transactions on Smart Grid*, 9(6):6986–6997, 2018.
- [36] Sangdon Park, Joohyung Lee, Sohee Bae, Ganguk Hwang, and Jun Kyun Choi. Contribution-based energy-trading mechanism in microgrids for future smart grid: A game theoretic approach. *IEEE Transactions on Industrial Electronics*, 63(7):4255–4265, 2016.
- [37] Pulkit Nahata, Alessio La Bella, Riccardo Scattolini, and Giancarlo Ferrari-Trecate. Hierarchical control in islanded dc microgrids with flexible structures. *IEEE transactions on control systems technology*, 29(6):2379–2392, 2020.
- [38] Christian A Hans, Pantelis Sopasakis, Jörg Raisch, Carsten Reincke-Collon, and Panagiotis Patrinos. Risk-averse model predictive operation control of islanded microgrids. *IEEE transactions on control systems technology*, 28(6):2136–2151, 2019.
- [39] Pietro Ferraro, Emanuele Crisostomi, Robert Shorten, and Federico Milano. Stochastic frequency control of grid-connected microgrids. *IEEE Transactions on Power Systems*, 33(5):5704–5713, 2018.
- [40] Xiaodong Yang, Haibo He, Youbing Zhang, Yu Chen, and Guoqing Weng. Interactive energy management for enhancing power balances in multi-microgrids. *IEEE Transactions on Smart Grid*, 10(6):6055–6069, 2019.
- [41] Yang Han, Ke Zhang, Hong Li, Ernane Antônio Alves Coelho, and Josep M Guerrero. Mas-based distributed coordinated control and optimization in microgrid and microgrid clusters: A comprehensive overview. *IEEE Transactions on Power Electronics*, 33(8):6488–6508, 2017.
- [42] Shichang Cui, Yan-Wu Wang, Yang Shi, and Jiang-Wen Xiao. A new and fair peer-to-peer energy sharing framework for energy buildings. *IEEE Transactions on Smart Grid*, 11(5):3817–3826, 2020.

- [43] Mohammad B Shadmand and Robert S Balog. Multi-objective optimization and design of photovoltaic-wind hybrid system for community smart dc microgrid. *IEEE Transactions on Smart Grid*, 5(5):2635–2643, 2014.
- [44] Lanre Olatomiwa, Saad Mekhilef, Mahmoud S Ismail, and Mahmoud Moghavvemi. Energy management strategies in hybrid renewable energy systems: A review. *Renewable and Sustainable Energy Reviews*, 62:821–835, 2016.
- [45] Jon Olauson and Mikael Bergkvist. Correlation between wind power generation in the european countries. *Energy*, 114:663–670, 2016.
- [46] Guorui Ren, Jie Wan, Jinfu Liu, and Daren Yu. Spatial and temporal correlation analysis of wind power between different provinces in china. *Energy*, 191:116514, 2020.
- [47] Crispin Gardiner. *Stochastic methods*, volume 4. Springer Berlin, 2009.
- [48] Fischer Black and Myron Scholes. The pricing of options and corporate liabilities. *Journal of political economy*, 81(3):637–654, 1973.
- [49] BPA Balancing Authority. Wind generation and total load in the bpa balancing authority, 2020. [Accessed 1 August 2020].
- [50] Bernhard Ernst, Y-H Wan, and Brendan Kirby. Short-term power fluctuation of wind turbines: Analyzing data from the german 250-mw measurement program from the ancillary services viewpoint. Technical report, National Renewable Energy Lab., Golden, CO (US), 1999.
- [51] Mary Natrella et al. Nist/sematech e-handbook of statistical methods. *Nist/Sematech*, 49, 2010.
- [52] Myles Hollander, Douglas A Wolfe, and Eric Chicken. *Nonparametric statistical methods*, volume 751. John Wiley & Sons, 2013.
- [53] Mukul C Chandorkar, Deepakraj M Divan, and Rambabu Adapa. Control of parallel connected inverters in standalone ac supply systems. *IEEE transactions on industry applications*, 29(1):136–143, 1993.
- [54] MA Hannan, M Faisal, Pin Jern Ker, RA Begum, ZY Dong, and C Zhang. Review of optimal methods and algorithms for sizing energy storage systems to achieve

- decarbonization in microgrid applications. *Renewable and Sustainable Energy Reviews*, 131:110022, 2020.
- [55] Morteza Zolfaghari, Navid Ghaffarzadeh, and Ali Jahanbani Ardakani. Optimal sizing of battery energy storage systems in off-grid micro grids using convex optimization. *Journal of Energy Storage*, 23:44–56, 2019.
 - [56] Yang Li, Zhen Yang, Guoqing Li, Dongbo Zhao, and Wei Tian. Optimal scheduling of an isolated microgrid with battery storage considering load and renewable generation uncertainties. *IEEE Transactions on Industrial Electronics*, 66(2):1565–1575, 2018.
 - [57] Shuai X Chen, Hoay Beng Gooi, and MingQiang Wang. Sizing of energy storage for microgrids. *IEEE transactions on smart grid*, 3(1):142–151, 2011.
 - [58] Faeza Hafiz, Anderson Rodrigo de Queiroz, Poria Fajri, and Iqbal Husain. Energy management and optimal storage sizing for a shared community: A multi-stage stochastic programming approach. *Applied energy*, 236:42–54, 2019.
 - [59] Mingtao Ma, Huijun Huang, Xiaoling Song, Feniosky Peña-Mora, Zhe Zhang, and Jie Chen. Optimal sizing and operations of shared energy storage systems in distribution networks: A bi-level programming approach. *Applied Energy*, 307:118170, 2022.
 - [60] Rahmat Khezri, Amin Mahmoudi, and Mohammed H Haque. Impact of optimal sizing of wind turbine and battery energy storage for a grid-connected household with/without an electric vehicle. *IEEE Transactions on Industrial Informatics*, 18(9):5838–5848, 2022.
 - [61] Rui Xie, Wei Wei, Ming-Feng Ge, Qiuwei Wu, and Shengwei Mei. Coordinate sizing of energy storage and transmission line for a remote renewable power plant. *IET Renewable Power Generation*, 16(12):2508–2520, 2022.
 - [62] Mohammad Reza Aghamohammadi and Hajar Abdolahinia. A new approach for optimal sizing of battery energy storage system for primary frequency control of islanded microgrid. *International Journal of Electrical Power & Energy Systems*, 54:325–333, 2014.

- [63] Chee Wei Tan, Tim C Green, and Carlos A Hernandez-Aramburo. A stochastic method for battery sizing with uninterruptible-power and demand shift capabilities in pv (photovoltaic) systems. *Energy*, 35(12):5082–5092, 2010.
- [64] Ye Yang, Hui Li, Andreas Aichhorn, Jianping Zheng, and Michael Greenleaf. Sizing strategy of distributed battery storage system with high penetration of photovoltaic for voltage regulation and peak load shaving. *IEEE Transactions on smart grid*, 5(2):982–991, 2013.
- [65] Mansur Khasanov, Salah Kamel, Claudia Rahmann, Hany M Hasanien, and Ahmed Al-Durra. Optimal distributed generation and battery energy storage units integration in distribution systems considering power generation uncertainty. *IET Generation, Transmission & Distribution*, 15(24):3400–3422, 2021.
- [66] Mahdi Habibi, Arman Oshnoei, Vahid Vahidinasab, and Soroush Oshnoei. Allocation and sizing of energy storage system considering wind uncertainty: An approach based on stochastic scuc. In *2018 Smart Grid Conference (SGC)*, pages 1–6. IEEE, 2018.
- [67] Yongsheng Cao, Demin Li, Yihong Zhang, Qinghua Tang, Amin Khodaei, Hongliang Zhang, and Zhu Han. Optimal energy management for multi-microgrid under a transactive energy framework with distributionally robust optimization. *IEEE Transactions on Smart Grid*, 13(1):599–612, 2021.
- [68] Xinrui Liu, Mingchao Zhang, Xiangpeng Xie, Liang Zhao, and Qiuye Sun. Consensus-based energy management of multi-microgrid: An improved soc-based power coordinated control method. *Applied Mathematics and Computation*, 425:127086, 2022.
- [69] Yijian Wang, Yang Cui, Yang Li, and Yang Xu. Collaborative optimization of multi-microgrids system with shared energy storage based on multi-agent stochastic game and reinforcement learning. *Energy*, page 128182, 2023.
- [70] Ioannis Karatzas and Steven Shreve. *Brownian motion and stochastic calculus*, volume 113. Springer Science & Business Media, 2012.
- [71] Steven E Shreve. *Stochastic calculus for finance II: Continuous-time models*, volume 11. Springer Science & Business Media, 2004.

- [72] Deepjyoti Deka, Vassilis Kekatos, and Guido Cavraro. Learning distribution grid topologies: A tutorial. *IEEE Transactions on Smart Grid*, 2023.
- [73] David E Whitehead, Kevin Owens, Dennis Gammel, and Jess Smith. Ukraine cyber-induced power outage: Analysis and practical mitigation strategies. In *2017 70th Annual Conference for Protective Relay Engineers (CPRE)*, pages 1–8. IEEE, 2017.
- [74] Mark Zeller. Myth or reality—does the aurora vulnerability pose a risk to my generator? In *2011 64th Annual Conference for Protective Relay Engineers*, pages 130–136. IEEE, 2011.
- [75] Jinsub Kim and Lang Tong. On topology attack of a smart grid: Undetectable attacks and countermeasures. *IEEE Journal on Selected Areas in Communications*, 31(7):1294–1305, 2013.
- [76] Jaime De La Ree, Virgilio Centeno, James S Thorp, and Arun G Phadke. Synchronized phasor measurement applications in power systems. *IEEE Transactions on smart grid*, 1(1):20–27, 2010.
- [77] Yih-Fang Huang, Stefan Werner, Jing Huang, Neelabh Kashyap, and Vijay Gupta. State estimation in electric power grids: Meeting new challenges presented by the requirements of the future grid. *IEEE Signal Processing Magazine*, 29(5):33–43, 2012.
- [78] Aditya Ashok, Manimaran Govindarasu, and Venkataramana Ajjarapu. Online detection of stealthy false data injection attacks in power system state estimation. *IEEE Transactions on Smart Grid*, 9(3):1636–1646, 2016.
- [79] Rakesh Bobba, Erich Heine, Himanshu Khurana, and Tim Yardley. Exploring a tiered architecture for naspinet. In *2010 Innovative Smart Grid Technologies (ISGT)*, pages 1–8. IEEE, 2010.
- [80] Xiaoyang Tong, Xingxing Dong, Sohail Manzoor, Yabing Wang, and Zibin Zhao. Robust fault location method for transmission lines using pmus. *IEEE Transactions on Power Delivery*, 2023.
- [81] Yu Christine Chen, Taposh Banerjee, Alejandro D Dominguez-Garcia, and Venugopal V Veeravalli. Quickest line outage detection and identification. *IEEE Transactions on Power Systems*, 31(1):749–758, 2015.

- [82] Mehebab Alam, Shubhrajyoti Kundu, Siddhartha Sankar Thakur, and Sumit Banerjee. Pmu based line outage identification using comparison of current phasor measurement technique. *International Journal of Electrical Power & Energy Systems*, 115:105501, 2020.
- [83] Hamid Mirshekali, Rahman Dashti, Ahmad Keshavarz, Amin J Torabi, and Hamid Reza Shaker. A novel fault location methodology for smart distribution networks. *IEEE Transactions on smart grid*, 12(2):1277–1288, 2020.
- [84] Zahra Soltani, Shanshan Ma, Mojdeh Khorsand, and Vijay Vittal. Simultaneous robust state estimation, topology error processing, and outage detection for unbalanced distribution systems. *IEEE Transactions on Power Systems*, 2022.
- [85] Anandini Gandluru, Shiva Poudel, and Anamika Dubey. Joint estimation of operational topology and outages for unbalanced power distribution systems. *IEEE Transactions on Power Systems*, 35(1):605–617, 2019.
- [86] Ying Zhang, Jianhui Wang, and Mohammad E Khodayar. Graph-based faulted line identification using micro-pmu data in distribution systems. *IEEE Transactions on Smart Grid*, 11(5):3982–3992, 2020.
- [87] Miao He and Junshan Zhang. A dependency graph approach for fault detection and localization towards secure smart grid. *IEEE Transactions on Smart Grid*, 2(2):342–351, 2011.
- [88] Deepjyoti Deka, Saurav Talukdar, Michael Chertkov, and Murti V Salapaka. Graphical models in meshed distribution grids: Topology estimation, change detection & limitations. *IEEE Transactions on Smart Grid*, 11(5):4299–4310, 2020.
- [89] Yuxuan Yuan, Yifei Guo, Kaveh Dehghanpour, Zhaoyu Wang, and Yanchao Wang. Learning-based real-time event identification using rich real pmu data. *IEEE Transactions on Power Systems*, 36(6):5044–5055, 2021.
- [90] David Ofosu Amoateng, Ruifeng Yan, Mehdi Mosadeghy, and Tapan Kumar Saha. Topology detection in power distribution networks: A pmu based deep learning approach. *IEEE Transactions on Power Systems*, 37(4):2771–2782, 2021.
- [91] Tin Rabuzin and Lars Nordström. Data-driven islanding detection using a principal subspace of voltage angle differences. *IEEE Transactions on Smart Grid*, 12(5):4250–4258, 2021.

- [92] Xiaoxue Wang, Yikang Zhao, and Yue Zhou. A data-driven topology and parameter joint estimation method in non-pmu distribution networks. *IEEE Transactions on Power Systems*, 2023.
- [93] Jian Zhao, Liang Li, Zhao Xu, Xiaoyu Wang, Haobo Wang, and Xianjun Shao. Full-scale distribution system topology identification using markov random field. *IEEE Transactions on Smart Grid*, 11(6):4714–4726, 2020.
- [94] Priyank Shah and Xiaowei Zhao. Network identification using μ -pmu and smart meter measurements. *IEEE Transactions on Industrial Informatics*, 18(11):7572–7586, 2022.
- [95] Jiafan Yu, Yang Weng, and Ram Rajagopal. Patopaem: A data-driven parameter and topology joint estimation framework for time-varying system in distribution grids. *IEEE Transactions on Power Systems*, 34(3):1682–1692, 2018.
- [96] Jonathan C Mayo-Maldonado, Jesus E Valdez-Resendiz, Daniel Guillen, Mohini Bariya, Alexandra von Meier, Ernesto A Salas-Esquivel, and Aminy Ostfeld. Data-driven framework to model identification, event detection, and topology change location using d-pmus. *IEEE Transactions on Instrumentation and Measurement*, 69(9):6921–6933, 2020.
- [97] Xuan Liu and Zuyi Li. Local topology attacks in smart grids. *IEEE Transactions on Smart Grid*, 8(6):2617–2626, 2016.
- [98] R James Ranjith Kumar and Biplab Sikdar. Detection of stealthy cyber-physical line disconnection attacks in smart grid. *IEEE Transactions on Smart Grid*, 12(5):4484–4493, 2021.
- [99] Rui Ma, Sagnik Basumallik, and Sara Eftekharnesjad. A pmu-based data-driven approach for classifying power system events considering cyberattacks. *IEEE Systems Journal*, 14(3):3558–3569, 2020.
- [100] Sayari Das and Bijaya Ketan Panigrahi. A pmu-based data-driven approach for enhancing situational awareness in building a resilient power systems. *IEEE Transactions on Industrial Informatics*, 18(7):4773–4784, 2022.

- [101] Junbo Zhao, Gexiang Zhang, Kaushik Das, George N Korres, Nikolaos M Manousakis, Avinash K Sinha, and Zhengyou He. Power system real-time monitoring by using pmu-based robust state estimation method. *IEEE Transactions on Smart Grid*, 7(1):300–309, 2015.
- [102] Allen J Wood, Bruce F Wollenberg, and Gerald B Sheblé. *Power generation, operation, and control*. John Wiley & Sons, 2013.
- [103] Yizheng Liao, Yang Weng, Guangyi Liu, and Ram Rajagopal. Urban mv and lv distribution grid topology estimation via group lasso. *IEEE Transactions on Power Systems*, 34(1):12–27, 2018.
- [104] Christopher M Bishop. Pattern recognition and machine learning. *Springer google schola*, 2:5–43, 2006.
- [105] Ethem Alpaydin. *Introduction to machine learning*. MIT press, 2020.
- [106] Arthur P Dempster, Nan M Laird, and Donald B Rubin. Maximum likelihood from incomplete data via the em algorithm. *Journal of the royal statistical society: series B (methodological)*, 39(1):1–22, 1977.
- [107] Richard A Redner and Homer F Walker. Mixture densities, maximum likelihood and the em algorithm. *SIAM review*, 26(2):195–239, 1984.
- [108] Peter J Rousseeuw. Silhouettes: a graphical aid to the interpretation and validation of cluster analysis. *Journal of computational and applied mathematics*, 20:53–65, 1987.
- [109] Arnab Dey, Soham Chakraborty, and Murti V Salapaka. An end-to-end cyber-physical infrastructure for smart grid control and monitoring. In *2022 IEEE Power & Energy Society Innovative Smart Grid Technologies Conference (ISGT)*, pages 1–5. IEEE, 2022.
- [110] Sarineh Hacopian Dolatabadi, Maedeh Ghorbanian, Pierluigi Siano, and Nikos D. Hatziargyriou. An enhanced iee 33 bus benchmark test system for distribution system studies. *IEEE Transactions on Power Systems*, 36(3):2565–2572, 2021.

- [111] Soham Chakraborty, Govind Saraswat, Nischal Guruwacharya, Arnab Dey, Richard Bryce, and Murti V Salapaka. Evaluation of distributed power apportioning with net load management engine in microgrids using power hardware-in-the-loop simulation. In *2023 North American Power Symposium (NAPS)*, pages 1–6. IEEE, 2023.
- [112] Electric Power Research Institute (EPRI). EPRI Test Circuits. https://sourceforge.net/p/electricdss/code/HEAD/tree/trunk/Distrib/EPRI_TestCircuits/Readme.pdf. [Online; accessed 1-March-2024].
- [113] Jaquelin Cochran, Mackay Miller, Michael Milligan, Erik Ela, Douglas Arent, Aaron Bloom, Matthew Futch, Juha Kiviluoma, Hannele Holtinnen, Antje Orths, et al. Market evolution: Wholesale electricity market design for 21st century power systems. *NREL report*, 2013.
- [114] Haifeng Liu, Leigh Tesfatsion, and AA Chowdhury. Locational marginal pricing basics for restructured wholesale power markets. In *2009 IEEE Power & Energy Society General Meeting*, pages 1–8. IEEE, 2009.
- [115] Qiwei Zhang, Fangxing Li, Qingxin Shi, Kevin Tomsovic, Jinyuan Sun, and Lingyu Ren. Profit-oriented false data injection on electricity market: Reviews, analyses, and insights. *IEEE Transactions on Industrial Informatics*, 17(9):5876–5886, 2020.
- [116] Qiwei Zhang, Fangxing Li, Hantao Cui, Rui Bo, and Lingyu Ren. Market-level defense against fdia and a new lmp-disguising attack strategy in real-time market operations. *IEEE Transactions on Power Systems*, 36(2):1419–1431, 2020.
- [117] Mateo R Mengis and Ali Tajer. Data injection attacks on electricity markets by limited adversaries: Worst-case robustness. *IEEE Transactions on Smart Grid*, 9(6):5710–5720, 2017.
- [118] Thusitha Dayaratne, Carsten Rudolph, Ariel Liebman, Mahsa Salehi, and Shan He. High impact false data injection attack against real-time pricing in smart grids. In *2019 IEEE PES Innovative Smart Grid Technologies Europe (ISGT-Europe)*, pages 1–5. IEEE, 2019.

- [119] Chensheng Liu, Min Zhou, Jing Wu, Chengnian Long, and Deepa Kundur. Financially motivated fdi on sced in real-time electricity markets: Attacks and mitigation. *IEEE Transactions on Smart Grid*, 10(2):1949–1959, 2017.
- [120] Liyan Jia, Jinsub Kim, Robert J Thomas, and Lang Tong. Impact of data quality on real-time locational marginal price. *IEEE Transactions on Power Systems*, 29(2):627–636, 2013.
- [121] Gaoqi Liang, Steven R Weller, Fengji Luo, Junhua Zhao, and Zhao Yang Dong. Generalized fdia-based cyber topology attack with application to the australian electricity market trading mechanism. *IEEE Transactions on Smart Grid*, 9(4):3820–3829, 2017.
- [122] Hanchen Xu, Yuzhang Lin, Xiangyu Zhang, and Fei Wang. Power system parameter attack for financial profits in electricity markets. *IEEE Transactions on Smart Grid*, 11(4):3438–3446, 2020.
- [123] Gaoqi Liang, Steven R Weller, Junhua Zhao, Fengji Luo, and Zhao Yang Dong. A framework for cyber-topology attacks: Line-switching and new attack scenarios. *IEEE Transactions on Smart Grid*, 10(2):1704–1712, 2017.
- [124] Stuart J Russell. *Artificial intelligence a modern approach*. Pearson Education, Inc., 2010.
- [125] Christopher JCH Watkins and Peter Dayan. Q-learning. *Machine learning*, 8(3):279–292, 1992.
- [126] Mangalore A Pai. *Energy function analysis for power system stability*. Springer Science & Business Media, 2012.
- [127] Reza Alayi, Farhad Zishan, Seyed Reza Seyednouri, Ravinder Kumar, Mohammad Hossein Ahmadi, and Mohsen Sharifpur. Optimal load frequency control of island microgrids via a pid controller in the presence of wind turbine and pv. *Sustainability*, 13(19):10728, 2021.
- [128] Ali Bidram and Ali Davoudi. Hierarchical structure of microgrids control system. *IEEE Transactions on Smart Grid*, 3(4):1963–1976, 2012.

- [129] Engang Tian and Chen Peng. Memory-based event-triggering h_∞ load frequency control for power systems under deception attacks. *IEEE Transactions on Cybernetics*, 50(11):4610–4618, 2020.
- [130] KS Xiahou, Yang Liu, and QH Wu. Robust load frequency control of power systems against random time-delay attacks. *IEEE Transactions on Smart Grid*, 12(1):909–911, 2020.
- [131] Alireza Abbaspour, Arman Sargolzaei, Parisa Forouzannezhad, Kang K Yen, and Arif I Sarwat. Resilient control design for load frequency control system under false data injection attacks. *IEEE Transactions on Industrial Electronics*, 67(9):7951–7962, 2019.
- [132] Yuancheng Li, Rong Huang, and Longqiang Ma. False data injection attack and defense method on load frequency control. *IEEE Internet of Things Journal*, 8(4):2910–2919, 2020.
- [133] Abdallah Farraj, Eman Hammad, and Deepa Kundur. A storage-based multiagent regulation framework for smart grid resilience. *IEEE Transactions on Industrial Informatics*, 14(9):3859–3869, 2018.
- [134] Mohammad Reza Khalghani, Jignesh Solanki, Sarika Solanki, Mohammad Hassan Khooban, and Arman Sargolzaei. Resilient frequency control design for microgrids under false data injection. *IEEE Transactions on Industrial Electronics*, 2020.
- [135] Jun Ye and Xiang Yu. Detection and estimation of false data injection attacks for load frequency control systems. *Journal of Modern Power Systems and Clean Energy*, 2021.
- [136] Wenjun Bi, Kaifeng Zhang, Kun Yuan, Ying Wang, Chunyu Chen, and Ke Wang. Observer-based attack detection and mitigation for load frequency control system. In *2019 IEEE Power & Energy Society General Meeting (PESGM)*, pages 1–5. IEEE, 2019.
- [137] Thanh Ngoc Pham, Amanullah Maung Than Oo, and Hieu Trinh. Detecting and isolating false data injection attacks on electric vehicles of smart grids using distributed functional observers. *IET Generation, Transmission & Distribution*, 15(4):762–779, 2021.

- [138] Athira M Mohan, Nader Meskin, and Hasan Mehrjerdi. A comprehensive review of the cyber-attacks and cyber-security on load frequency control of power systems. *Energies*, 13(15):3860, 2020.
- [139] Müller Hou and PC Muller. Design of observers for linear systems with unknown inputs. *IEEE Transactions on automatic control*, 37(6):871–875, 1992.
- [140] BPA. Transmission services. <https://www.bpa.gov/transmission>. [Online; accessed Jan-2022].
- [141] NREL. Solar resource data. <https://www.nrel.gov/gis/data-tools.html>. [Online; accessed Jan-2022].
- [142] Sven Christian Müller, Hanno Georg, James J Nutaro, Edmund Widl, Yi Deng, Peter Palensky, Muhammad U Awais, Moustafa Chenine, Markus Küch, Matthias Stifter, et al. Interfacing power system and ict simulators: Challenges, state-of-the-art, and case studies. *IEEE Transactions on Smart Grid*, 9(1):14–24, 2016.
- [143] Ceeman B Vellaithurai, Saugata S Biswas, and Anurag K Srivastava. Development and application of a real-time test bed for cyber–physical system. *IEEE Systems Journal*, 11(4):2192–2203, 2015.
- [144] Carlos Queiroz, Abdun Mahmood, and Zahir Tari. Scadasim—a framework for building scada simulations. *IEEE Transactions on Smart Grid*, 2(4):589–597, 2011.
- [145] Tamara Becejac, Crystal Eppinger, Aditya Ashok, Urmila Agrawal, and James O’Brien. Prime: a real-time cyber-physical systems testbed: from wide-area monitoring, protection, and control prototyping to operator training and beyond. *IET Cyber-Physical Systems: Theory & Applications*, 5(2):186–195, 2020.
- [146] Hantao Cui, Fangxing Li, and Kevin Tomsovic. Cyber-physical system testbed for power system monitoring and wide-area control verification. *IET Energy Systems Integration*, 2(1):32–39, 2020.
- [147] Heqin Tong, Ming Ni, Lili Zhao, and Manli Li. Flexible hardware-in-the-loop testbed for cyber physical power system simulation. *IET Cyber-Physical Systems: Theory & Applications*, 4(4):374–381, 2019.

- [148] Vivek Kumar Singh, Rishi Sharma, and Manimaran Govindarasu. Testbed-based performance evaluation of attack resilient control for wind farm scada system. In *2020 IEEE Power & Energy Society General Meeting (PESGM)*, pages 1–5. IEEE, 2020.
- [149] Ge Cao, Wei Gu, Chenxiao Gu, Wanxing Sheng, Jing Pan, Rui Li, and Lijing Sun. Real-time cyber- physical system co-simulation testbed for microgrids control. *IET Cyber-Physical Systems: Theory & Applications*, 4(1):38–45, 2019.
- [150] Michael Starke, Bailu Xiao, Guodong Liu, Ben Ollis, Philip Irminger, Dan King, Andrew Herron, and Yaosuo Xue. Architecture and implementation of micro-grid controller. In *2016 IEEE Power & Energy Society Innovative Smart Grid Technologies Conference (ISGT)*, pages 1–5. IEEE, 2016.
- [151] Austin Nelson, Sudipta Chakraborty, Dexin Wang, Pawan Singh, Qiang Cui, Li- uqing Yang, and Siddharth Suryanarayanan. Cyber-physical test platform for microgrids: Combining hardware, hardware-in-the-loop, and network-simulator-in-the-loop. In *2016 IEEE Power and Energy Society General Meeting (PESGM)*, pages 1–5. IEEE, 2016.
- [152] Minglei You, Xinruo Zhang, Gan Zheng, Jing Jiang, and Hongjian Sun. A versatile software defined smart grid testbed: artificial intelligence enhanced real-time co-evaluation of ict systems and power systems. *IEEE Access*, 8:88651–88663, 2020.
- [153] Farah Abdel Khalek, Jad Nassar, Vincent Lefevre, Flovic Gosselin, and Nicolas Gouvry. A smart grid wsn research testbed. In *2020 16th International Conference on Wireless and Mobile Computing, Networking and Communications (WiMob)(50308)*, pages 388–391. IEEE, 2020.
- [154] Marco Pignati, Miroslav Popovic, Sergio Barreto, Rachid Cherkaoui, German Dario Flores, Jean-Yves Le Boudec, Maaz Mohiuddin, Mario Paolone, Paolo Romano, Styliani Sarri, et al. Real-time state estimation of the epfl-campus medium-voltage grid by using pmus. In *2015 IEEE Power & Energy Society Innovative Smart Grid Technologies Conference (ISGT)*, pages 1–5. IEEE, 2015.
- [155] Wayes Tushar, Chau Yuen, Bo Chai, Shisheng Huang, Kristin L Wood, See Gim Kerk, and Zaiyue Yang. Smart grid testbed for demand focused energy management in end user environments. *IEEE Wireless Communications*, 23(6):70–80, 2016.

- [156] Wen-Tai Li, Chau Yuen, Naveed Hassan, Wayes Tushar, Chao-Kai Wen, Kristin L Wood, Kun Hu, and Xiang Liu. Demand response management for residential smart grid: From theory to practice. *IEEE Access*, 3:2431–2440, 2015.
- [157] Deepjyoti Deka, Saurav Talukdar, Michael Chertkov, and Murti V Salapaka. Graphical models in meshed distribution grids: Topology estimation, change detection & limitations. *IEEE Transactions on Smart Grid*, 11(5):4299–4310, 2020.
- [158] Lawrence C Evans. *Partial differential equations*, volume 19. American Mathematical Soc., 2010.
- [159] Bernt Oksendal. *Stochastic differential equations: an introduction with applications*. Springer Science & Business Media, 2013.

Appendix A

A.1 Proof of Proposition 1

Recall $\Delta_k = T_f/2^k$ and $\varepsilon_k = \varepsilon/2^k$. Let $(\{a_{it}^*\}, b_t^*)_{t \in [0, T_f]}$ denote the solution to (2.5). Note that $(\{a_{it}^*\}, b_t^*)_{t \in [0, T_f]}$ is a feasible solution to both (2.6) and (2.8). Thus, $\Psi_k \leq \sum_{i=1}^{N_m} [a_{i0}^* P_{gi}(0)] + b_0^* P_b$, and $\xi_k \geq \sum_{i=1}^{N_m} [a_{i0}^* P_{gi}(0)] + b_0^* P_b$ for all k . Therefore, $\Psi_k \leq \xi_k$ for all k . Note that, as $k \rightarrow \infty$, $\varepsilon_k \rightarrow 0$, and since the probability of having a surplus decreases with the decrease in the initial value of the portfolio, the solution set of (2.5) and (2.6) coincide as $k \rightarrow \infty$. Further, Ψ_k and ξ_k increases and decreases with k (as the feasible set decreases). Thus, $\Psi_k \leq \sum_{i=1}^{N_m} [a_{i0}^* P_{gi}(0)] + b_0^* P_b \leq \xi_k$. Therefore, letting $k \rightarrow \infty$ gives the desired result.

A.2 Proof of Theorem 1

The dynamics of $P_{gi}(t)$ is governed by (2.1). Since P_b is constant, $dP_b = 0$. In the subsequent development, explicit dependency of variables on t is omitted for brevity of notation. Applying differentiation operator to γ_i , given in (2.9), and, using the constraint (2.10) for $t \in [0, T_f]$ and $i \in \mathcal{I}$, we have,

$$d\gamma_i = a_{it} dP_{gi} + b_{it} dP_b = a_{it} (\mu_{gi} P_{gi} dt + \sigma_{gi} P_{gi} dW_{ti}), \quad (\text{A.1})$$

Applying Ito's lemma [47],

$$\begin{aligned} d\gamma_i &= \frac{\partial \gamma_i}{\partial t} dt + \frac{\partial \gamma_i}{\partial P_{gi}} dP_{gi} + \frac{1}{2} \frac{\partial^2 \gamma_i}{\partial P_{gi}^2} (dP_{gi}^2) \\ &= \frac{\partial \gamma_i}{\partial t} dt + \frac{\partial \gamma_i}{\partial P_{gi}} (\mu_{gi} dt + \sigma_{gi} dW_{ti}) P_{gi} + \frac{\sigma_{gi}^2 P_{gi}^2}{2} \frac{\partial^2 \gamma_i}{\partial P_{gi}^2} dt. \end{aligned} \quad (\text{A.2})$$

Using, (A.1) and (A.2) we get,

$$\begin{aligned} & \left[\frac{\partial \gamma_i}{\partial t} + \mu_{gi} P_{gi} \frac{\partial \gamma_i}{\partial P_{gi}} + \frac{1}{2} \sigma_{gi}^2 P_{gi}^2 \frac{\partial^2 \gamma_i}{\partial P_{gi}^2} - a_{it} \mu_{gi} P_{gi} \right] dt \\ & + \left[\sigma_{gi} P_{gi} \frac{\partial \gamma_i}{\partial P_{gi}} - a_{it} \sigma_{gi} P_{gi} \right] dW_{ti} = 0. \end{aligned} \quad (\text{A.3})$$

Note that, if $a_{it} = \frac{\partial \gamma_i}{\partial P_{gi}}$ then $[\sigma_{gi} P_{gi} \frac{\partial \gamma_i}{\partial P_{gi}} - a_{it} \sigma_{gi} P_{gi}] = 0$, which eliminates the uncertainty. Thus,

$$\frac{\partial \gamma_i}{\partial t} + \frac{1}{2} \sigma_{gi}^2 P_{gi}^2 \frac{\partial^2 \gamma_i}{\partial P_{gi}^2} = 0. \quad (\text{A.4})$$

Now, to ensure that the critical demand is met *almost surely* at the requested time T_f , we need to solve (A.4) with the terminal condition (2.12) and find a_{it} , b_{it} .

Let $\tau = T_f - t$, $x_i = \ln(P_{gi})$ for all $i \in \mathcal{I}$. Therefore, $\frac{\partial \gamma_i}{\partial t} = -\frac{\partial \gamma_i}{\partial \tau}$, $\frac{\partial \gamma_i}{\partial P_{gi}} = \frac{1}{P_{gi}} \frac{\partial \gamma_i}{\partial x_i}$, $\frac{\partial^2 \gamma_i}{\partial P_{gi}^2} = \frac{1}{P_{gi}^2} \left[\frac{\partial^2 \gamma_i}{\partial x_i^2} - \frac{\partial \gamma_i}{\partial x_i} \right]$. From (A.4),

$$\frac{\partial \gamma_i}{\partial \tau} = \frac{1}{2} \sigma_{gi}^2 \frac{\partial^2 \gamma_i}{\partial x_i^2} - \frac{1}{2} \sigma_{gi}^2 \frac{\partial \gamma_i}{\partial x_i} = A_i \frac{\partial^2 \gamma_i}{\partial x_i^2} + B_i \frac{\partial \gamma_i}{\partial x_i}, \quad (\text{A.5})$$

where, $A_i = \frac{1}{2} \sigma_{gi}^2$ ($A_i > 0$), $B_i = -\frac{1}{2} \sigma_{gi}^2$. From (2.12), the final condition on $\gamma_i(P_{gi}(t), t = T_f)$, or equivalently, initial condition on $\gamma_i(x_i, \tau = 0)$ is given as:

$$\gamma_i(x_i, 0) = \begin{cases} 0 & \text{if } x_i \geq \ln(D_{ci}) \\ D_{ci} - e^{x_i} & \text{if } x_i < \ln(D_{ci}). \end{cases} \quad (\text{A.6})$$

Let $\gamma_i(x_i, \tau) = e^{-(\alpha_i x_i + \beta_i \tau)} u_i(x_i, \tau)$, where $\alpha_i, \beta_i \in \mathbb{R}$. Then,

$$\frac{\partial \gamma_i}{\partial \tau} = e^{-(\alpha_i x_i + \beta_i \tau)} \left[\frac{\partial u_i}{\partial \tau} - \beta_i u_i \right].$$

$$\begin{aligned} \text{Similarly, } \quad \frac{\partial \gamma_i}{\partial x_i} &= e^{-(\alpha_i x_i + \beta_i \tau)} \left[\frac{\partial u_i}{\partial x_i} - \alpha_i u_i \right], \\ \frac{\partial^2 \gamma_i}{\partial x_i^2} &= e^{-(\alpha_i x_i + \beta_i \tau)} \left[\frac{\partial^2 u_i}{\partial x_i^2} - 2\alpha_i \frac{\partial u_i}{\partial x_i} + \alpha_i^2 u_i \right]. \end{aligned}$$

Therefore, using (A.5), we get,

$$\frac{\partial u_i}{\partial \tau} - \beta_i u_i = A_i \left[\frac{\partial^2 u_i}{\partial x_i^2} - 2\alpha_i \frac{\partial u_i}{\partial x_i} + \alpha_i^2 u_i \right] + B_i \left[\frac{\partial u_i}{\partial x_i} - \alpha_i u_i \right],$$

and thus,

$$\frac{\partial u_i}{\partial \tau} = A_i \frac{\partial^2 u_i}{\partial x_i^2} + [B_i - 2\alpha_i A_i] \frac{\partial u_i}{\partial x_i} + [\beta_i + \alpha_i^2 A_i - \alpha_i B_i] u_i.$$

Choosing $\alpha_i = \frac{B_i}{2A_i}$, $\beta_i = \frac{B_i^2}{4A_i}$, we get, $\frac{\partial u_i}{\partial \tau} = A_i \frac{\partial^2 u_i}{\partial x_i^2}$. Solution to this partial differential equation [158] is given by

$$u_i(x_i, \tau) = \frac{1}{\sqrt{4\pi A_i \tau}} \int_{-\infty}^{\infty} u_i(y, 0) e^{-\frac{(x_i - y)^2}{4A_i \tau}} dy.$$

Since, $\gamma_i(y, 0) = e^{-(\alpha_i y)} u_i(y, 0)$, and $\alpha_i = \frac{B_i}{2A_i}$, $\beta_i = \frac{B_i^2}{4A_i}$,

$$\begin{aligned} u_i(x_i, \tau) &= \frac{1}{\sqrt{4\pi A_i \tau}} \int_{-\infty}^{\infty} e^{-\frac{(x_i - y)^2}{4A_i \tau}} e^{\alpha_i y} \gamma_i(y, 0) dy \\ &= \frac{1}{\sqrt{4\pi A_i \tau}} \int_{-\infty}^{\infty} e^{-\left[\frac{(y - B_i \tau - x_i)^2}{4A_i \tau}\right]} e^{(\alpha_i x_i + \beta_i \tau)} \gamma_i(y, 0) dy. \end{aligned} \quad (\text{A.7})$$

Therefore, $\gamma_i(x_i, \tau)$ is given by

$$\begin{aligned} \gamma_i(x_i, \tau) &= e^{-(\alpha_i x_i + \beta_i \tau)} u_i(x_i, \tau) \\ &= \frac{1}{\sqrt{4\pi A_i \tau}} \int_{-\infty}^{\infty} e^{-\left[\frac{(y - B_i \tau - x_i)^2}{4A_i \tau}\right]} \gamma_i(y, 0) dy \\ &= \frac{1}{\sigma_{gi} \sqrt{2\pi \tau}} \int_{-\infty}^{\infty} e^{-\frac{1}{2} \left[\frac{(y + \frac{\sigma_{gi}^2}{2} \tau - x_i)^2}{\sigma_{gi}^2 \tau} \right]} \gamma_i(y, 0) dy. \end{aligned}$$

Applying the initial condition (A.6) to $\gamma_i(y, 0)$ we get:

$$\begin{aligned} \gamma_i(x_i, \tau) &= \left(\frac{D_{ci}}{\sigma_{gi} \sqrt{2\pi \tau}} \left[\int_{-\infty}^{\ln(D_{ci})} e^{-\frac{1}{2} \left[\frac{(y + \frac{\sigma_{gi}^2}{2} \tau - x_i)^2}{\sigma_{gi}^2 \tau} \right]} dy \right] \right)_{:=I_1} \\ &\quad - \left(\frac{1}{\sigma_{gi} \sqrt{2\pi \tau}} \left[\int_{-\infty}^{\ln(D_{ci})} e^{-\frac{1}{2} \left[\frac{(y + \frac{\sigma_{gi}^2}{2} \tau - x_i)^2}{\sigma_{gi}^2 \tau} \right]} e^y dy \right] \right)_{:=I_2}. \end{aligned}$$

Let, $z_i = \frac{1}{\sigma_{gi} \sqrt{\tau}} (y + (\frac{\sigma_{gi}^2}{2}) \tau - x_i) \implies dz_i = \frac{1}{\sigma_{gi} \sqrt{\tau}} dy$. Then,

$$I_1 = \frac{D_{ci}}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\ln(D_{ci}) + \frac{\sigma_{gi}^2}{2} \tau - x_i}{\sigma_{gi} \sqrt{\tau}}} e^{-\frac{z_i^2}{2}} dz_i = D_{ci} \Phi \left[\frac{\ln \left(\frac{D_{ci}}{\sigma_{gi}} \right) + \frac{\sigma_{gi}^2}{2} \tau}{\sigma_{gi} \sqrt{\tau}} \right],$$

where, $\Phi(\cdot)$ is the Cumulative Distribution Function (CDF) of the standard Gaussian random variable $\sim \mathcal{N}(0, 1)$. Similarly,

$$I_2 = \frac{e^x}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\ln(D_{ci}) + \frac{\sigma_{gi}^2}{2} \tau - x_i}{\sigma_{gi} \sqrt{\tau}}} e^{-\frac{1}{2} [z_i - \sigma_{gi} \sqrt{\tau}]^2} dz_i = P_{gi} \Phi \left[\frac{\ln(\frac{D_{ci}}{P_{gi}}) - \frac{\sigma_{gi}^2}{2} \tau}{\sigma_{gi} \sqrt{\tau}} \right].$$

where, $\Phi(\cdot)$ is the CDF of the standard Gaussian random variable $\sim \mathcal{N}(0, 1)$. Therefore, substituting the value of τ , we obtain,

$$\begin{aligned} \gamma_i(P_{gi}(t), t) &= (D_{ci}) \Phi \left[\frac{\ln \left(\frac{D_{ci}}{P_{gi}(t)} \right) + (\sigma_{gi}^2/2)(T_f - t)}{\sigma_{gi} \sqrt{(T_f - t)}} \right] \\ &\quad - P_{gi}(t) \Phi \left[\frac{\ln \left(\frac{D_{ci}}{P_{gi}(t)} \right) - (\sigma_{gi}^2/2)(T_f - t)}{\sigma_{gi} \sqrt{(T_f - t)}} \right]. \end{aligned}$$

We get the desired results from the relations $a_{it} = \frac{\partial \gamma_i}{\partial P_{gi}}$ and $b_{it} = (1/P_b)(\gamma_i - a_{it} P_{gi}(t))$. This completes the proof.

A.3 Derivation of stochastic differential equation of Γ_t in IMM for RESs uncertainty mitigation

From (2.1), and for constant P_b , and under constraint (2.4), applying Ito's lemma [47] on differential of Γ_t we get,

$$\frac{\partial \Gamma_t}{\partial t} dt + \sum_{i=1}^{N_m} \frac{\partial \Gamma_t}{\partial P_{gi}} dP_{gi} + \sum_{i=1}^{N_m} \sum_{j=1}^{N_m} \frac{\partial^2 \Gamma_t}{2 \partial P_{gi} \partial P_{gj}} dP_{gi} dP_{gj} = \sum_{i=1}^{N_m} a_{it} dP_{gi}.$$

Using (2.1), (2.2), we get,

$$\begin{aligned} &\left[\frac{\partial \Gamma_t}{\partial t} + \sum_{i=1}^{N_m} \mu_{gi} P_{gi} \frac{\partial \Gamma_t}{\partial P_{gi}} + \frac{1}{2} \sum_{i=1}^{N_m} \sum_{j=1}^{N_m} \sigma_{gi} \sigma_{gj} \rho_{ij} \frac{\partial^2 \Gamma_t}{\partial P_{gi} \partial P_{gj}} P_{gi} P_{gj} \right. \\ &\quad \left. - \sum_{i=1}^{N_m} a_{it} \mu_{gi} P_{gi} \right] dt = \sum_{i=1}^{N_m} \left[a_{it} \sigma_{gi} P_{gi} - \frac{\partial \Gamma_t}{\partial P_{gi}} \sigma_{gi} P_{gi} \right] dW_{ti}. \end{aligned}$$

Note, if $a_{it} = \frac{\partial \Gamma_t}{\partial P_{gi}}$, $i \in \mathcal{I}$, uncertainty of all RESs generation is eliminated, and we have:

$$\frac{\partial \Gamma_t}{\partial t} + \frac{1}{2} \sum_{i=1}^{N_m} \sum_{j=1}^{N_m} \sigma_{gi} \sigma_{gj} \rho_{ij} P_{gi} P_{gj} \frac{\partial^2 \Gamma_t}{\partial P_{gi} \partial P_{gj}} = 0. \quad (\text{A.8})$$

Note that, if there are numerous MGs present in the IMM ($N_m \gg 1$), theoretical solution of (A.8) to find a_{it} , b_t , under the terminal condition (2.13), becomes intractable.

A.4 Proof of Theorem 2

Given the dynamics of RESs (2.1), we can find the value of Γ_t , $t \in [0, T_f]$, given the terminal condition (2.13), by changing the probability measure to $\mathbb{Q}$. From (2.1), for all $i \in \mathcal{I}$,

$$dP_{gi}(t) = \mu_{gi} P_{gi}(t) dt + \sigma_{gi} P_{gi}(t) (d\bar{W}_{ti} - \eta_{gi} dt) = \sigma_{gi} P_{gi}(t) d\bar{W}_{ti}.$$

Applying differentiation operator to (2.3) and since P_b is constant,

$$d\Gamma_t = \sum_{i=1}^{N_m} a_{it} dP_{gi} = \sum_{i=1}^{N_m} a_{it} \sigma_{gi} P_{gi} d\bar{W}_{ti}.$$

Thus, for any $t \in [0, T_f]$, $\Gamma_{T_f} = \Gamma_t + \int_t^{T_f} \sum_{i=1}^{N_m} a_{it} \sigma_{gi} P_{gi} d\bar{W}_{ti}$. Hence, given $\mathbf{P}_g(t)$, under the transformed probability measure, the expected value of Γ_{T_f} , calculated at $t \in [0, T_f]$ is,

$$\bar{\mathbb{E}}[\Gamma_{T_f} | \mathbf{P}_g] = \Gamma_t + \int_t^{T_f} \sum_{i=1}^{N_m} a_{it} \sigma_{gi} P_{gi} \bar{\mathbb{E}}[d\bar{W}_{ti} | \mathbf{P}_g] = \Gamma_t,$$

where, the last equality follows from the fact that $\bar{\mathbb{E}}[d\bar{W}_{ti} | \mathbf{P}_g] = 0$, under the transformed probability measure $\mathbb{Q}$. Thus, from (2.13),

$$\Gamma_t = \bar{\mathbb{E}}[\max(\sum_{i=1}^{N_m} (D_{ci} - P_{gi}(T_f)), 0)].$$

This completes the proof.

A.5 Proof of Theorem 3

From the discrete-time process $P_{gi}^{\Delta t}(t_n)$, described in Section 2.5.2, we obtain an approximate continuous-time process $\tilde{P}_{gi}(t)$, $t \in [0, T_f]$, for all $i \in \{1, 2, \dots, N_m\}$ as follows:

$$\begin{aligned}\tilde{P}_{gi}(t_n) &= P_{gi}^{\Delta t}(t_n), \text{ for all } n \in \{0, 1, \dots, N\}, \\ \tilde{P}_{gi}(t) &= \tilde{P}_{gi}(t_n) + \int_{t_n}^t \sigma_{gi} \tilde{P}_{gi}(t) d\bar{W}_i(t), \text{ for } t_n \leq t < t_{n+1}.\end{aligned}$$

Let $s = T_f - t$ and $s_n = T_f - t_n$. Let the power portfolios $\Gamma(P_{g1}^{\Delta t}(s), \dots, P_{gN_m}^{\Delta t}(s), s)$ and $\Gamma(\tilde{P}_{g1}(s), \dots, \tilde{P}_{gN_m}(s), s)$ be denoted by $\Gamma_s^{\Delta t}$ and $\tilde{\Gamma}_s'$ respectively for brevity of notation. From (A.8) and Theorem 2, we have

$$\begin{aligned}\epsilon(t_n) &= \epsilon(s_n) = |\mathbb{E}[\Gamma_0^{\Delta t} | \mathbf{P}_g(s_n)] - \mathbb{E}[\Gamma_0' | \mathbf{P}_g(s_n)]| \\ &= |\mathbb{E}[\Gamma_0^{\Delta t} | \mathbf{P}_g(s_n)] - \Gamma_{s_n}'|.\end{aligned}$$

Since by definition, $\mathbf{P}_g^{\Delta t}(s_n) = \mathbf{P}_g(s_n)$, we have,

$$\begin{aligned}\epsilon(s_n) &= \left| \mathbb{E} \left[\sum_{k=n}^1 \left(\Gamma_{s_{k-1}}^{\Delta t} - \Gamma_{s_k}^{\Delta t} \right) | \mathbf{P}_g^{\Delta t}(s_n) \right] \right| \\ &= \left| \sum_{k=n}^1 \mathbb{E} \left[\Gamma_{s_{k-1}}^{\Delta t} - \Gamma_{s_k}^{\Delta t} | \mathbf{P}_g^{\Delta t}(s_n) \right] \right| = \left| \sum_{k=n}^1 \mathbb{E} \left[e_k | \mathbf{P}_g^{\Delta t}(s_n) \right] \right|,\end{aligned}\tag{A.9}$$

where $e_k := \Gamma_{s_{k-1}}^{\Delta t} - \Gamma_{s_k}^{\Delta t}$. Since for all n , $\mathbf{P}_g^{\Delta t}(s_n) = \tilde{\mathbf{P}}_g(s_n)$, we have, $\mathbb{E}[e_k | \mathbf{P}_g^{\Delta t}(s_n)] = \mathbb{E}[\tilde{\Gamma}_{s_{k-1}}' - \tilde{\Gamma}_{s_k}' | \tilde{\mathbf{P}}_g(s_n)]$. Applying Ito's Lemma on the differential of $\tilde{\Gamma}_{s_k}'$, we obtain,

$$\begin{aligned}\tilde{\Gamma}_{s_{k-1}}' &= \tilde{\Gamma}_{s_k}' \\ &+ \int_{s_k}^{s_{k-1}} \left(\frac{\partial \tilde{\Gamma}_s'}{\partial s} - \frac{1}{2} \sum_{i,j=1}^{N_m} \sigma_{gi} \sigma_{gj} \rho_{ij} \frac{\partial^2 \tilde{\Gamma}_s'}{\partial \tilde{P}_{gi} \partial \tilde{P}_{gj}} \tilde{P}_{gi}(s) \tilde{P}_{gj}(s) \right) ds \\ &+ \int_{s_k}^{s_{k-1}} \sum_{i=1}^{N_m} \sigma_{gi} \tilde{P}_{gi}(s) \frac{\partial \tilde{\Gamma}_s'}{\partial \tilde{P}_{gi}} d\bar{W}_i(s).\end{aligned}$$

Since $\bar{W}_i(s)$ is a Wiener process, $\mathbb{E}[d\bar{W}_i(s)] = 0$. Thus,

$$\begin{aligned}\mathbb{E}[e_k | \mathbf{P}_g^{\Delta t}(s_n)] &= \mathbb{E} \left[\int_{s_k}^{s_{k-1}} \left(\frac{\partial \tilde{\Gamma}_s'}{\partial s} \right. \right. \\ &\left. \left. - \frac{1}{2} \sum_{i,j=1}^{N_m} \sigma_{gi} \sigma_{gj} \rho_{ij} \frac{\partial^2 \tilde{\Gamma}_s'}{\partial \tilde{P}_{gi} \partial \tilde{P}_{gj}} \tilde{P}_{gi}(s) \tilde{P}_{gj}(s) \right) ds | \tilde{\mathbf{P}}_g(s_n) \right].\end{aligned}$$

We define,

$$\begin{aligned}\tilde{g}_{1_s} &:= \frac{\partial \tilde{\Gamma}_s'}{\partial s} \\ \tilde{g}_{2_s} &:= -\frac{1}{2} \sum_{i,j=1}^{N_m} \sigma_{gi} \sigma_{gj} \rho_{ij} \frac{\partial^2 \tilde{\Gamma}_s'}{\partial \tilde{P}_{gi} \partial \tilde{P}_{gj}} \tilde{P}_{gi}(s) \tilde{P}_{gj}(s).\end{aligned}$$

From (A.8), $\tilde{g}_{1_{s_k}} + \tilde{g}_{2_{s_k}} = 0$ for all $k \in \{0, 1, \dots, N-1\}$. Thus,

$$\mathbb{E}[e_k | \mathbf{P}_g^{\Delta t}(s_n)] = \mathbb{E}[\int_{s_k}^{s_{k-1}} ((\tilde{g}_{1_s} - \tilde{g}_{1_{s_k}}) + (\tilde{g}_{2_s} - \tilde{g}_{2_{s_k}})) ds | \tilde{\mathbf{P}}_g(s_n)].$$

Define $e_{km} := \int_{s_k}^{s_{k-1}} (\tilde{g}_{m_s} - \tilde{g}_{m_{s_k}}) ds$, $m = 1, 2$. Then,

$$\mathbb{E}[e_k | \mathbf{P}_g^{\Delta t}(s_n)] = \sum_{m=1}^2 \mathbb{E}[e_{km} | \tilde{\mathbf{P}}_g(s_n)]. \quad (\text{A.10})$$

Applying Ito's Lemma on the differential of $\tilde{g}_{m_{s_k}}$, we get,

$$\begin{aligned}\tilde{g}_{m_s} - \tilde{g}_{m_{s_k}} &= \int_{s_k}^s \left(\frac{\partial \tilde{g}_{m_x}}{\partial x} - \frac{1}{2} \sum_{i,j=1}^{N_m} \sigma_{gi} \sigma_{gj} \rho_{ij} \frac{\partial^2 \tilde{g}_{m_x}}{\partial \tilde{P}_{gi} \partial \tilde{P}_{gj}} \tilde{P}_{gi}(x) \tilde{P}_{gj}(x) \right) dx \\ &\quad + \int_{s_k}^s \sum_{i=1}^{N_m} \sigma_{gi} \tilde{P}_{gi}(x) \frac{\partial \tilde{g}_{m_x}}{\partial \tilde{P}_{gi}} d\bar{W}_i(x).\end{aligned}$$

Since $\bar{W}_i(x)$ is a Wiener process, $\mathbb{E}[d\bar{W}_j(x)] = 0$ for all $x \in [0, T_f]$. Therefore, we have,

$$\begin{aligned}\mathbb{E}[e_{km} | \tilde{\mathbf{P}}_g(s_n)] &= \mathbb{E}[\mathbb{E}[e_{km} | \tilde{\mathbf{P}}_g(s_k)] | \tilde{\mathbf{P}}_g(s_n)] = \mathbb{E}[\mathbb{E}[\int_{s_k}^{s_{k-1}} \int_{s_k}^s \left(\frac{\partial \tilde{g}_{m_x}}{\partial x} - \right. \\ &\quad \left. \frac{1}{2} \sum_{i,j=1}^{N_m} \sigma_{gi} \sigma_{gj} \rho_{ij} \frac{\partial^2 \tilde{g}_{m_x}}{\partial \tilde{P}_{gi} \partial \tilde{P}_{gj}} \tilde{P}_{gi}(x) \tilde{P}_{gj}(x) \right) dx ds | \tilde{\mathbf{P}}_g(s_k)] | \tilde{\mathbf{P}}_g(s_n)].\end{aligned}$$

Since $\tilde{P}_{gi}(s)$, $s \in [0, T_f]$ is continuous, $a_i(s, \tilde{P}_{gi}(s))$ is uniformly continuous in $\tilde{P}_{gi}(s)$. Further, $\sigma_{gi} < \infty$ for $i \in \{1, 2, \dots, N_m\}$. Thus, there exists a constant $L' < \infty$ independent of Δt such that $|\mathbb{E}[\frac{\partial \tilde{g}_{m_x}}{\partial x} - \frac{1}{2} \sum_{i,j=1}^{N_m} \sigma_{gi} \sigma_{gj} \rho_{ij} \frac{\partial^2 \tilde{g}_{m_x}}{\partial \tilde{P}_{gi} \partial \tilde{P}_{gj}} \tilde{P}_{gi}(x) \tilde{P}_{gj}(x)]| \leq L'$. Thus,

$|\overline{\mathbb{E}}[e_{km}|\tilde{\mathbf{P}}_{\mathbf{g}}(s_n)]| \leq \int_{s_k}^{s_{k-1}} \int_{s_k}^s L' dx ds = L' \Delta t^2 / 2$ and from (A.9) and (A.10),

$$\begin{aligned} \epsilon(t_n) &= \left| \sum_{k=n}^1 \overline{\mathbb{E}}[e_k | \mathbf{P}_{\mathbf{g}}^{\Delta t}(s_n)] \right| = \left| \sum_{k=n}^1 \sum_{m=1}^2 \overline{\mathbb{E}}[e_{km} | \tilde{\mathbf{P}}_{\mathbf{g}}(s_n)] \right| \\ &\leq \sum_{k=n}^1 \sum_{m=1}^2 \left| \overline{\mathbb{E}}[e_{km} | \tilde{\mathbf{P}}_{\mathbf{g}}(s_n)] \right| \leq n L' \Delta t^2 \\ &= (T_f - s_n) L' \Delta t = L \Delta t. \end{aligned}$$

This completes the proof.

Appendix B

B.1 Proof of Lemma 1

We have $X(t) = \sigma W(t) + \alpha N$. Recall that for $a \in \mathbb{R}$, $T(a) = \inf\{t \geq 0 : X(t) = a\} = \inf\{t \geq 0 : W(t) = \frac{a - \alpha N}{\sigma}\}$. We define $\beta := \frac{a - \alpha N}{\sigma}$. It follows from the properties of the Weiner process, that $T(N)$ is a stopping time [71].

Let $M(t) := e^{cW(t) - \frac{1}{2}c^2t}$ for any $c \neq 0$. Note that, the stochastic process $M(t)$ is a martingale [70]. Therefore, for all $t \geq 0$, we have $\mathbb{E}[M(t)] = M(0) = 1$. Further, we have $\mathbb{E}[M(T(a))] = 1$ [70].

We first consider the case, $a = N$. Here, we have,

$$\begin{aligned} \mathbb{E}[M(T(N))] &= 1 \\ \implies \mathbb{E}[e^{\frac{c}{\sigma}W(T(N)) - \frac{1}{2}c^2T(N)}] &= 1 \\ \implies \mathbb{E}[e^{\frac{c}{\sigma}X(T(N)) - \frac{1}{2}c^2T(N)}] &= e^{\frac{c\alpha N}{\sigma}}. \end{aligned} \tag{B.1}$$

Note that, in the case of $a = N$, we have, $\beta = \frac{N(1-\alpha)}{\sigma} \geq 0$. Therefore, we choose a positive constant c . Let $c = \sqrt{2s}$ for some $s \in \mathbb{R}^+$. Then, from (B.1), we have

$$\mathbb{E}[e^{\frac{c}{\sigma}X(T(N)) - sT(N)}] = e^{\frac{c\alpha N}{\sigma}}.$$

By definition of $T(N)$, we obtain,

$$\begin{aligned} \mathbb{E}[e^{\frac{cN}{\sigma} - sT(N)}] &= e^{\frac{c\alpha N}{\sigma}} \\ \implies \mathbb{E}[e^{-sT(N)}] &= e^{-\frac{\sqrt{2s}N(1-\alpha)}{\sigma}}. \end{aligned}$$

Further, note that for any $s > 0$, we have, $\mathbb{P}[T(N) \leq T_f] = \mathbb{P}[e^{-sT(N)} \geq e^{-sT_f}]$. Thus,

applying Markov's inequality we get,

$$\begin{aligned}\mathbb{P}[T(N) \leq T_f] &= \mathbb{P}[e^{-sT(N)} \geq e^{-sT_f}] \\ &\leq e^{sT_f} \mathbb{E}[e^{-sT(N)}] \\ &= e^{-\frac{\sqrt{2s}N(1-\alpha)}{\sigma} + sT_f}\end{aligned}$$

Thus,

$$\mathbb{P}[T(N) \leq T_f] \leq \inf_{s>0} e^{-\frac{\sqrt{2s}N(1-\alpha)}{\sigma} + sT_f}. \quad (\text{B.2})$$

It can be shown that for any given $N \geq 0$ and $\alpha \in [0, 1]$, the function $e^{-\frac{\sqrt{2s}N(1-\alpha)}{\sigma} + sT_f}$ reaches global minima at $s^* = \frac{N^2(1-\alpha)^2}{2\sigma^2T_f}$. Plugging in the expression of s^* in (B.2), we obtain the following:

$$\mathbb{P}[T(N) \leq T_f] \leq e^{-\frac{N^2(1-\alpha)^2}{2\sigma^2T_f}}. \quad (\text{B.3})$$

Carrying out a similar procedure for the case $a = 0$, to obtain an upper bound for $\mathbb{P}[T(0) \leq T_f]$, we can derive the following:

$$\mathbb{P}[T(0) \leq T_f] \leq e^{-\frac{\alpha^2 N^2}{2\sigma^2T_f}}. \quad (\text{B.4})$$

Combining both (B.3) and (B.4), we obtain the desired result. This completes the proof.

B.2 Proof of Theorem 4

Recall that

$$f(N, \alpha) := e^{-\frac{N^2(1-\alpha)^2}{2\sigma^2T_f}} + e^{-\frac{N^2\alpha^2}{2\sigma^2T_f}}.$$

For $\alpha = \frac{1}{2}$, we have $f(N, \frac{1}{2}) = 2e^{-\frac{N^2}{8\sigma^2T_f}}$. Note that, $f(N, \frac{1}{2})$ is strictly monotonic decreasing in N , for $N > 0$. Thus, for $\alpha = \frac{1}{2}$, the solution to (3.11) is obtained when the constraint $f(N, \frac{1}{2}) \leq \delta$ is an equality. Solving $f(N, \frac{1}{2}) = \delta$ we obtain $N(\frac{1}{2}) = N_o = \sqrt{8\sigma^2T_f \ln(2/\delta)}$.

We define $k := \frac{2}{\delta}$. Thus, for any $\alpha \in (0, 1)$, we have,

$$f(N_0, \alpha) = e^{-4\ln(k)(1-\alpha)^2} + e^{-4\ln(k)\alpha^2}.$$

We first prove that $f(N_0, \alpha)$ achieves global minima at $\alpha = \frac{1}{2}$. Differentiating with respect to α , we have,

$$\frac{\partial f(N_0, \alpha)}{\partial \alpha} = 8(1-\alpha) \ln(k) (k)^{-4(1-\alpha)^2} - 8\alpha \ln(k) (k)^{-4\alpha^2}. \quad (\text{B.5})$$

Note, for any $\alpha \in (0, 1)$ that satisfies $\frac{\partial f(N_0, \alpha)}{\partial \alpha} = 0$, we have the following:

$$(k)^{4(1-2\alpha)} = \frac{1-\alpha}{\alpha}. \quad (\text{B.6})$$

Note that $k = \frac{2}{\delta} \geq 2$. Clearly, $\alpha = \frac{1}{2}$ satisfies (B.6). Double differentiation of $f(N_0, \alpha)$ with respect to α leads to the following:

$$\begin{aligned} \frac{\partial^2 f(N_0, \alpha)}{\partial \alpha^2} &= (8(1-\alpha) \ln(k))^2 (k)^{-4(1-\alpha)^2} \\ &\quad - 8 \ln(k) (k)^{-4(1-\alpha)^2} \\ &\quad + (8\alpha \ln(k))^2 (k)^{-4\alpha^2} - 8 \ln(k) (k)^{-4\alpha^2}. \end{aligned}$$

For any α that satisfies $\frac{\partial f(N_0, \alpha)}{\partial \alpha} = 0$, from (B.5), we get,

$$\begin{aligned} \frac{\partial^2 f(N_0, \alpha)}{\partial \alpha^2} &= (8\alpha \ln(k))^2 (k)^{8(1-\alpha)^2 - 8\alpha^2} (k)^{-4(1-\alpha)^2} \\ &\quad - 8 \ln(k) (k)^{-4(1-\alpha)^2} + (8\alpha \ln(k))^2 (k)^{-4\alpha^2} - 8 \ln(k) (k)^{-4\alpha^2} \\ &= (8\alpha \ln(k))^2 (k)^{-4\alpha^2} (1 + (k)^{4(1-\alpha)^2 - 4\alpha^2}) \\ &\quad - 8 \ln(k) (k)^{-4\alpha^2} (1 + (k)^{4\alpha^2 - 4(1-\alpha)^2}) \\ &= (8\alpha \ln(k))^2 (k)^{-4\alpha^2} (1 + (k)^{4(1-2\alpha)}) \\ &\quad - 8 \ln(k) (k)^{-4\alpha^2} (1 + (k)^{4(2\alpha-1)}) \\ &= (8\alpha \ln(k))^2 (k)^{-4\alpha^2} (1 + \frac{1-\alpha}{\alpha}) - 8 \ln(k) (k)^{-4\alpha^2} (1 + \frac{\alpha}{1-\alpha}) \\ &= (8 \ln(k))^2 (k)^{-4\alpha^2} \alpha - 8 \ln(k) (k)^{-4\alpha^2} (\frac{1}{1-\alpha}) \\ &= 8 \ln(k) (k)^{-4\alpha^2} (8 \ln(k) \alpha - \frac{1}{1-\alpha}). \end{aligned} \quad (\text{B.7})$$

Note that, for $\alpha = \frac{1}{2}$, (B.7) can be simplified to $16 \ln(k) (\frac{1}{k}) (2 \ln(k) - 1)$ which is positive, since $k = \frac{2}{\delta} > 2$. Thus, $f(N_0, \alpha)$ achieves minima at $\alpha = \frac{1}{2}$.

Next, we show that $f(N_0, \alpha)$ achieves global minima at $\alpha = \frac{1}{2}$, that is for $\alpha \neq \frac{1}{2}, \alpha \in$

$(0, 1)$, we have $\frac{\partial^2 f(N_0, \alpha)}{\partial \alpha^2} < 0$, or equivalently, from (B.7), $(8 \ln(k)\alpha - \frac{1}{1-\alpha}) < 0$. We first define $g(\alpha) := 2 \ln(\frac{1}{\alpha}) - 2 \ln(\frac{1}{1-\alpha}) - \frac{1}{\alpha} + \frac{1}{1-\alpha}$. Here we get, $\frac{\partial g(\alpha)}{\partial \alpha} = (1-2\alpha)(\frac{1}{\alpha^2} - \frac{1}{(1-\alpha)^2})$. In the region, $\alpha \in (0, \frac{1}{2})$, we have, $0 < \frac{1}{1-\alpha} < \frac{1}{\alpha}$ and $(1-2\alpha) > 0$. Thus, $\frac{\partial g(\alpha)}{\partial \alpha} > 0$, hence, $g(\alpha)$ is strictly increasing in $\alpha \in (0, \frac{1}{2})$. Similarly, in the region $\alpha \in (\frac{1}{2}, 1)$, we have $(1-2\alpha) < 0$, and $0 < \frac{1}{\alpha} < \frac{1}{1-\alpha}$. Thus, $\frac{\partial g(\alpha)}{\partial \alpha} > 0$, hence $g(\alpha)$ is increasing in $\alpha \in (\frac{1}{2}, 1)$. Since, $g(\alpha)$ is a continuous function of $\alpha \in (0, 1)$, we have

$$\begin{aligned} g(\alpha) &< g(\frac{1}{2}) = 0, \quad \alpha \in (0, \frac{1}{2}), \\ g(\alpha) &> g(\frac{1}{2}) = 0, \quad \alpha \in (\frac{1}{2}, 1). \end{aligned} \tag{B.8}$$

Next, we will show that $(8 \ln(k)\alpha - \frac{1}{1-\alpha}) < 0, \alpha \in (0, 1) \setminus \{\frac{1}{2}\}$. First consider the case $\alpha \in (0, \frac{1}{2})$. From (B.6), we have $8 \ln(k) = \frac{2}{1-2\alpha} \ln(\frac{1-\alpha}{\alpha})$, and since $\alpha \in (0, \frac{1}{2})$, we have $(1-2\alpha) > 0$. Now, suppose, $(8 \ln(k)\alpha - \frac{1}{1-\alpha}) \geq 0$. Here we have,

$$\begin{aligned} 8 \ln(k)\alpha - \frac{1}{1-\alpha} &\geq 0 \\ \implies \frac{2\alpha}{1-2\alpha} \ln(\frac{1-\alpha}{\alpha}) - \frac{1}{1-\alpha} &\geq 0 \\ \implies 2 \ln(\frac{1-\alpha}{\alpha}) - \frac{1-2\alpha}{\alpha(1-\alpha)} &\geq 0 \\ \implies 2 \ln(\frac{1}{\alpha}) - 2 \ln(\frac{1}{1-\alpha}) - \frac{1}{\alpha} + \frac{1}{1-\alpha} &\geq 0 \\ \implies g(\alpha) &\geq 0. \end{aligned}$$

However, from (B.8), we have $g(\alpha) < 0$ when $\alpha \in (0, \frac{1}{2})$, and thus, $(8 \ln(k)\alpha - \frac{1}{1-\alpha}) < 0$.

Next, consider the region $\alpha \in (\frac{1}{2}, 1)$. Here, we have $(1-2\alpha) < 0$. Suppose $(8 \ln(k)\alpha - \frac{1}{1-\alpha}) \geq 0$. Here, we have,

$$\begin{aligned} 8 \ln(k)\alpha - \frac{1}{1-\alpha} &\geq 0 \\ \implies \frac{2\alpha}{1-2\alpha} \ln(\frac{1-\alpha}{\alpha}) - \frac{1}{1-\alpha} &\geq 0 \\ \implies 2 \ln(\frac{1-\alpha}{\alpha}) - \frac{1-2\alpha}{\alpha(1-\alpha)} &\leq 0 \\ \implies 2 \ln(\frac{1}{\alpha}) - 2 \ln(\frac{1}{1-\alpha}) - \frac{1}{\alpha} + \frac{1}{1-\alpha} &\leq 0 \\ \implies g(\alpha) &\leq 0. \end{aligned}$$

However, from (B.8), we have $g(\alpha) > 0$ when $\alpha \in (\frac{1}{2}, 1)$, and thus, $(8 \ln(k)\alpha - \frac{1}{1-\alpha}) < 0$. Therefore, for any $\alpha \in (0, 1) \setminus \{\frac{1}{2}\}$ that satisfies (B.6), $\frac{\partial^2 f(N_0, \alpha)}{\partial \alpha^2} < 0$. Thus, $f(N_0, \alpha)$ achieves global minima at $\alpha = \frac{1}{2}$. Hence, for any $\alpha \neq \frac{1}{2}$, we have $f(N_0, \alpha) > f(N_0, \frac{1}{2}) = 2e^{-\ln(k)} = \delta$. Clearly, for an $\alpha \in (0, 1) \setminus \{\frac{1}{2}\}$, the tuple (α, N_o) is not a feasible solution

to (3.11). Further, note that, $f(N, \alpha)$ is strictly monotonic decreasing in N . Thus for any $\alpha \in (0, 1) \setminus \{\frac{1}{2}\}$, $M \leq N_o$, $f(M, \alpha) \geq f(N_o, \alpha) > \delta$; thus such a M remains infeasible for $\alpha \neq \frac{1}{2}$. Hence, for any $\alpha \in (0, 1) \setminus \{\frac{1}{2}\}$, we must have $N(\alpha) \geq N(\frac{1}{2})$. This completes the proof.

B.3 Proof of Theorem 5

Note that as (N^*, α^*, π^*) solves (3.15), we have $(N^*, \alpha^*, \pi^*) \in S_1 \subseteq F_1$, and thus (3.14) is satisfied, $\pi^* \in \Pi$, and $P(N^*, \alpha^*, \pi^*) \geq 1 - \delta$. Further, as $N^* \leq N^*$, it follows that $(N^*, \pi^*) \in F_2$ as well and thus $P(N^*, \pi^*, \alpha^*) \leq \nu$, as ν is the optimal value of the maximization problem (3.16). Further, as $(\hat{N}, \hat{\pi})$ solves (3.16), we have $P(\hat{N}, \alpha^*, \hat{\pi}) = \nu \geq P(N^*, \alpha^*, \pi^*) \geq 1 - \delta$. Thus, in summary, we have $P(\hat{N}, \alpha^*, \hat{\pi}) \geq 1 - \delta$ and as $(\hat{N}, \hat{\pi}) \in S_2 \subseteq F_2$, $\hat{\pi} \in \Pi$, $\hat{N} \leq N^*$, and (3.14) is satisfied. Hence, $(\hat{N}, \alpha^*, \hat{\pi}) \in F_1$ as well. Note that, as (3.15) is a minimization problem, it follows that $\hat{N} \geq N^*$. However, as $(\hat{N}, \hat{\pi}) \in S_2 \subseteq F_2$, we get $\hat{N} \leq N^*$. Thus $\hat{N} = N^*$.

We have established that, $(\hat{N}, \alpha^*, \hat{\pi}) \in F_1$, and $\hat{N} = N^*$, thus $(\hat{N}, \alpha, \hat{\pi})$ solves (3.15) as well, that is $(\hat{N}, \alpha^*, \hat{\pi}) \in S_1$. This completes the proof.

B.4 Power transfer policy determination

We discretize the interval $[0, T_f]$ in K sub-intervals $0 = t_0 < t_1 < \dots < t_K = T_f$, with $t_m = m\Delta t$, $m \in \{0, 1, \dots, K\}$, where $\Delta t = \frac{T_f}{K}$. We choose Δt in such a way that $N - 2\bar{P}\Delta t > 0$. In discrete time the policy π is a sequence of K power transfer control laws, that is $\tilde{\pi}^{(0)} := [\pi(t_0) \ \pi(t_1) \ \dots \ \pi(t_{K-1})]$, with $P_{12}(t_m) = \pi(t_m)$ for all $t \in [t_m, t_{m+1})$, $m \in \{0, \dots, K-1\}$. In discrete time, the problem is as follows:

$$\begin{aligned} & \max_{\tilde{\pi}^{(0)} \in \Pi} \mathbb{P}[\cap_{m=0}^K \{\cap_{i=1}^2 \{X_i(t_m) \in (0, N)\}\}] \\ & \text{s. t. } X_i \text{ follows (3.14), } i \in \{1, 2\}. \end{aligned} \tag{B.9}$$

Let $\mathbf{X}(t) := [X_1(t) \ X_2(t)]^T$ denote the vector of the available energy in the BESs of the MGs at time $t \in [0, T_f]$, and let $\mathbf{W}(t) := [W_1(t) \ W_2(t)]^T$ denote the vector of Wiener processes. For sake of notational convenience, we define $B^{(2)} := [0, N] \times [0, N]$. Further, for $\mathbf{x} = [x_1 \ x_2]^T$, we define $\mathbb{1}_{B^{(2)}}(\mathbf{x}) := 1$, if $\mathbf{x} \in B^{(2)}$ and 0 otherwise. Note that given the initial states $\mathbf{X}(0) = \mathbf{x} \notin B^{(2)}$, the objective function of (B.9) is zero. Here, in

vector notation, the objective function of problem (B.9) can be written as follows:

$$\begin{aligned}
& \mathbb{P}[\cap_{m=0}^K \{\mathbf{X}(t_m) \in B^{(2)}\}] \\
&= \mathbb{P}[\cap_{m=1}^K \{\mathbf{X}(t_m) \in B^{(2)}\} | \{\mathbf{X}(0) \in B^{(2)}\}] \mathbb{P}[\mathbf{X}(0) \in B^{(2)}] \\
&= \mathbb{1}_{B^{(2)}}(\mathbf{X}(0)) \mathbb{P}[\cap_{m=1}^K \{\mathbf{X}(t_m) \in B^{(2)}\} | \{\mathbf{X}(0) \in B^{(2)}\}] \quad (\text{B.10})
\end{aligned}$$

Now, consider any time step $t_m = m\Delta t, m \in \{0, 1, \dots, K-1\}$ and let $\tilde{\pi}^{(m)} := [\pi(t_m), \dots, \pi(t_{K-1})]$ denote the sequence of control laws that is to be decided for time $t_m, t_{m+1}, \dots, t_{K-1}$. We define the cost-to-go function for $m \in \{0, 1, \dots, K-1\}$ as follows:

$$J_m(\tilde{\pi}^{(m)}, \mathbf{x}) := \mathbb{1}_{B^{(2)}}(\mathbf{x}) \mathbb{P}[\cap_{s=m+1}^K \{\mathbf{X}(t_s) \in B^{(2)}\} | \mathbf{X}(t_m) = \mathbf{x}].$$

We further define the value function for $m \in \{0, 1, \dots, K-1\}$ as follows:

$$V_m(\mathbf{x}) := \max_{\tilde{\pi}^{(m)} \in \Pi} J_m(\tilde{\pi}^{(m)}, \mathbf{x}).$$

Note that, from (B.10), we have $J_0(\tilde{\pi}^{(0)}, \mathbf{x}) = \mathbb{P}[\cap_{m=0}^K \{\mathbf{X}(t_m) \in B^{(2)}\}]$. Hence, it follows that $V_0(\mathbf{x}) = \max_{\tilde{\pi}^{(0)} \in \Pi} J_0(\tilde{\pi}^{(0)}, \mathbf{x})$ is the solution to (B.9). In the subsequent sections, we derive the optimal policy $\tilde{\pi}^{*(0)} := [\pi^*(0) \ \pi^*(t_1) \ \dots \ \pi^*(t_{K-1})] = \arg \max_{\tilde{\pi}^{(0)} \in \Pi} J_0(\tilde{\pi}^{(0)}, \mathbf{x})$ that achieves the optimal value $V_0(\mathbf{x})$ using dynamic programming.

We introduce the following notation for ease of exposition. Let ξ and Z be two random variables. We define the conditional expectation of a function $f(\xi)$ of the random variable ξ , $\hat{\mathbb{E}}_{\xi|Z=z}[f(\xi)] := \int_{\xi} f(\mathbf{y}) \mathbb{P}[\xi \in d\mathbf{y} | Z = z]$.

Below, we show that the following recursion solves the optimization problem (B.9):

$$\begin{aligned}
V_K(\mathbf{x}) &= \mathbb{1}_{B^{(2)}}(\mathbf{x}), \\
V_m(\mathbf{x}) &= \max_{\pi(t_m) \in \Pi} \mathbb{1}_{B^{(2)}}(\mathbf{x}) \hat{\mathbb{E}}_{\mathbf{X}(t_{m+1}) | \mathbf{X}(t_m) = \mathbf{x}} [V_{m+1}(\mathbf{X}(t_{m+1}))], m \in \{0, 1, \dots, K-1\}.
\end{aligned} \quad (\text{B.11})$$

We establish the above recursion using the induction principle. Here, we start with the base case with $V_{K-1}(\mathbf{x})$.

Base case: For t_{K-1} , we have,

$$\begin{aligned}
V_{K-1}(\mathbf{x}) &= \max_{\tilde{\pi}^{(K-1)} \in \Pi} J_{K-1}(\tilde{\pi}^{(K-1)}, \mathbf{x}) \\
&= \max_{\pi(t_{K-1}) \in \Pi} \mathbf{1}_{B^{(2)}}(\mathbf{x}) \mathbb{P}[\mathbf{X}(t_K) \in B^{(2)} | \mathbf{X}(t_{K-1}) = \mathbf{x}] \\
&= \max_{\pi(t_{K-1}) \in \Pi} \mathbf{1}_{B^{(2)}} \int \mathbf{1}_{B^{(2)}}(\mathbf{y}) \mathbb{P}[\mathbf{X}(t_K) \in d\mathbf{y} | \mathbf{X}(t_{K-1}) = \mathbf{x}] \\
&= \max_{\pi(t_{K-1}) \in \Pi} \mathbf{1}_{B^{(2)}} \hat{\mathbb{E}}_{\mathbf{X}(t_K) | \mathbf{X}(t_{K-1}) = \mathbf{x}}[\mathbf{1}_{B^{(2)}}(\mathbf{X}(t_K))] \\
&= \max_{\pi(t_{K-1}) \in \Pi} \mathbf{1}_{B^{(2)}} \hat{\mathbb{E}}_{\mathbf{X}(t_K) | \mathbf{X}(t_{K-1}) = \mathbf{x}}[V_K(\mathbf{X}(t_K))],
\end{aligned}$$

where the last equality follows since $V_K(\mathbf{x}) = \mathbf{1}_{B^{(2)}}(\mathbf{x})$ by definition.

Now, consider any $m \in \{0, 1, \dots, K-2\}$ and presume that $V_{m+1}(\mathbf{x})$ is known. Here we have the following,

$$\begin{aligned}
V_m(\mathbf{x}) &= \max_{\tilde{\pi}^{(m)} \in \Pi} J_m(\tilde{\pi}^{(m)}, \mathbf{x}) \\
&= \max_{\tilde{\pi}^{(m)} \in \Pi} \mathbf{1}_{B^{(2)}}(\mathbf{x}) \mathbb{P}[\cap_{s=m+1}^K \{\mathbf{X}(t_s) \in B^{(2)}\} | \mathbf{X}(t_m) = \mathbf{x}] \\
&= \max_{\tilde{\pi}^{(m)} \in \Pi} \mathbf{1}_{B^{(2)}}(\mathbf{x}) \mathbb{P}[\cap_{s=m+2}^K \{\mathbf{X}(t_s) \in B^{(2)}\} \cap \{\mathbf{X}(t_{m+1}) \in B^{(2)}\} | \mathbf{X}(t_m) = \mathbf{x}] \\
&= \max_{\tilde{\pi}^{(m)} \in \Pi} \mathbf{1}_{B^{(2)}}(\mathbf{x}) \mathbb{P}[\cap_{s=m+2}^K \{\mathbf{X}(t_s) \in B^{(2)}\} | \{\mathbf{X}(t_{m+1}) \in B^{(2)}\}, \mathbf{X}(t_m) = \mathbf{x}] \\
&\quad \mathbb{P}[\{\mathbf{X}(t_{m+1}) \in B^{(2)}\} | \mathbf{X}(t_m) = \mathbf{x}].
\end{aligned}$$

By the Markov property of the process $\mathbf{X}(\cdot)$, we have

$$\begin{aligned}
&\mathbb{P}[\cap_{s=m+2}^K \{\mathbf{X}(t_s) \in B^{(2)}\} | \{\mathbf{X}(t_{m+1}) \in B^{(2)}\}, \mathbf{X}(t_m) = \mathbf{x}] \\
&= \mathbb{P}[\cap_{s=m+2}^K \{\mathbf{X}(t_s) \in B^{(2)}\} | \mathbf{X}(t_{m+1}) \in B^{(2)}].
\end{aligned}$$

Therefore, it follows that,

$$\begin{aligned}
V_m(\mathbf{x}) &= \max_{\tilde{\pi}^{(m)} \in \Pi} \mathbf{1}_{B^{(2)}}(\mathbf{x}) \mathbb{P}[\cap_{s=m+2}^K \{\mathbf{X}(t_s) \in B^{(2)}\} | \mathbf{X}(t_{m+1}) \in B^{(2)}] \\
&\quad \mathbb{P}[\{\mathbf{X}(t_{m+1}) \in B^{(2)}\} | \mathbf{X}(t_m) = \mathbf{x}].
\end{aligned}$$

Now recall the definition $\tilde{\pi}^{(m)} = [\pi(t_m), \dots, \pi(t_{K-1})]$. Here, note, since the process $\mathbf{X}(\cdot)$ is Markov, given the event $\{\mathbf{X}(t_{m+1}) \in B^{(2)}\}$, $\mathbb{P}[\cap_{s=m+2}^K \{\mathbf{X}(t_s) \in B^{(2)}\} | \mathbf{X}(t_{m+1}) \in$

$B^{(2)}]$ depends only on the future control laws, that is $\tilde{\pi}^{(m+1)}$. Hence, it follows that

$$\begin{aligned}
V_m(\mathbf{x}) &= \max_{\tilde{\pi}^{(m)} \in \Pi} \mathbb{1}_{B^{(2)}}(\mathbf{x}) \mathbb{P}[\cap_{s=m+2}^K \{\mathbf{X}(t_s) \in B^{(2)}\} | \mathbf{X}(t_{m+1}) \in B^{(2)}] \\
&\quad \mathbb{P}[\{\mathbf{X}(t_{m+1}) \in B^{(2)}\} | \mathbf{X}(t_m) = \mathbf{x}] \\
&= \max_{\pi^{(m)} \in \Pi} \mathbb{1}_{B^{(2)}}(\mathbf{x}) \left[\max_{\tilde{\pi}^{(m+1)} \in \Pi} \mathbb{P}[\cap_{s=m+2}^K \{\mathbf{X}(t_s) \in B^{(2)}\} | \mathbf{X}(t_{m+1}) \in B^{(2)}] \right] \\
&\quad \mathbb{P}[\{\mathbf{X}(t_{m+1}) \in B^{(2)}\} | \mathbf{X}(t_m) = \mathbf{x}] \\
&= \max_{\pi^{(m)} \in \Pi} \mathbb{1}_{B^{(2)}}(\mathbf{x}) \int_{\mathbf{X}(t_{m+1})} \mathbb{1}_{B^{(2)}}(\mathbf{y}) \\
&\quad \max_{\tilde{\pi}^{(m+1)} \in \Pi} \mathbb{P}[\cap_{s=m+2}^K \{\mathbf{X}(t_s) \in B^{(2)}\} | \mathbf{X}(t_{m+1}) = \mathbf{y}] \mathbb{P}[\mathbf{X}(t_{m+1}) \in d\mathbf{y} | \mathbf{X}(t_m) = \mathbf{x}] \\
&= \max_{\pi^{(m)} \in \Pi} \mathbb{1}_{B^{(2)}}(\mathbf{x}) \int_{\mathbf{X}(t_{m+1})} \max_{\tilde{\pi}^{(m+1)} \in \Pi} \mathbb{1}_{B^{(2)}}(\mathbf{y}) \\
&\quad \mathbb{P}[\cap_{s=m+2}^K \{\mathbf{X}(t_s) \in B^{(2)}\} | \mathbf{X}(t_{m+1}) = \mathbf{y}] \mathbb{P}[\mathbf{X}(t_{m+1}) \in d\mathbf{y} | \mathbf{X}(t_m) = \mathbf{x}] \\
&= \max_{\pi^{(m)} \in \Pi} \mathbb{1}_{B^{(2)}}(\mathbf{x}) \int_{\mathbf{X}(t_{m+1})} \max_{\tilde{\pi}^{(m+1)} \in \Pi} J(\tilde{\pi}^{(m+1)}, \mathbf{y}) \\
&\quad \mathbb{P}[\mathbf{X}(t_{m+1}) \in d\mathbf{y} | \mathbf{X}(t_m) = \mathbf{x}] \\
&= \max_{\pi^{(m)} \in \Pi} \mathbb{1}_{B^{(2)}}(\mathbf{x}) \int_{\mathbf{X}(t_{m+1})} V_{m+1}(\mathbf{y}) \mathbb{P}[\mathbf{X}(t_{m+1}) \in d\mathbf{y} | \mathbf{X}(t_m) = \mathbf{x}] \\
&= \max_{\pi^{(m)} \in \Pi} \mathbb{1}_{B^{(2)}}(\mathbf{x}) \hat{\mathbb{E}}_{\mathbf{X}(t_{m+1}) | \mathbf{X}(t_m) = \mathbf{x}} [V_{m+1}(\mathbf{X}(t_{m+1}))].
\end{aligned}$$

Thus, by induction, we have that, using the recursion,

$$\begin{aligned}
V_K(\mathbf{x}) &= \mathbb{1}_{B^{(2)}}(\mathbf{x}), \\
V_m(\mathbf{x}) &= \max_{\pi^{(m)} \in \Pi} \mathbb{1}_{B^{(2)}}(\mathbf{x}) \hat{\mathbb{E}}_{\mathbf{X}(t_{m+1}) | \mathbf{X}(t_m) = \mathbf{x}} [V_{m+1}(\mathbf{X}(t_{m+1}))], m \in \{0, 1, \dots, K-1\},
\end{aligned}$$

we can find $V_0(\mathbf{x})$, which is the solution to (B.9).

Next, to derive the optimal policy, we first analyze the function $V_m(\mathbf{x})$, $m \in \{0, 1, \dots, K\}$.

Suppose $V_m(\mathbf{x})$ satisfies the following properties:

$$\begin{aligned}
&\text{Property 1: Let the vectors } \hat{\mathbf{x}} = [\hat{x}_1 \ \hat{x}_2]^T \in B^{(2)}, \text{ and } \mathbf{x}' = [x'_1 \ x'_2]^T \in B^{(2)}, \\
&\text{be such that } |\hat{x}_1 - \hat{x}_2| \leq |x'_1 - x'_2|, \text{ then } V_m(\hat{\mathbf{x}}) \geq V_m(\mathbf{x}'). \tag{B.12}
\end{aligned}$$

$$\begin{aligned}
&\text{Property 2: Let the vectors } \hat{\mathbf{x}} = [\hat{x}_1 \ \hat{x}_2]^T \in B^{(2)}, \text{ and } \mathbf{x}' = [x'_1 \ x'_2]^T \in B^{(2)}, \\
&\text{be such that } \hat{x}_1 = x'_2, \ \hat{x}_2 = x'_1, \text{ thus satisfying } \hat{x}_1 - \hat{x}_2 = -(x'_1 - x'_2), \\
&\text{then } V_m(\hat{\mathbf{x}}) = V_m(\mathbf{x}'). \tag{B.13}
\end{aligned}$$

Note that, $V_K(\mathbf{x})$ satisfies the aforementioned properties. Now, presume that $V_{m+1}(\mathbf{x})$

also satisfy the aforementioned property for $m \in \{0, 1, \dots, K-1\}$. In the next section, we establish that $\pi^*(t_m)$ is of bang-bang type, and $V_m(\mathbf{x})$ also satisfies the properties (B.12)-(B.13).

DP Recursion: Bang-Bang Policy:

Recall from (3.14) that, given $\mathbf{X}(t_m) = \mathbf{x} = [x_1 \ x_2]^T$, the components of the vector $\mathbf{X}(t_{m+1})$, that is $X_i(t_{m+1}), i \in \{1, 2\}$ can be written as follows:

$$\begin{aligned} X_i(t_{m+1}) &= X_i(t_m) + \sigma(W_i(t_{m+1}) - W_i(t_m)) + \bar{B}(-1)^i P_{12,m} \Delta t \\ &= x_i + \sigma(W_i(t_{m+1}) - W_i(t_m)) + \bar{B}(-1)^i P_{12,m} \Delta t. \end{aligned}$$

Since, $W_i(t_{m+1}) - W_i(t_m) \sim \mathcal{N}(0, \Delta t)$, we have $X_i(t_{m+1}) \sim \mathcal{N}(x_i + \bar{B}(-1)^i P_{12,m} \Delta t, \sigma^2 \Delta t)$. Further, note that given $\mathbf{X}(t_m) = \mathbf{x}$, $X_1(t_{m+1})$ and $X_2(t_{m+1})$ are independent of each other. Therefore, we have $\mathbb{P}[\mathbf{X}(t_{m+1}) \in d\mathbf{y} | \mathbf{X}(t_m) = \mathbf{x}] = \prod_{i=1}^2 \mathbb{P}[X_i(t_{m+1}) \in dy_i | X_i(t_m) = x_i]$. Hence, from the recursion (B.11), for $m \in \{0, 1, \dots, K-1\}$, we have

$$\begin{aligned} V_m(\mathbf{x}) &= \max_{\pi(t_m) \in \Pi} \mathbb{1}_{B^{(2)}}(\mathbf{x}) \mathbb{E}[V_{m+1}(\mathbf{X}(t_{m+1}))] \\ &= \max_{\pi(t_m) \in \Pi} \mathbb{1}_{B^{(2)}}(\mathbf{x}) \int \int V_{m+1}(\mathbf{y}) \prod_{i=1}^2 \mathbb{P}[X_i(t_{m+1}) \in dy_i | X_i(t_m) = x_i] \\ &= \max_{\pi(t_m) \in \Pi} \mathbb{1}_{B^{(2)}}(\mathbf{x}) \int \int V_{m+1}(\mathbf{y}) \prod_{i=1}^2 \phi\left(\frac{y_i - x_i - \bar{B}(-1)^i P_{12,m} \Delta t}{\sigma \sqrt{\Delta t}}\right) dy_i, \quad (\text{B.14}) \end{aligned}$$

where $\phi(\cdot)$ denotes the probability density function of a standard normal random variable.

From (B.14), considering $\mathbf{x} \in B^{(2)}$, we have

$$\begin{aligned} V_m(\mathbf{x}) &= \max_{\pi(t_m) \in \Pi} \int \int V_{m+1}(\mathbf{y}) \phi\left(\frac{y_1 - x_1 + \bar{B} P_{12,m} \Delta t}{\sigma \sqrt{\Delta t}}\right) \phi\left(\frac{y_2 - x_2 - \bar{B} P_{12,m} \Delta t}{\sigma \sqrt{\Delta t}}\right) dy_1 dy_2 \\ &= \max_{\pi(t_m) \in \Pi} \hat{J}_m(\pi(t_m), \mathbf{x}), \quad (\text{B.15}) \end{aligned}$$

where we define

$$\hat{J}_m(\pi(t_m), \mathbf{x}) := \int \int V_{m+1}(\mathbf{y}) \phi\left(\frac{y_1 - x_1 + \bar{B} P_{12,m} \Delta t}{\sigma \sqrt{\Delta t}}\right) \phi\left(\frac{y_2 - x_2 - \bar{B} P_{12,m} \Delta t}{\sigma \sqrt{\Delta t}}\right) dy_1 dy_2, \quad (\text{B.16})$$

for brevity of notation. In the subsequent paragraphs, we analyze $\hat{J}_m(\cdot)$.

We consider the following transformation of variables $z^- := y_1 - y_2$, and $z^+ := y_1 + y_2$. Recall from (B.11) that $V_{m+1}(\mathbf{y}) = 0$ if $\mathbf{y} \notin B^{(2)}$. Therefore, considering $\mathbf{y} \in B^{(2)}$, we have, $z^+ \in (0, 2N)$ and $z^- \in D(z^+)$, where $D(z^+) := (-\min(z^+, 2N - z^+), \min(z^+, 2N - z^+))$. Here, in matrix form, $[z^- \ z^+]^T = \mathbf{T}[y_1 \ y_2]^T$, where $\mathbf{T} =$

$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Note that, $y_1 = \frac{1}{2}(z^+ + z^-)$, $y_2 = \frac{1}{2}(z^+ - z^-)$, and $\det(\mathbf{T}) = 2$. Further, denote $V'_{m+1}(z^-, z^+) := V_{m+1}(\frac{1}{2}(z^+ + z^-), \frac{1}{2}(z^+ - z^-))$. By the aforementioned change of variables, from (B.16), we have the following:

$$\begin{aligned} \hat{J}_m(\cdot) &= \int_{z^+ \in (0, 2N)} \int_{z^- \in D(z^+)} V_{m+1}(\frac{1}{2}(z^+ + z^-), \frac{1}{2}(z^+ - z^-)) \\ &\quad \det(\mathbf{T}) \phi\left(\frac{\frac{1}{2}(z^+ + z^-) - x_1 + \bar{B}P_{12,m}\Delta t}{\sigma\sqrt{\Delta t}}\right) \phi\left(\frac{\frac{1}{2}(z^+ - z^-) - x_2 - \bar{B}P_{12,m}\Delta t}{\sigma\sqrt{\Delta t}}\right) dz^- dz^+ \\ &= \int_{z^+ \in (0, 2N)} \int_{z^- \in D(z^+)} V'_{m+1}(z^-, z^+) \\ &\quad 2\phi\left(\frac{\frac{1}{2}(z^+ + z^-) - x_1 + \bar{B}P_{12,m}\Delta t}{\sigma\sqrt{\Delta t}}\right) \phi\left(\frac{\frac{1}{2}(z^+ - z^-) - x_2 - \bar{B}P_{12,m}\Delta t}{\sigma\sqrt{\Delta t}}\right) dz^- dz^+. \quad (\text{B.17}) \end{aligned}$$

Note that, since $V'_{m+1}(z^-, z^+) = V_{m+1}(\frac{1}{2}(z^+ + z^-), \frac{1}{2}(z^+ - z^-))$, from the properties (B.12)-(B.13) of $V_{m+1}(\cdot)$, it follows that $V'_{m+1}(z^-, \cdot)$ has the following properties:

$$\begin{aligned} V'_{m+1}(\hat{z}^-, z^+) &\geq V'_{m+1}(z'^-, z^+), \text{ if } |\hat{z}^-| < |z'^-|, \text{ for any given } z^+, \text{ and,} \\ V'_{m+1}(z^-, z^+) &= V'_{m+1}(-z^-, z^+), \text{ for any given } z^+. \quad (\text{B.18}) \end{aligned}$$

We first analyze the product $\phi\left(\frac{\frac{1}{2}(z^+ + z^-) - x_1 + \bar{B}P_{12,m}\Delta t}{\sigma\sqrt{\Delta t}}\right) \phi\left(\frac{\frac{1}{2}(z^+ - z^-) - x_2 - \bar{B}P_{12,m}\Delta t}{\sigma\sqrt{\Delta t}}\right)$ that appears in the integrand of (B.17). Here, it can be shown that,

$$\begin{aligned} &\phi\left(\frac{\frac{1}{2}(z^+ + z^-) - x_1 + \bar{B}P_{12,m}\Delta t}{\sigma\sqrt{\Delta t}}\right) \phi\left(\frac{\frac{1}{2}(z^+ - z^-) - x_2 - \bar{B}P_{12,m}\Delta t}{\sigma\sqrt{\Delta t}}\right) \\ &= \phi\left(\frac{z^+ - (x_1 + x_2)}{\sigma\sqrt{2\Delta t}}\right) \phi\left(\frac{z^- - (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t)}{\sigma\sqrt{2\Delta t}}\right). \end{aligned}$$

Thus, from (B.17), we have the following:

$$\begin{aligned} \hat{J}_m(\cdot) &= \int_{z^+ \in (0, 2N)} \int_{z^- \in D(z^+)} 2V'_{m+1}(z^-, z^+) \phi\left(\frac{z^+ - (x_1 + x_2)}{\sigma\sqrt{2\Delta t}}\right) \phi\left(\frac{z^- - (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t)}{\sigma\sqrt{2\Delta t}}\right) dz^- dz^+. \quad (\text{B.19}) \end{aligned}$$

Note that $(x_1 + x_2)$ does not depend on $P_{12,m}$. Hence, we can derive the following:

$$\begin{aligned}
& \frac{\partial \hat{J}_m(\cdot)}{\partial P_{12,m}} \\
&= \int_{z^+ \in (0, 2N)} \int_{z^- \in D(z^+)} 2V'_{m+1}(z^-, z^+) \phi\left(\frac{z^+ - (x_1 + x_2)}{\sigma\sqrt{2\Delta t}}\right) \left[\frac{\partial \phi\left(\frac{z^- - (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t)}{\sigma\sqrt{2\Delta t}}\right)}{\partial P_{12,m}} \right] dz^- dz^+ \\
&= \int_{z^+ \in (0, 2N)} \phi\left(\frac{z^+ - (x_1 + x_2)}{\sigma\sqrt{2\Delta t}}\right) \left[\int_{z^- \in D(z^+)} 2V'_{m+1}(z^-, z^+) \right. \\
&\quad \left. \left[\frac{\partial \phi\left(\frac{z^- - (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t)}{\sigma\sqrt{2\Delta t}}\right)}{\partial P_{12,m}} \right] dz^- \right] dz^+ \\
&= \int_{z^+ \in (0, 2N)} \phi\left(\frac{z^+ - (x_1 + x_2)}{\sigma\sqrt{2\Delta t}}\right) I_-(z^+, P_{12,m}) dz^+, \tag{B.20}
\end{aligned}$$

where we define the inner integral by $I_-(\cdot)$, that is,

$$I_-(z^+, P_{12,m}) := \int_{z^- \in D(z^+)} 2V'_{m+1}(z^-, z^+) \left[\frac{\partial \phi\left(\frac{z^- - (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t)}{\sigma\sqrt{2\Delta t}}\right)}{\partial P_{12,m}} \right] dz^-.$$

Note that,

$$\begin{aligned}
& \frac{\partial \phi\left(\frac{z^- - (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t)}{\sigma\sqrt{2\Delta t}}\right)}{\partial P_{12,m}} \\
&= -\left(\frac{z^- - (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t)}{\sigma\sqrt{2\Delta t}}\right) \phi\left(\frac{z^- - (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t)}{\sigma\sqrt{2\Delta t}}\right) \frac{\partial\left(\frac{z^- - (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t)}{\sigma\sqrt{2\Delta t}}\right)}{\partial P_{12,m}} \\
&= -\left(\frac{z^- - (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t)}{\sigma\sqrt{2\Delta t}}\right) \phi\left(\frac{z^- - (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t)}{\sigma\sqrt{2\Delta t}}\right) \left(\frac{\bar{B}\sqrt{2\Delta t}}{\sigma}\right).
\end{aligned}$$

Therefore we have,

$$\begin{aligned}
& I_-(z^+, P_{12,m}) \\
&= -\left(\frac{\bar{B}\sqrt{2\Delta t}}{\sigma}\right) \int_{z^- \in D(z^+)} 2V'_{m+1}(z^-, z^+) \\
&\quad \left(\frac{z^- - (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t)}{\sigma\sqrt{2\Delta t}}\right) \phi\left(\frac{z^- - (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t)}{\sigma\sqrt{2\Delta t}}\right) dz^-, \tag{B.21}
\end{aligned}$$

which can further be written as:

$$\begin{aligned}
& I_-(z^+, P_{12,m}) \\
&= -\left(\frac{\bar{B}\sqrt{2\Delta t}}{\sigma}\right) \left[\int_{z^- \in D(z^+) \cap [x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t, \infty)} 2V'_{m+1}(z^-, z^+) \left(\frac{z^- - (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t)}{\sigma\sqrt{2\Delta t}}\right) \right. \\
&\quad \left. \phi\left(\frac{z^- - (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t)}{\sigma\sqrt{2\Delta t}}\right) dz^- \right. \\
&\quad \left. + \int_{z^- \in D(z^+) \cap (-\infty, x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t]} 2V'_{m+1}(z^-, z^+) \left(\frac{z^- - (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t)}{\sigma\sqrt{2\Delta t}}\right) \right. \\
&\quad \left. \phi\left(\frac{z^- - (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t)}{\sigma\sqrt{2\Delta t}}\right) dz^- \right].
\end{aligned}$$

Using the change of variables, $k^- = z^- - (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t)$, and defining $D'(z^+) := (-\min(z^+, 2N - z^+) - (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t), \min(z^+, 2N - z^+) - (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t))$, we have the following:

$$\begin{aligned}
& I_-(z^+, P_{12,m}) \\
&= -\left(\frac{\bar{B}\sqrt{2\Delta t}}{\sigma}\right) \\
&\left[\int_{k^- \in D'(z^+) \cap [0, \infty)} 2V'_{m+1}(k^- + (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t), z^+) \left(\frac{k^-}{\sigma\sqrt{2\Delta t}}\right) \phi\left(\frac{k^-}{\sigma\sqrt{2\Delta t}}\right) dk^- \right. \\
&\quad \left. + \int_{k^- \in D'(z^+) \cap (-\infty, 0]} 2V'_{m+1}(k^- + (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t), z^+) \left(\frac{k^-}{\sigma\sqrt{2\Delta t}}\right) \phi\left(\frac{k^-}{\sigma\sqrt{2\Delta t}}\right) dk^- \right] \\
&= -\left(\frac{\bar{B}\sqrt{2\Delta t}}{\sigma}\right) [I_1 + I_2], \tag{B.22}
\end{aligned}$$

where,

$$\begin{aligned}
I_1 &:= \int_{k^- \in D'(z^+) \cap [0, \infty)} 2V'_{m+1}(k^- + (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t), z^+) \left(\frac{k^-}{\sigma\sqrt{2\Delta t}}\right) \phi\left(\frac{k^-}{\sigma\sqrt{2\Delta t}}\right) dk^-, \\
I_2 &:= \int_{k^- \in D'(z^+) \cap (-\infty, 0]} 2V'_{m+1}(k^- + (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t), z^+) \left(\frac{k^-}{\sigma\sqrt{2\Delta t}}\right) \phi\left(\frac{k^-}{\sigma\sqrt{2\Delta t}}\right) dk^-.
\end{aligned}$$

Now, note that,

$$\begin{aligned}
I_2 &= \int_{k^- \in D''(z^+) \cap (-\infty, 0]} 2V'_{m+1}(k^- + (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t), z^+) \left(\frac{k^-}{\sigma\sqrt{2\Delta t}}\right) \phi\left(\frac{k^-}{\sigma\sqrt{2\Delta t}}\right) dk^- \\
&= \int_{k^- \in D''(z^+) \cap [0, \infty)} 2V'_{m+1}(-k^- + (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t), z^+) \left(\frac{-k^-}{\sigma\sqrt{2\Delta t}}\right) \phi\left(\frac{-k^-}{\sigma\sqrt{2\Delta t}}\right) dk^-,
\end{aligned}$$

where $D''(z^+) := (-\min(z^+, 2N - z^+) + (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t), \min(z^+, 2N - z^+) + (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t))$. Further, since $\phi(\frac{-k^-}{\sigma\sqrt{2\Delta t}}) = \phi(\frac{k^-}{\sigma\sqrt{2\Delta t}})$, and $V'_{m+1}(-k^- + (x_1 -$

$x_2 - 2\bar{B}P_{12,m}\Delta t, \cdot) = V'_{m+1}(k^- - (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t), \cdot)$, we have,

$$I_2 = - \int_{k^- \in D''(z^+) \cap [0, \infty)} 2V'_{m+1}(k^- - (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t), z^+) \left(\frac{k^-}{\sigma\sqrt{2\Delta t}}\right) \phi\left(\frac{k^-}{\sigma\sqrt{2\Delta t}}\right) dk^-.$$

Therefore, from (B.22), it follows that

$$\begin{aligned} I_-(z^+, P_{12,m}) &= -\left(\frac{\bar{B}\sqrt{2\Delta t}}{\sigma}\right) \\ &\left[\int_{k^- \in D'(z^+) \cap [0, \infty)} 2V'_{m+1}(k^- + (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t), z^+) \left(\frac{k^-}{\sigma\sqrt{2\Delta t}}\right) \phi\left(\frac{k^-}{\sigma\sqrt{2\Delta t}}\right) dk^- \right. \\ &\quad \left. - \int_{k^- \in D''(z^+) \cap [0, \infty)} 2V'_{m+1}(k^- - (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t), z^+) \left(\frac{k^-}{\sigma\sqrt{2\Delta t}}\right) \phi\left(\frac{k^-}{\sigma\sqrt{2\Delta t}}\right) dk^- \right]. \end{aligned} \quad (\text{B.23})$$

We now consider two cases; case 1: $x_1 - x_2 > 2\bar{B}\bar{P}\Delta t$, and case 2: $x_1 - x_2 < -2\bar{B}\bar{P}\Delta t$, and show that for case 1, $I_-(\cdot) > 0$, and for case 2, $I_-(\cdot) < 0$.

First, consider the case 1, $x_1 - x_2 > 2\bar{B}\bar{P}\Delta t$. Here, we have $x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t > 0$, since $-\bar{P} \leq P_{12,m} \leq \bar{P}$. Now, recall that,

$$\begin{aligned} D'(z^+) &= (-\min(z^+, 2N - z^+) - (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t), \\ &\quad \min(z^+, 2N - z^+) - (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t)) \\ D''(z^+) &= (-\min(z^+, 2N - z^+) + (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t), \\ &\quad \min(z^+, 2N - z^+) + (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t)) \end{aligned}$$

In the case of $x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t > 0$, we have, $\{D'(z^+) \cap [0, \infty)\} \subset \{D''(z^+) \cap [0, \infty)\}$. Thus, from (B.23), we have,

$$\begin{aligned} I_-(z^+, P_{12,m}) &\geq -\left(\frac{\bar{B}\sqrt{2\Delta t}}{\sigma}\right) \\ &\left[\int_{k^- \in D'(z^+) \cap [0, \infty)} 2V'_{m+1}(k^- + (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t), z^+) \left(\frac{k^-}{\sigma\sqrt{2\Delta t}}\right) \phi\left(\frac{k^-}{\sigma\sqrt{2\Delta t}}\right) dk^- \right. \\ &\quad \left. - \int_{k^- \in D''(z^+) \cap [0, \infty)} 2V'_{m+1}(k^- - (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t), z^+) \left(\frac{k^-}{\sigma\sqrt{2\Delta t}}\right) \phi\left(\frac{k^-}{\sigma\sqrt{2\Delta t}}\right) dk^- \right] \\ &= -\left(\frac{2\bar{B}\sqrt{2\Delta t}}{\sigma}\right) \left[\int_{k^- \in D'(z^+) \cap [0, \infty)} \left(\frac{k^-}{\sigma\sqrt{2\Delta t}}\right) \phi\left(\frac{k^-}{\sigma\sqrt{2\Delta t}}\right) \right. \\ &\quad \left. \left[V'_{m+1}(k^- + (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t), \cdot) - V'_{m+1}(k^- - (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t), \cdot) \right] dk^- \right]. \end{aligned} \quad (\text{B.24})$$

Note that, in the region $\{D'(z^+) \cap [0, \infty)\}$, $k \geq 0$. Further, since under the case being considered we have $x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t > 0$, it follows that $k^- + (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t) > k^- - (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t)$. Furthermore, in the case of $k^- + (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t) > k^- - (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t) \geq 0$, we have $|k^- + (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t)| > |k^- - (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t)|$, and in the case of $k^- + (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t) > 0 > k^- - (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t)$, we have $k^- + (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t) > -(k^- - (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t)) > 0$ (since $k^- \geq 0 \geq -k^-$), which further implies $|k^- + (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t)| > |-(k^- - (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t))| = |(k^- - (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t))|$. Thus, from the property (B.18) of the function $V'_{m+1}(\cdot)$, it follows that, $V'_{m+1}(k^- + (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t), \cdot) \leq V'_{m+1}(k^- - (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t), \cdot)$. Therefore, from (B.24), we have $I_-(\cdot) > 0$.

Similarly, for case 2, $x_1 - x_2 < -2\bar{B}P_{12,m}\Delta t$, we have $x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t < 0$, since $-\bar{P} \leq P_{12,m} \leq \bar{P}$. In this case, we have, $\{D''(z^+) \cap [0, \infty)\} \subset \{D'(z^+) \cap [0, \infty)\}$. Thus, from (B.24), it follows that

$$\begin{aligned}
& I_-(z^+, P_{12,m}) \\
& \leq -\left(\frac{\bar{B}\sqrt{2\Delta t}}{\sigma}\right) \\
& \left[\int_{k^- \in D''(z^+) \cap [0, \infty)} 2V'_{m+1}(k^- + (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t), z^+) \left(\frac{k^-}{\sigma\sqrt{2\Delta t}}\right) \phi\left(\frac{k^-}{\sigma\sqrt{2\Delta t}}\right) dk^- \right. \\
& \quad \left. - \int_{k^- \in D''(z^+) \cap [0, \infty)} 2V'_{m+1}(k^- - (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t), z^+) \left(\frac{k^-}{\sigma\sqrt{2\Delta t}}\right) \phi\left(\frac{k^-}{\sigma\sqrt{2\Delta t}}\right) dk^- \right] \\
& = -\left(\frac{2\bar{B}\sqrt{2\Delta t}}{\sigma}\right) \left[\int_{k^- \in D''(z^+) \cap [0, \infty)} \left(\frac{k^-}{\sigma\sqrt{2\Delta t}}\right) \phi\left(\frac{k^-}{\sigma\sqrt{2\Delta t}}\right) \right. \\
& \quad \left. \left[V'_{m+1}(k^- + (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t), \cdot) - V'_{m+1}(k^- - (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t), \cdot) \right] dk^- \right] \tag{B.25}
\end{aligned}$$

In the region $\{D''(z^+) \cap [0, \infty)\}$, $k \geq 0$. Further, since under this case $x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t < 0$, we have $k^- - (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t) > k^- + (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t)$, and hence, $V'_{m+1}(k^- - (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t), \cdot) \leq V'_{m+1}(k^- + (x_1 - x_2 - 2\bar{B}P_{12,m}\Delta t), \cdot)$. Therefore, from (B.25), we have $I_-(\cdot) < 0$.

Now, recall from (B.20), that $\frac{\partial \hat{J}_m(\cdot)}{\partial P_{12,m}} = \int_{z^+ \in (0, 2N)} \phi\left(\frac{z^+ - (x_1 + x_2)}{\sigma\sqrt{2\Delta t}}\right) I_-(z^+, P_{12,m}) dz^+$. Since $\phi\left(\frac{z^+ - (x_1 + x_2)}{\sigma\sqrt{2\Delta t}}\right) > 0$, in summary, we have the following:

$$\begin{aligned}
& \frac{\partial \hat{J}_m(\cdot)}{\partial P_{12,m}} > 0, \text{ if } x_1 - x_2 > 2\bar{B}\bar{P}\Delta t, \\
& \frac{\partial \hat{J}_m(\cdot)}{\partial P_{12,m}} < 0, \text{ if } x_1 - x_2 < -2\bar{B}\bar{P}\Delta t. \tag{B.26}
\end{aligned}$$

Since $P_{12,m} \in [-\bar{P}, \bar{P}]$, it follows that under the case $x_1 - x_2 > 2\bar{B}\bar{P}\Delta t$, $\hat{J}_m(\cdot)$ is maximized when $P_{12,m} = \bar{P}$, and under the case $x_1 - x_2 < -2\bar{B}\bar{P}\Delta t$, $\hat{J}_m(\cdot)$ is maximized when $P_{12,m} = -\bar{P}$.

Recall from (B.15), that $V_m(\mathbf{x}) = \max_{\pi(t_m) \in \Pi} \hat{J}_m(\pi(t_m), \mathbf{x})$. Let

$$\pi^*(t_m) := \arg \max_{\pi(t_m) \in \Pi} \hat{J}_m(\pi(t_m), \mathbf{x}),$$

and let $P_{12,m}^* = \pi^*(t_m)$. From, our previous analysis, it follows that under the case $x_1 - x_2 > 2\bar{B}\bar{P}\Delta t$, $P_{12,m}^* = \bar{P}$, and under the case $x_1 - x_2 < -2\bar{B}\bar{P}\Delta t$, $\hat{J}_m(\cdot)$, $P_{12,m}^* = -\bar{P}$. When, $|x_1 - x_2| \in [0, 2\bar{B}\bar{P}\Delta t]$, we set $P_{12,m}^* = \frac{x_1 - x_2}{2\bar{B}\Delta t}$, and here note that $\frac{\partial \hat{J}_m(\cdot)}{\partial P_{12,m}} = 0$. Thus, we have,

$$\pi^*(t_m) = P_{12,m}^* = \begin{cases} \bar{P}, & \text{if } x_1 - x_2 > 2\bar{B}\bar{P}\Delta t, \\ -\bar{P}, & \text{if } x_1 - x_2 < -2\bar{B}\bar{P}\Delta t, \\ \frac{x_1 - x_2}{2\bar{B}\Delta t}, & \text{if } |x_1 - x_2| \in [0, 2\bar{B}\bar{P}\Delta t]. \end{cases} \quad (\text{B.27})$$

Since $V_m(\mathbf{x}) = \hat{J}_m(\pi^*(t_m), \mathbf{x}) = \max_{\pi(t_m) \in \Pi} \hat{J}_m(\pi(t_m), \mathbf{x})$, from (B.19), $V_m(\mathbf{x})$ can be written as follows:

$$\begin{aligned} & V_m(x_1, x_2) \\ &= \int_{z^+ \in (0, 2N)} \int_{z^- \in D(z^+)} 2V'_{m+1}(z^-, z^+) \phi\left(\frac{z^+ - (x_1 + x_2)}{\sigma\sqrt{2\Delta t}}\right) \phi\left(\frac{z^- - (x_1 - x_2 - 2P_{12,m}^*\Delta t)}{\sigma\sqrt{2\Delta t}}\right) dz^- dz^+. \end{aligned} \quad (\text{B.28})$$

In the subsequent development, we show that $V_m(x_1, x_2)$ also satisfies the properties (B.12)-(B.13), thereby closing the induction step.

Here, we can derive the following

$$\begin{aligned} & \frac{\partial V_m(\cdot)}{\partial (x_1 - x_2)} \\ &= \int_{z^+ \in (0, 2N)} \int_{z^- \in D(z^+)} 2V'_{m+1}(z^-, z^+) \phi\left(\frac{z^+ - (x_1 + x_2)}{\sigma\sqrt{2\Delta t}}\right) \left[\frac{\partial \phi\left(\frac{z^- - (x_1 - x_2 - 2\bar{B}P_{12,m}^*\Delta t)}{\sigma\sqrt{2\Delta t}}\right)}{\partial (x_1 - x_2)} \right] dz^- dz^+ \\ &= \int_{z^+ \in (0, 2N)} \phi\left(\frac{z^+ - (x_1 + x_2)}{\sigma\sqrt{2\Delta t}}\right) \\ & \quad \left[\int_{z^- \in D(z^+)} 2V'_{m+1}(z^-, z^+) \left[\frac{\partial \phi\left(\frac{z^- - (x_1 - x_2 - 2\bar{B}P_{12,m}^*\Delta t)}{\sigma\sqrt{2\Delta t}}\right)}{\partial (x_1 - x_2)} \right] dz^- \right] dz^+ \\ &= \int_{z^+ \in (0, 2N)} \phi\left(\frac{z^+ - (x_1 + x_2)}{\sigma\sqrt{2\Delta t}}\right) \hat{I}_-(z^+, x_1, x_2) dz^+, \end{aligned} \quad (\text{B.29})$$

where we denote the inner integral by $\hat{I}_-(z^+, x_1, x_2)$, that is,

$$\hat{I}_-(z^+, x_1, x_2) := \int_{z^- \in D(z^+)} 2V'_{m+1}(z^-, z^+) \left[\frac{\partial \phi\left(\frac{z^- - (x_1 - x_2 - 2\bar{B}P_{12,m}^* \Delta t)}{\sigma\sqrt{2\Delta t}}\right)}{\partial(x_1 - x_2)} \right] dz^-.$$

Note that,

$$\begin{aligned} \frac{\partial \phi\left(\frac{z^- - (x_1 - x_2 - 2\bar{B}P_{12,m}^* \Delta t)}{\sigma\sqrt{2\Delta t}}\right)}{\partial(x_1 - x_2)} &= - \left(\frac{z^- - (x_1 - x_2 - 2\bar{B}P_{12,m}^* \Delta t)}{\sigma\sqrt{2\Delta t}} \right) \phi\left(\frac{z^- - (x_1 - x_2 - 2\bar{B}P_{12,m}^* \Delta t)}{\sigma\sqrt{2\Delta t}}\right) \\ &\quad \frac{\partial\left(\frac{z^- - (x_1 - x_2 - 2\bar{B}P_{12,m}^* \Delta t)}{\sigma\sqrt{2\Delta t}}\right)}{\partial(x_1 - x_2)} \\ &= \left(\frac{z^- - (x_1 - x_2 - 2\bar{B}P_{12,m}^* \Delta t)}{\sigma\sqrt{2\Delta t}} \right) \phi\left(\frac{z^- - (x_1 - x_2 - 2\bar{B}P_{12,m}^* \Delta t)}{\sigma\sqrt{2\Delta t}}\right) \left(\frac{1}{\sigma\sqrt{2\Delta t}} \right). \end{aligned}$$

Therefore, we have

$$\begin{aligned} \hat{I}_-(z^+, x_1, x_2) &= \left(\frac{1}{\sigma\sqrt{2\Delta t}} \right) \int_{z^- \in D(z^+)} 2V'_{m+1}(z^-, z^+) \\ &\quad \left(\frac{z^- - (x_1 - x_2 - 2\bar{B}P_{12,m}^* \Delta t)}{\sigma\sqrt{2\Delta t}} \right) \phi\left(\frac{z^- - (x_1 - x_2 - 2\bar{B}P_{12,m}^* \Delta t)}{\sigma\sqrt{2\Delta t}}\right) dz^-. \end{aligned}$$

Recall the definition of $I_-(\cdot)$ (Eq. (B.21)), and note that, the only differences between $I_-(\cdot)$ and $\hat{I}_-(\cdot)$ are (i) in $I_-(\cdot)$ the constant factor outside the integral, $-\frac{\bar{B}\sqrt{2\Delta t}}{\sigma}$, is negative, whereas in $\hat{I}_-(\cdot)$, the constant factor, $\frac{1}{\sigma\sqrt{2\Delta t}}$, is positive, (ii) $P_{12,m}$ in $I_-(\cdot)$ are replaced by $P_{12,m}^*$ in $\hat{I}_-(\cdot)$. Hence, following the similar development as done in the previous analysis, we can show the following:

$$\begin{aligned} \frac{\partial V_m(\cdot)}{\partial(x_1 - x_2)} &< 0, \text{ if } x_1 - x_2 > 2\bar{B}\bar{P}\Delta t, \\ \frac{\partial V_m(\cdot)}{\partial(x_1 - x_2)} &> 0, \text{ if } x_i - x_j < -2\bar{B}\bar{P}\Delta t. \end{aligned} \quad (\text{B.30})$$

Thus, it follows that, for two vectors $\hat{\mathbf{x}}$, and $\mathbf{x}'$, such that $2\bar{B}\bar{P}\Delta t < \hat{x}_1 - \hat{x}_2 < x'_1 - x'_2$, we have $V_m(\hat{\mathbf{x}}) \geq V_m(\mathbf{x}')$. Similarly, if $x'_1 - x'_2 < \hat{x}_1 - \hat{x}_2 < -2\bar{B}\bar{P}\Delta t$, we have $V_m(\hat{\mathbf{x}}) \geq V_m(\mathbf{x}')$. Now, note that in the case of $|x_1 - x_2| \in [0, 2\bar{B}\bar{P}\Delta t]$, we have $P_{12,m}^* = \frac{x_1 - x_2}{2\bar{B}\Delta t}$, leading to $\frac{\partial V_m(\cdot)}{\partial(x_1 - x_2)} = 0$. Let $V_m(\cdot) = c$ in this case, where c is a constant. In the case of $|\hat{x}_1 - \hat{x}_2| \leq 2\bar{B}\bar{P}\Delta t < |x'_1 - x'_2|$, from (B.30), we have $c = V_m(\hat{\mathbf{x}}) \leq V_m(\mathbf{x}')$, and in the case of $|\hat{x}_1 - \hat{x}_2| < |x'_1 - x'_2| \leq 2\bar{B}\bar{P}\Delta t$, we have $V_m(\hat{\mathbf{x}}) = V_m(\mathbf{x}') = c$. Therefore, $V_m(\mathbf{x})$ satisfies the property (B.12).

Next, we show that, for two vectors, $\hat{\mathbf{x}}$ and $\mathbf{x}'$, such that $\hat{x}_1 = x'_2$, and $\hat{x}_2 = x'_1$, thus

satisfying $(\hat{x}_1 - \hat{x}_2) = -(x'_1 - x'_2)$, we have $V_m(\hat{\mathbf{x}}) = V_m(\mathbf{x}')$.

Let $\hat{\pi}^*(t_m) := \arg \max_{\pi(t_m) \in \Pi} \hat{J}_m(\pi(t_m), \hat{\mathbf{x}})$, and $\pi'^*(t_m) := \arg \max_{\pi(t_m) \in \Pi} \hat{J}_m(\pi(t_m), \mathbf{x}')$, and further let $\hat{P}_{12,m}^* = \hat{\pi}^*(t_m)$, and $P'_{12,m} = \pi'^*(t_m)$. Since $\hat{x}_1 = x'_2$, and $\hat{x}_2 = x'_1$ we have $\hat{P}_{12,m}^* = P'_{21,m} = -P'_{12,m}$. Further, recall from (B.16) that,

$$\begin{aligned} V_m(\hat{\mathbf{x}}) &= \hat{J}_m(\hat{\pi}^*(t_m), \hat{\mathbf{x}}) = \int \int V_{m+1}(\mathbf{y}) \phi\left(\frac{y_1 - \hat{x}_1 + \bar{B}\hat{P}_{12,m}^* \Delta t}{\sigma^2 \Delta t}\right) \phi\left(\frac{y_2 - \hat{x}_2 - \bar{B}\hat{P}_{12,m}^* \Delta t}{\sigma^2 \Delta t}\right) dy_1 dy_2 \\ &= \int \int V_{m+1}(\mathbf{y}) \phi\left(\frac{y_1 - x'_2 + \bar{B}\hat{P}_{12,m}^* \Delta t}{\sigma^2 \Delta t}\right) \phi\left(\frac{y_2 - x'_1 - \bar{B}\hat{P}_{12,m}^* \Delta t}{\sigma^2 \Delta t}\right) dy_1 dy_2 \\ &= \int \int V_{m+1}(\mathbf{y}) \phi\left(\frac{y_1 - x'_2 - \bar{B}P'_{12,m} \Delta t}{\sigma^2 \Delta t}\right) \phi\left(\frac{y_2 - x'_1 + \bar{B}P'_{12,m} \Delta t}{\sigma^2 \Delta t}\right) dy_1 dy_2 \\ &= \hat{J}(\pi'^*(t_m), \mathbf{x}') \\ &= V_m(\mathbf{x}'). \end{aligned}$$

Therefore, $V_m(\mathbf{x})$ satisfies the property (B.13).

In summary, we have the following: presuming $V_{m+1}(\mathbf{x})$ satisfies the properties (B.12)-(B.13), the optimal policy is (B.27), and under the optimal policy (B.27), $V_m(\mathbf{x})$ satisfies the properties (B.12)-(B.13). Thus, by induction, we establish that for $m \in \{0, 1, \dots, K-1\}$ the optimal policy is given by

$$\pi^*(t_m) = P_{12,m}^* = \begin{cases} \bar{P}, & \text{if } x_1 - x_2 > 2\bar{B}\bar{P}\Delta t, \\ -\bar{P}, & \text{if } x_1 - x_2 < -2\bar{B}\bar{P}\Delta t, \\ \frac{x_1 - x_2}{2\bar{B}\Delta t}, & \text{if } |x_1 - x_2| \in [0, 2\bar{B}\bar{P}\Delta t]. \end{cases}$$

Note that, as $\Delta t \rightarrow 0$, the region $[0, 2\bar{B}\bar{P}\Delta t]$ shrinks. Taking the limit $\Delta t \rightarrow 0$, and denoting the optimal policy by $P'_{12}(t)$, we can show that,

$$P'_{12}(t) = \begin{cases} \bar{P}, & \text{if } X_1(t) > X_2(t), \\ -\bar{P}, & \text{if } X_1(t) < X_2(t), \\ 0, & \text{if } X_1(t) = X_2(t). \end{cases}$$

B.5 Gaussian Tail Inequality

Let $\Phi(\cdot)$ denote the cumulative distribution function of a standard normal random variable. For $x \geq 0$, we have

$$(1 - \Phi(x)) \leq \frac{1}{2}e^{-\frac{x^2}{2}}. \quad (\text{B.31})$$

Here, we show the proof of the inequality (B.31). Note that,

$$\begin{aligned} e^{\frac{x^2}{2}}(1 - \Phi(x)) &= e^{\frac{x^2}{2}} \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{s^2}{2}} ds \\ &= \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{s^2-x^2}{2}} ds \\ &= \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{(s-x)(s+x)}{2}} ds \\ &= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{u(2x+u)}{2}} du \\ &= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-ux} e^{-\frac{u^2}{2}} du. \end{aligned}$$

Since, $x \geq 0$, we have,

$$\begin{aligned} e^{\frac{x^2}{2}}(1 - \Phi(x)) &\leq \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{u^2}{2}} du \\ &= \frac{1}{2} \\ \implies (1 - \Phi(x)) &\leq \frac{1}{2}e^{-\frac{x^2}{2}}. \end{aligned}$$

B.6 Inequalities involving Wiener process with drift

Consider a process $X(t), t \in [0, T_f]$ defined as

$$X(t) := X(0) + \sigma W(t) + \mu t,$$

where $W(t)$ is a standard Wiener process and μ is the drift parameter and $\sigma > 0$ is the volatility parameter, and $X(0)$ is the given initial value of the process. Consider $\beta > X(0)$. We show that,

$$\mathbb{P}[\sup_{0 \leq t \leq T_f} X(t) \geq \beta] \leq e^{\frac{\mu(\beta-X(0))}{\sigma^2} - \frac{(\beta-X(0))^2}{2\sigma^2 T_f}} \quad (\text{B.32})$$

We apply Girsanov's theorem to prove the inequality (B.32).

Note that,

$$\begin{aligned}
& \mathbb{P}[\sup_{0 \leq t \leq T_f} X(t) \geq \beta] \\
&= \mathbb{P}[\sup_{0 \leq t \leq T_f} (X(0) + \sigma W(t) + \mu t) \geq \beta] \\
&= \mathbb{P}[\sup_{0 \leq t \leq T_f} W(t) + \frac{\mu}{\sigma} t \geq \frac{\beta - X(0)}{\sigma}].
\end{aligned}$$

Let $\beta' := \frac{\beta - X(0)}{\sigma}$ and $\mu' := \frac{\mu}{\sigma}$. Then, it follows that,

$$\begin{aligned}
& \mathbb{P}[\sup_{0 \leq t \leq T_f} X(t) \geq \beta] \\
&= \mathbb{P}[\sup_{0 \leq t \leq T_f} (W(t) + \mu' t) \geq \beta'] \\
&= \mathbb{P}[\sup_{0 \leq t \leq T_f} W^Q(t) \geq \beta'], \tag{B.33}
\end{aligned}$$

where $W^Q(t) := W(t) + \mu' t$. Let $\tau_a := \inf\{t \geq 0 : W^Q(t) = a\}$ denote the first time $W^Q(t)$ reaches the level a . Here, we have,

$$\mathbb{P}[\sup_{0 \leq t \leq T_f} W^Q(t) \geq \beta'] = \mathbb{P}[\tau_{\beta'} \leq T_f]. \tag{B.34}$$

Let, $Z(t) := e^{-\mu' W(t) - \frac{1}{2} \mu'^2 t}$, $t \in [0, T_f]$. By Girsanov's theorem ([70]), $W^Q(t)$ is a Wiener process under a probability measure $\mathbb{Q}$, defined by $d\mathbb{Q} = Z(T_f)d\mathbb{P}$, and further

$$\mathbb{P}[\tau_{\beta'} \leq T_f] = \mathbb{E}^Q[\mathbb{1}_{\{\tau_{\beta'} \leq T_f\}} e^{\mu' \beta' - \frac{1}{2} \mu'^2 \tau_{\beta'}}], \tag{B.35}$$

where $\mathbb{E}^Q[\cdot]$ denotes the expectation under the probability measure $\mathbb{Q}$. Note that,

$$\begin{aligned}
& \mathbb{E}^Q[\mathbb{1}_{\{\tau_{\beta'} \leq T_f\}} e^{\mu' \beta' - \frac{1}{2} \mu'^2 \tau_{\beta'}}] \\
&= e^{\mu' \beta'} \mathbb{E}^Q[\mathbb{1}_{\{\tau_{\beta'} \leq T_f\}} e^{-\frac{1}{2} \mu'^2 \tau_{\beta'}}] \\
&\leq e^{\mu' \beta'} \mathbb{E}^Q[\mathbb{1}_{\{\tau_{\beta'} \leq T_f\}}] \\
&= e^{\mu' \beta'} \mathbb{Q}[\tau_{\beta'} \leq T_f], \tag{B.36}
\end{aligned}$$

where the last inequality follows from the fact that $\tau_{\beta'}$ is non-negative by definition. Since, $W^Q(t)$ is a standard Brownian motion under $\mathbb{Q}$, $\tau_{\beta'}$ is the first time a standard Brownian motion reaches a level β' , and thus, $\mathbb{Q}[\tau_{\beta'} \leq T_f] = 2(1 - \Phi(\frac{\beta'}{\sqrt{T_f}}))$, where

$\Phi(\cdot)$ is the CDF of standard normal. From (B.31), we have

$$(1 - \Phi(\frac{\beta'}{\sqrt{T_f}})) \leq \frac{1}{2} e^{-\frac{1}{2} \left(\frac{\beta'}{\sqrt{T_f}} \right)^2}.$$

Thus, from (B.33)-(B.36), we conclude,

$$\mathbb{P}[\sup_{0 \leq t \leq T_f} (W_d(t) + \mu' t) \geq \beta'] \leq e^{\mu' \beta' - \frac{\beta'^2}{2T_f}}.$$

Since, $\beta' = \frac{\beta - X(0)}{\sigma}$, and $\mu' = \frac{\mu}{\sigma}$, we have

$$\begin{aligned} & \mathbb{P}[\sup_{0 \leq t \leq T_f} (W_d(t) + \mu' t) \geq \beta'] \\ & \leq e^{\frac{\mu(\beta - X(0))}{\sigma^2} - \frac{(\beta - X(0))^2}{2\sigma^2 T_f}}, \end{aligned}$$

and hence,

$$\mathbb{P}[\sup_{0 \leq t \leq T_f} X(t) \geq \beta] \leq e^{\frac{\mu(\beta - X(0))}{\sigma^2} - \frac{(\beta - X(0))^2}{2\sigma^2 T_f}}.$$

B.7 Proof of Proposition 2

Let $A_d := \{\sup_{0 \leq t \leq T_f} |X_d^{\pi'}(t)| < \beta\}$ and its complement, $A'_d := \{\sup_{0 \leq t \leq T_f} |X_d^{\pi'}(t)| \geq \beta\}$. Marginalizing over the event sets A_d and A'_d , we derive the following:

$$\begin{aligned} \mathbb{P}[A_1 \cup A_2 \cup A_3 \cup A_4] &= \mathbb{P}[(A_1 \cup A_2 \cup A_3 \cup A_4) \cap A_d] \\ &\quad + \mathbb{P}[(A_1 \cup A_2 \cup A_3 \cup A_4) \cap A'_d]. \end{aligned} \tag{B.37}$$

First, we focus on the first term on the right-hand side of (B.37), and derive the following:

$$\begin{aligned} & \mathbb{P}[(A_1 \cup A_2 \cup A_3 \cup A_4) \cap A_d] \\ & \leq \mathbb{P}[(A_1 \cup A_3) \cap A_d] + \mathbb{P}[(A_2 \cup A_4) \cap A_d] \\ & = \mathbb{P}[(A_1 \cap A_d) \cup (A_3 \cap A_d)] \\ & \quad + \mathbb{P}[(A_2 \cap A_d) \cup (A_4 \cap A_d)]. \end{aligned} \tag{B.38}$$

Now, note that,

$$\begin{aligned}
& \sup_{0 \leq t \leq T_f} |X_d^{\pi'}(t)| < \beta \\
\implies & |X_d^{\pi'}(t)| < \beta, \text{ for all } t \in [0, T_f] \\
\implies & -\beta < X_d^{\pi'}(t) < \beta, \text{ for all } t \in [0, T_f] \\
\implies & -\beta < X_2^{\pi'}(t) - X_1^{\pi'}(t) < \beta, \text{ for all } t \in [0, T_f] \\
\implies & 2X_1^{\pi'}(t) - \beta < X_1^{\pi'}(t) + X_2^{\pi'}(t) \\
& < 2X_1^{\pi'} + \beta, \text{ for all } t \in [0, T_f] \\
\implies & \sup_{0 \leq t \leq T_f} 2X_1^{\pi'}(t) - \beta \\
& < \sup_{0 \leq t \leq T_f} X_c^{\pi'}(t) \\
& < \sup_{0 \leq t \leq T_f} 2X_1^{\pi'} + \beta.
\end{aligned}$$

Therefore, note that,

$$\begin{aligned}
& \sup_{0 \leq t \leq T_f} X_1^{\pi'}(t) \geq N \\
\implies & \sup_{0 \leq t \leq T_f} X_c^{\pi'} \geq 2N - \beta.
\end{aligned}$$

Recall that $X_{c,T_f}^{\pi',sup} := \sup_{0 \leq t \leq T_f} X_c^{\pi'}(t)$. Thus denoting $A_c := \{X_{c,T_f}^{\pi',sup} \geq 2N - \beta\}$, we have $(A_1 \cap A_d) \subseteq A_c$. In a similar way, we can show that $(A_3 \cap A_d) \subseteq A_c$, which leads to

$$\mathbb{P}[(A_1 \cap A_d) \cup (A_3 \cap A_d)] \leq \mathbb{P}[A_c]. \quad (\text{B.39})$$

In a similar manner, we define $\tilde{A}_c := \{X_{c,T_f}^{\pi',inf} \leq \beta\}$, and derive the following:

$$\mathbb{P}[(A_2 \cap A_d) \cup (A_4 \cap A_d)] \leq \mathbb{P}[\tilde{A}_c]. \quad (\text{B.40})$$

Hence, from (B.38) and combining (B.39) and (B.40), we have

$$\mathbb{P}[(A_1 \cup A_2 \cup A_3 \cup A_4) \cap A_d] \leq \mathbb{P}[A_c] + \mathbb{P}[\tilde{A}_c]. \quad (\text{B.41})$$

Thus, from (B.41) and (B.37), we have

$$\begin{aligned}
& \mathbb{P}[A_1 \cup A_2 \cup A_3 \cup A_4] \\
& \leq \mathbb{P}[A_c] + \mathbb{P}[\tilde{A}_c] + P[(A_1 \cup A_2 \cup A_3 \cup A_4) \cap A'_d] \\
& \leq \mathbb{P}[A_c] + \mathbb{P}[\tilde{A}_c] + P[A'_d].
\end{aligned} \tag{B.42}$$

This completes the proof.

B.8 Proof of Theorem 6

Note that

$$\begin{aligned}
\mathbb{P}[A_1 \cup A_2 \cup A_3 \cup A_4] & \geq \mathbb{P}[A_1] \\
& = \mathbb{P}[\sup_{0 \leq t \leq T_f} X_1^{\pi'}(t) \geq \tilde{N}_\infty].
\end{aligned} \tag{B.43}$$

Further recall, from (3.14), under the power transfer policy $P'_{12}(t) = -\bar{P} \text{sgn}(X_d^{\pi'}(t))$ we have:

$$X_1^{\pi'}(t) = \frac{1}{2}\tilde{N}_\infty + \sigma W_1(t) + 2\bar{B}\bar{P} \int_0^t \text{sgn}(X_d^{\pi'}(s)) ds.$$

Note, for all $s \in [0, T_f]$ we have,

$$-2\bar{P} \leq 2\bar{P} \text{sgn}(X_d^{\pi'}(s)) \leq 2\bar{P},$$

and hence,

$$\begin{aligned}
& \frac{1}{2}\tilde{N}_\infty + \sigma W_1(t) - 2\bar{B}\bar{P}t \\
& \leq \frac{1}{2}\tilde{N}_\infty + \sigma W_1(t) + 2\bar{B}\bar{P} \int_0^t \text{sgn}(X_d^{\pi'}(s)) ds \\
& \leq \frac{1}{2}\tilde{N}_\infty + \sigma W_1(t) + 2\bar{B}\bar{P}t,
\end{aligned}$$

for all $t \in [0, T_f]$. Since, $X_1^{\pi'}(t) = \frac{1}{2}\tilde{N}_\infty + \sigma W_1(t) + 2\bar{B}\bar{P} \int_0^t \text{sgn}(X_d^{\pi'}(s)) ds$, we have

$$\begin{aligned}
& \frac{1}{2}\tilde{N}_\infty + \sigma W_1(t) - 2\bar{B}\bar{P}t \\
& \leq X_1^{\pi'}(t) \leq \frac{1}{2}\tilde{N}_\infty + \sigma W_1(t) + 2\bar{B}\bar{P}t.
\end{aligned}$$

Therefore,

$$\begin{aligned} & \sup_{0 \leq t \leq T_f} (\frac{1}{2}\tilde{N}_\infty + \sigma W_1(t) - 2\overline{BP}t) \geq \tilde{N}_\infty \\ \implies & \sup_{0 \leq t \leq T_f} X_1^{\pi'}(t) \geq \tilde{N}_\infty, \end{aligned}$$

and thus,

$$\begin{aligned} & \mathbb{P}[\sup_{0 \leq t \leq T_f} X_1^{\pi'}(t) \geq \tilde{N}_\infty] \\ & \geq \mathbb{P}[\sup_{0 \leq t \leq T_f} (\frac{1}{2}\tilde{N}_\infty + \sigma W_1(t) - 2\overline{BP}t) \geq \tilde{N}_\infty] \\ & = \mathbb{P}[\sup_{0 \leq t \leq T_f} (W_1(t) - \frac{2\overline{BP}}{\sigma}t) \geq \frac{\tilde{N}_\infty}{2\sigma}] \end{aligned}$$

We define $W_1^Q(t) := W_1(t) - \frac{2\overline{BP}}{\sigma}t$, and let $\tau := \inf\{t \geq 0 | W_1^Q(t) \geq \frac{\tilde{N}_\infty}{2\sigma}\}$ denote the first time the process $W_1^Q(t)$ reaches the level $\frac{\tilde{N}_\infty}{2\sigma}$. Defining $Z_1(t) = e^{\frac{2\overline{BP}}{\sigma}W_1(t) - \frac{1}{2}(\frac{2\overline{BP}}{\sigma})^2t}$, and using Girsanov's theorem ([70]), as shown in Appendix B.6, we can derive the following:

$$\begin{aligned} & \mathbb{P}[\sup_{0 \leq t \leq T_f} (W_1(t) - \frac{2\overline{BP}}{\sigma}t) \geq \frac{\tilde{N}_\infty}{2\sigma}] \\ & = \mathbb{P}[\tau \leq T_f] \\ & = e^{-\frac{\overline{BP}\tilde{N}_\infty}{\sigma^2}} \mathbb{E}^Q[\mathbb{1}_{\{\tau \leq T_f\}} e^{-\frac{1}{2}(\frac{2\overline{BP}}{\sigma})^2\tau}], \end{aligned} \tag{B.44}$$

where $\mathbb{E}^Q[\cdot]$ defines the expectation under a probability measure $\mathbb{Q}$ defined as $d\mathbb{Q} = Z_1(T_f)d\mathbb{P}$ ([70]). From (B.44), we further have the following:

$$e^{-\frac{\overline{BP}\tilde{N}_\infty}{\sigma^2}} \mathbb{E}^Q[\mathbb{1}_{\{\tau \leq T_f\}} e^{-\frac{1}{2}(\frac{2\overline{BP}}{\sigma})^2\tau}] \tag{B.45}$$

$$\geq e^{-\frac{\overline{BP}\tilde{N}_\infty}{\sigma^2} - \frac{1}{2}(\frac{2\overline{BP}}{\sigma})^2T_f} \mathbb{E}^Q[\mathbb{1}_{\{\tau \leq T_f\}}] \tag{B.46}$$

$$= e^{-\frac{\overline{BP}\tilde{N}_\infty}{\sigma^2} - \frac{1}{2}(\frac{2\overline{BP}}{\sigma})^2T_f} \mathbb{Q}[\mathbb{1}_{\{\tau \leq T_f\}}]. \tag{B.47}$$

By Girsanov's theorem, we have that $W_1^Q(t)$ is a standard Wiener process under the probability measure $\mathbb{Q}$, and thus we have ([70])

$$\mathbb{Q}[\mathbb{1}_{\{\tau \leq T_f\}}] = 2\left(1 - \Phi\left(\frac{\tilde{N}_\infty}{2\sigma\sqrt{T_f}}\right)\right), \tag{B.48}$$

where $\Phi(\cdot)$ denotes the cumulative distribution function of a standard normal random variable. A known lower bound of the quantity $(1 - \Phi(x))$ is given as follows

$$(1 - \Phi(x)) \geq \frac{1}{\sqrt{2\pi}} \frac{x}{1+x^2} e^{-\frac{1}{2}x^2}, \quad (\text{B.49})$$

for $x \geq 0$. Combining (B.43)- (B.49), we have

$$\begin{aligned} & \mathbb{P}[\sup_{0 \leq t \leq T_f} X_1^{\pi'}(t) \geq \tilde{N}_\infty] \\ & \geq e^{-\frac{\overline{B}\overline{P}\tilde{N}_\infty}{\sigma^2} - \frac{1}{2}(\frac{2\overline{B}\overline{P}}{\sigma})^2 T_f} \frac{2}{\sqrt{2\pi}} \frac{(\frac{\tilde{N}_\infty}{2\sigma\sqrt{T_f}})}{1+(\frac{\tilde{N}_\infty}{2\sigma\sqrt{T_f}})^2} e^{-\frac{1}{2}(\frac{\tilde{N}_\infty}{2\sigma\sqrt{T_f}})^2}. \end{aligned} \quad (\text{B.50})$$

Since, $\overline{P} = c_p T_f^\xi$ and $\tilde{N}_\infty = c_\infty \sqrt{T_f}$, from (B.50), we have

$$\begin{aligned} & \mathbb{P}[\sup_{0 \leq t \leq T_f} X_1^{\pi'}(t) \geq \tilde{N}_\infty] \\ & \geq e^{-\frac{\overline{B}c_p c_\infty}{\sigma^2} T_f^{\frac{1}{2}+\xi} - \frac{2\overline{B}^2 c_p^2}{\sigma^2} T_f^{2\xi+1}} \frac{2}{\sqrt{2\pi}} \frac{(\frac{c_\infty}{2\sigma})}{1+(\frac{c_\infty}{2\sigma})^2} e^{-\frac{1}{2}(\frac{c_\infty}{2\sigma})^2}. \end{aligned} \quad (\text{B.51})$$

As $\xi < -\frac{1}{2}$, we have $\xi + \frac{1}{2} < 0$, and also $2\xi + 1 < 0$. Thus,

$$\lim_{T_f \rightarrow \infty} e^{-\frac{\overline{B}c_p c_\infty}{\sigma^2} T_f^{\frac{1}{2}+\xi} - \frac{2\overline{B}^2 c_p^2}{\sigma^2} T_f^{2\xi+1}} = 1. \quad (\text{B.52})$$

Thus, we have

$$\begin{aligned} & \lim_{T_f \rightarrow \infty} e^{-\frac{\overline{B}c_p c_\infty}{\sigma^2} T_f^{\frac{1}{2}+\xi} - \frac{2\overline{B}^2 c_p^2}{\sigma^2} T_f^{2\xi+1}} \frac{2}{\sqrt{2\pi}} \frac{(\frac{c_\infty}{2\sigma})}{1+(\frac{c_\infty}{2\sigma})^2} e^{-\frac{1}{2}(\frac{c_\infty}{2\sigma})^2} \\ & = \frac{2}{\sqrt{2\pi}} \frac{(\frac{c_\infty}{2\sigma})}{1+(\frac{c_\infty}{2\sigma})^2} e^{-\frac{1}{2}(\frac{c_\infty}{2\sigma})^2}. \end{aligned} \quad (\text{B.53})$$

Recall that, $c_\infty = 2\sigma\sqrt{\ln(2/\delta)}$, and thus,

$$\begin{aligned} & \frac{2}{\sqrt{2\pi}} \frac{(\frac{c_\infty}{2\sigma})}{1+(\frac{c_\infty}{2\sigma})^2} e^{-\frac{1}{2}(\frac{c_\infty}{2\sigma})^2} \\ & = \frac{2}{\sqrt{2\pi}} \frac{\sqrt{\ln(2/\delta)}}{1+\ln(2/\delta)} e^{-\frac{1}{2}\ln(2/\delta)} \\ & = \frac{2}{\sqrt{2\pi}} \frac{\sqrt{\ln(2/\delta)}}{1+\ln(2/\delta)} \sqrt{\frac{\delta}{2}}. \end{aligned} \quad (\text{B.54})$$

We can numerically verify that there exists a $\delta^* \in (0, 1)$ such that for all $\delta \leq \delta^*$,

$$\frac{2}{\sqrt{2\pi}} \frac{\sqrt{\ln(2/\delta)}}{1+\ln(2/\delta)} \sqrt{\frac{\delta}{2}} - \delta > 0.$$

Thus for all $\delta \leq \delta^*$, combining (B.50)- (B.54), we have

$$\lim_{T_f \rightarrow \infty} \mathbb{P}[\sup_{0 \leq t \leq T_f} X_1^{\pi'}(t) \geq \tilde{N}_\infty] > \delta,$$

and thus from (B.43), we have the desired result.

B.9 Proof of Lemma 2

We have,

$$\begin{aligned} & \mathbb{P}[\hat{\tau}_{\hat{\beta},i}^+ \leq \hat{\tau}_{\hat{\beta},i-1}^+ + T] \\ &= \mathbb{P}[\sup_{\hat{\tau}_{\hat{\beta},i-1}^+ \leq t \leq \hat{\tau}_{\hat{\beta},i-1}^+ + T} X_d^{\pi'}(t) \geq T_f^\gamma] \\ &= \mathbb{P}[\sup_{0 \leq h \leq T} X_d^{\pi'}(h + \hat{\tau}_{\hat{\beta},i-1}^+) \geq T_f^\gamma] \end{aligned} \tag{B.55}$$

Note that, for all $h \geq 0$, we have

$$\begin{aligned} X_d^{\pi'}(h + \hat{\tau}_{\hat{\beta},i-1}^+) &= \sigma_d W_d(h + \hat{\tau}_{\hat{\beta},i-1}^+) \\ &\quad - 2\bar{P} \int_0^{h + \hat{\tau}_{\hat{\beta},i-1}^+} \text{sgn}(X_d^{\pi'}(s)) ds \\ &= \sigma_d W_d(h + \hat{\tau}_{\hat{\beta},i-1}^+) - \sigma_d W_d(\hat{\tau}_{\hat{\beta},i-1}^+) + \sigma_d W_d(\hat{\tau}_{\hat{\beta},i-1}^+) \\ &\quad - 2\bar{P} \int_0^{\hat{\tau}_{\hat{\beta},i-1}^+} \text{sgn}(X_d^{\pi'}(s)) ds \\ &\quad - 2\bar{P} \int_{\hat{\tau}_{\hat{\beta},i-1}^+}^{h + \hat{\tau}_{\hat{\beta},i-1}^+} \text{sgn}(X_d^{\pi'}(s)) ds \\ &= \sigma_d W_d(h + \hat{\tau}_{\hat{\beta},i-1}^+) - \sigma_d W_d(\hat{\tau}_{\hat{\beta},i-1}^+) + X_d^{\pi'}(\hat{\tau}_{\hat{\beta},i-1}^+) \\ &\quad - 2\bar{P} \int_{\hat{\tau}_{\hat{\beta},i-1}^+}^{h + \hat{\tau}_{\hat{\beta},i-1}^+} \text{sgn}(X_d^{\pi'}(s)) ds \\ &= X_d^{\pi'}(\hat{\tau}_{\hat{\beta},i-1}^+) + \sigma_d [W_d(h + \hat{\tau}_{\hat{\beta},i-1}^+) - W_d(\hat{\tau}_{\hat{\beta},i-1}^+)] \\ &\quad - 2\bar{P} \int_0^h \text{sgn}(X_d^{\pi'}(v + \hat{\tau}_{\hat{\beta},i-1}^+)) dv. \end{aligned}$$

By the strong Markov property of Brownian motion, we have $\tilde{W}_d(h) := W_d(h + \hat{\tau}_{\hat{\beta}, i-1}^+) - W_d(\hat{\tau}_{\hat{\beta}, i-1}^+)$, $h \geq 0$, is again a Brownian motion [[159] Section 7.2, pp. 114]. Further by definition, $X_d^{\pi'}(\hat{\tau}_{\hat{\beta}, i-1}^+) = 0$. Therefore,

$$\begin{aligned} X_d^{\pi'}(h + \hat{\tau}_{\hat{\beta}, i-1}^+) &= \sigma_d \tilde{W}_d(h) \\ &\quad - 2\overline{P} \int_0^h \text{sgn}(X_d^{\pi'}(v + \hat{\tau}_{\hat{\beta}, i-1}^+)) dv. \end{aligned} \quad (\text{B.56})$$

Note, for all $v \geq 0$ we have,

$$-2\overline{P} \leq 2\overline{P} \text{sgn}(X_d^{\pi'}(v + \hat{\tau}_{\hat{\beta}, i-1}^+)) \leq 2\overline{P},$$

and hence,

$$\begin{aligned} \sigma_d \tilde{W}_d(h) - 2\overline{BP}h &\leq \sigma_d \tilde{W}_d(h) - 2\overline{BP} \int_0^h \text{sgn}(X_d^{\pi'}(v + \hat{\tau}_{\hat{\beta}, i-1}^+)) dv \\ &\leq \sigma_d \tilde{W}_d(h) + 2\overline{BP}h, \end{aligned}$$

for all $h \in [0, T]$. From (B.56), we have $X_d^{\pi'}(h + \hat{\tau}_{\hat{\beta}, i-1}^+) = \sigma_d \tilde{W}_d(h) - 2\overline{BP} \int_0^h \text{sgn}(X_d^{\pi'}(v + \hat{\tau}_{\hat{\beta}, i-1}^+)) dv$, and thus, for all $h \in [0, T]$,

$$\begin{aligned} \sigma_d \tilde{W}_d(h) - 2\overline{BP}h &\leq X_d^{\pi'}(h + \hat{\tau}_{\hat{\beta}, i-1}^+) \leq \sigma_d \tilde{W}_d(h) + 2\overline{BP}h. \end{aligned}$$

Therefore,

$$\begin{aligned} \sup_{0 \leq h \leq T} X_d^{\pi'}(h + \hat{\tau}_{\hat{\beta}, i-1}^+) &\geq \hat{\beta} \\ \implies \sup_{0 \leq h \leq T} (\sigma_d \tilde{W}_d(h) + 2\overline{BP}h) &\geq \hat{\beta}, \end{aligned}$$

and thus,

$$\begin{aligned} \mathbb{P}[\sup_{0 \leq h \leq T} X_d^{\pi'}(h + \hat{\tau}_{\hat{\beta}, i-1}^+) \geq \hat{\beta}] &\leq \mathbb{P}[\sup_{0 \leq h \leq T} (\sigma_d \tilde{W}_d(h) + 2\overline{BP}h) \geq \hat{\beta}]. \end{aligned} \quad (\text{B.57})$$

Let $\bar{\tau}_{\hat{\beta}} := \inf\{h \geq 0 : \sigma_d \tilde{W}_d(h) + 2\overline{BP}h \geq \hat{\beta}\}$ denote the first time $\sigma_d \tilde{W}_d(h) + 2\overline{BP}h$ reaches the level $\hat{\beta}$. Then, we have $\mathbb{P}[\sup_{0 \leq h \leq T} (\sigma_d \tilde{W}_d(h) + 2\overline{BP}h) \geq \hat{\beta}] = \mathbb{P}[\bar{\tau}_{\hat{\beta}} \leq T]$. Combining (B.55), and (B.57), we have

$$\mathbb{P}[\hat{\tau}_{\hat{\beta},i} \leq \hat{\tau}_{\hat{\beta},i-1}^+ + T] \leq \mathbb{P}[\bar{\tau}_{\gamma} \leq T]. \quad (\text{B.58})$$

Note that the process, $\sigma_d \tilde{W}_d(h) + 2\overline{BP}h, h \in [0, T]$ is a Wiener process with positive drift of $2\overline{BP}$. Here using Girsanov's theorem, we can show that,

$$\mathbb{P}[\bar{\tau}_{\gamma} \leq T] \leq e^{\frac{2\overline{BP}\hat{\beta}}{\sigma_d^2} - \frac{\hat{\beta}^2}{2\sigma_d^2 T}}. \quad (\text{B.59})$$

Since, $\overline{P} = c_p T_f^{-\frac{1}{2}+\epsilon}$, from (B.59), we have,

$$\mathbb{P}[\bar{\tau}_{\gamma} \leq T] \leq e^{-\left(\frac{\hat{\beta}^2}{2\sigma_d^2 T} - \frac{2\overline{B}\hat{\beta}c_p T_f^{-\frac{1}{2}+\epsilon}}{\sigma_d^2}\right)}. \quad (\text{B.60})$$

Combining (B.58), and (B.60), we have

$$\mathbb{P}[\hat{\tau}_{\hat{\beta},i} \leq \hat{\tau}_{\hat{\beta},i-1}^+ + T] \leq e^{-\left(\frac{\hat{\beta}^2}{2\sigma_d^2 T} - \frac{2\overline{B}\hat{\beta}c_p T_f^{-\frac{1}{2}+\epsilon}}{\sigma_d^2}\right)}.$$

This completes the proof.

B.10 Proof of Corollary 1

Let the random variable $L_i := \hat{\tau}_{\hat{\beta},i}^+ - \hat{\tau}_{\hat{\beta},i-1}^+$ denote the length of epoch $i \in \mathbb{N}$. Note that, $\hat{\tau}_{\hat{\beta},i} \leq \hat{\tau}_{\hat{\beta},i}^+$ by the definition of $\hat{\tau}_{\hat{\beta},i}^+$. Therefore,

$$\begin{aligned} \hat{\tau}_{\hat{\beta},i} - \hat{\tau}_{\hat{\beta},i-1}^+ &\leq \hat{\tau}_{\hat{\beta},i}^+ - \hat{\tau}_{\hat{\beta},i-1}^+ \\ \implies \hat{\tau}_{\hat{\beta},i} - \hat{\tau}_{\hat{\beta},i-1}^+ &\leq L_i. \end{aligned}$$

Thus for a given $T > 0$, we have $L_i \leq T \implies \hat{\tau}_{\hat{\beta},i} - \hat{\tau}_{\hat{\beta},i-1}^+ \leq T$, and hence,

$$\mathbb{P}[L_i \leq T] \leq \mathbb{P}[\hat{\tau}_{\hat{\beta},i} - \hat{\tau}_{\hat{\beta},i-1}^+ \leq T].$$

Further, from Lemma 2, we have

$$\begin{aligned}\mathbb{P}[\hat{\tau}_{\hat{\beta},i} \leq \hat{\tau}_{\hat{\beta},i-1}^+ + T] &= \mathbb{P}[\hat{\tau}_{\hat{\beta},i} - \hat{\tau}_{\hat{\beta},i-1}^+ \leq T] \\ &\leq e^{-\left(\frac{\hat{\beta}^2}{2\sigma_d^2 T} - \frac{2\bar{B}\hat{\beta}c_p T_f^{-\frac{1}{2}+\epsilon}}{\sigma_d^2}\right)}.\end{aligned}$$

Therefore, it follows that,

$$\mathbb{P}[L_i \leq T] \leq e^{-\left(\frac{\hat{\beta}^2}{2\sigma_d^2 T} - \frac{2\bar{B}\hat{\beta}c_p T_f^{-\frac{1}{2}+\epsilon}}{\sigma_d^2}\right)}. \quad (\text{B.61})$$

Note that, if the length of every epoch is greater than T , the number of epochs occurring within the time interval $[0, T_f]$ is less than $K_{max} = \frac{T_f}{T}$. Therefore,

$$\mathbb{P}[K \leq K_{max}] \geq \mathbb{P}[\cap_{i=0}^{K_{max}-1} \{L_i > T\}]. \quad (\text{B.62})$$

Further,

$$\begin{aligned}\mathbb{P}[\cap_{i=0}^{K_{max}-1} \{L_i > T\}] &= 1 - \mathbb{P}[\cup_{i=0}^{K_{max}-1} \{L_i \leq T\}] \\ &\geq 1 - \sum_{i=0}^{K_{max}-1} \mathbb{P}[L_i \leq T] \\ &\geq 1 - K_{max} e^{-\left(\frac{\hat{\beta}^2}{2\sigma_d^2 T} - \frac{2\bar{B}\hat{\beta}c_p T_f^{-\frac{1}{2}+\epsilon}}{\sigma_d^2}\right)},\end{aligned} \quad (\text{B.63})$$

where the last line follows from (B.61). Hence, from (B.62) and (B.63), we have,

$$\begin{aligned}\mathbb{P}[K \leq K_{max}] &\geq 1 - K_{max} e^{-\left(\frac{\hat{\beta}^2}{2\sigma_d^2 T} - \frac{2\bar{B}\hat{\beta}c_p T_f^{-\frac{1}{2}+\epsilon}}{\sigma_d^2}\right)} \\ \implies \mathbb{P}[K > K_{max}] &\leq K_{max} e^{-\left(\frac{\hat{\beta}^2}{2\sigma_d^2 T} - \frac{2\bar{B}\hat{\beta}c_p T_f^{-\frac{1}{2}+\epsilon}}{\sigma_d^2}\right)}.\end{aligned}$$

This completes the proof.

B.11 Proof of Lemma 3

We have,

$$\begin{aligned}
& \mathbb{P}[\hat{\tau}_{\beta,i} \leq \hat{\tau}_{\beta,i}^+, \hat{\tau}_{\beta,i}^+ \leq T_f] \\
&= \mathbb{P}[\hat{\tau}_{\beta,i} \leq \hat{\tau}_{\beta,i}^+ | \hat{\tau}_{\beta,i}^+ \leq T_f] \mathbb{P}[\hat{\tau}_{\beta,i}^+ \leq T_f] \\
&= \mathbb{P}[\sup_{\hat{\tau}_{\beta,i} \leq t \leq \hat{\tau}_{\beta,i}^+} X_d^{\pi'}(t) \geq \beta | \hat{\tau}_{\beta,i}^+ \leq T_f] \mathbb{P}[\hat{\tau}_{\beta,i}^+ \leq T_f] \\
&= \mathbb{P}[\sup_{0 \leq h \leq \hat{\tau}_{\beta,i}^+ - \hat{\tau}_{\beta,i}} X_d^{\pi'}(h + \hat{\tau}_{\beta,i}) \geq \beta | \hat{\tau}_{\beta,i}^+ \leq T_f] \\
&\quad \mathbb{P}[\hat{\tau}_{\beta,i}^+ \leq T_f]. \tag{B.64}
\end{aligned}$$

Note that, for all $h \geq 0$, we have

$$\begin{aligned}
& X_d^{\pi'}(h + \hat{\tau}_{\beta,i}) = \sigma_d W_d(h + \hat{\tau}_{\beta,i}) \\
&\quad - 2\bar{P} \int_0^{h + \hat{\tau}_{\beta,i}} \text{sgn}(X_d^{\pi'}(s)) ds \\
&= \sigma_d W_d(h + \hat{\tau}_{\beta,i}) - \sigma_d W_d(\hat{\tau}_{\beta,i}) + \sigma_d W_d(\hat{\tau}_{\beta,i}) \\
&\quad - 2\bar{P} \int_0^{\hat{\tau}_{\beta,i}} \text{sgn}(X_d^{\pi'}(s)) ds \\
&\quad - 2\bar{P} \int_{\hat{\tau}_{\beta,i}}^{h + \hat{\tau}_{\beta,i}} \text{sgn}(X_d^{\pi'}(s)) ds \\
&= \sigma_d W_d(h + \hat{\tau}_{\beta,i}) - \sigma_d W_d(\hat{\tau}_{\beta,i}) + X_d^{\pi'}(\hat{\tau}_{\beta,i}) \\
&\quad - 2\bar{P} \int_{\hat{\tau}_{\beta,i}}^{h + \hat{\tau}_{\beta,i}} \text{sgn}(X_d^{\pi'}(s)) ds \\
&= X_d^{\pi'}(\hat{\tau}_{\beta,i}) + \sigma_d [W_d(h + \hat{\tau}_{\beta,i}) - W_d(\hat{\tau}_{\beta,i})] \\
&\quad - 2\bar{P} \int_0^h \text{sgn}(X_d^{\pi'}(v + \hat{\tau}_{\beta,i})) dv.
\end{aligned}$$

By the strong Markov property of Brownian motion, we have $\tilde{W}_d(h) := W_d(h + \hat{\tau}_{\beta,i}) - W_d(\hat{\tau}_{\beta,i})$, $h \geq 0$, is again a Brownian motion [[159] Section 7.2, pp. 114]. Further by definition, $X_d^{\pi'}(\hat{\tau}_{\beta,i}) = \hat{\beta}$. Therefore,

$$\begin{aligned}
& X_d^{\pi'}(h + \hat{\tau}_{\beta,i}) = \hat{\beta} + \sigma_d \tilde{W}_d(h) \\
&\quad - 2\bar{P} \int_0^h \text{sgn}(X_d^{\pi'}(v + \hat{\tau}_{\beta,i})) dv. \tag{B.65}
\end{aligned}$$

By definition of $\hat{\tau}_{\beta,i}^+$, we have, for all $t \in [\hat{\tau}_{\beta,i}, \hat{\tau}_{\beta,i}^+]$, $X_d^{\pi'}(t) \geq 0$, or equivalently, for all $h \in [0, \hat{\tau}_{\beta,i}^+ - \hat{\tau}_{\beta,i}]$, $X_d^{\pi'}(h + \hat{\tau}_{\beta,i}) \geq 0$. Hence, $\text{sgn}(X_d^{\pi'}(h + \hat{\tau}_{\beta,i})) = 1$ for all $h \in$

$[0, \min(T_f, \hat{\tau}_{\hat{\beta},i}^+) - \hat{\tau}_{\hat{\beta},i}]$. Thus, from (B.65), we have

$$X_d^{\pi'}(h + \hat{\tau}_{\hat{\beta},i}) = \hat{\beta} + \sigma_d \tilde{W}_d(h) - 2\bar{P}h, \quad (\text{B.66})$$

for all $h \in [0, \hat{\tau}_{\hat{\beta},i}^+ - \hat{\tau}_{\hat{\beta},i}]$.

We define,

$$\begin{aligned} Y_d(0) &:= \hat{\beta} \\ Y_d(h) &:= Y_d(0) + \sigma_d \tilde{W}_d(h) - 2\bar{P}h, \quad h \geq 0. \end{aligned} \quad (\text{B.67})$$

Further, let,

$$\begin{aligned} \tau_\beta^Y &:= \inf\{h \geq 0 : Y_d(h) = \beta\}, \\ \tau^{Y+} &:= \inf\{h \geq 0 : Y_d(h) = 0\}, \end{aligned}$$

denote the first time the process $Y_d(h)$ reaches the level $\beta > \hat{\beta}$, and zero respectively, starting from $Y_{di}(0) = \hat{\beta}$.

From (B.66), and (B.67), we have, for all $h \in [0, \hat{\tau}_{\hat{\beta},i}^+ - \hat{\tau}_{\hat{\beta},i}]$, $X_d(h + \hat{\tau}_{\hat{\beta},i}) = Y_d(h)$ almost surely. Further, note that, for all $s \in [\hat{\tau}_{\hat{\beta},i}, \hat{\tau}_{\hat{\beta},i}^+)$, $X_d^{\pi'}(s) > 0$, or equivalently, for all $h \in [0, \hat{\tau}_{\hat{\beta},i}^+ - \hat{\tau}_{\hat{\beta},i})$, $X(h + \hat{\tau}_{\hat{\beta},i}) > 0$, and hence $Y_d(h) > 0$. At $h = \hat{\tau}_{\hat{\beta},i}^+ - \hat{\tau}_{\hat{\beta},i}$, we have $X_d^{\pi'}(h + \hat{\tau}_{\hat{\beta},i}^+ - \hat{\tau}_{\hat{\beta},i}) = X_d^{\pi'}(\hat{\tau}_{\hat{\beta},i}^+) = 0$ and hence $Y_d(h) = 0$ for the first time. Since, $X_d(h + \hat{\tau}_{\hat{\beta},i}) = Y_d(h)$ a.s. for all $h \in [0, \hat{\tau}_{\hat{\beta},i}^+ - \hat{\tau}_{\hat{\beta},i}]$, we have $\tau^{Y+} = \hat{\tau}_{\hat{\beta},i}^+ - \hat{\tau}_{\hat{\beta},i}$ a.s.

Therefore we have,

$$\begin{aligned} &\mathbb{P}[\sup_{0 \leq h \leq \hat{\tau}_{\hat{\beta},i}^+ - \hat{\tau}_{\hat{\beta},i}} X_d^{\pi'}(h + \hat{\tau}_{\hat{\beta},i}) \geq \beta | \hat{\tau}_{\hat{\beta},i}^+ \leq T_f] \\ &\quad \mathbb{P}[\hat{\tau}_{\hat{\beta},i}^+ \leq T_f] \\ &= \mathbb{P}[\sup_{0 \leq h \leq \tau^{Y+}} Y_d(h) \geq \beta | \hat{\tau}_{\hat{\beta},i}^+ \leq T_f] \mathbb{P}[\hat{\tau}_{\hat{\beta},i}^+ \leq T_f] \\ &= \mathbb{P}[\sup_{0 \leq h \leq \tau^{Y+}} Y_d(h) \geq \beta, \hat{\tau}_{\hat{\beta},i}^+ \leq T_f] \\ &= \mathbb{P}[\sup_{0 \leq h \leq \tau^{Y+}} Y_d(h) \geq \beta, \tau^{Y+} + \hat{\tau}_{\hat{\beta},i} \leq T_f] \\ &= \mathbb{P}[\tau_\beta^Y \leq \tau^{Y+}, \tau^{Y+} + \hat{\tau}_{\hat{\beta},i} \leq T_f], \end{aligned} \quad (\text{B.68})$$

where the last equality follows from the definition of τ_β^Y . Note that,

$$\{\tau_\beta^Y \leq \tau^{Y+}\} \cap \{\tau^{Y+} + \hat{\tau}_{\hat{\beta},i} \leq T_f\} \subseteq \{\tau_\beta^Y \leq T_f\},$$

since $\hat{\tau}_{\hat{\beta},i}$ is a non-negative random variable. Thus,

$$\mathbb{P}[\tau_\beta^Y \leq \tau^{Y+}, \tau^{Y+} + \hat{\tau}_{\hat{\beta},i} \leq T_f] \leq \mathbb{P}[\tau_\beta^Y \leq T_f]. \quad (\text{B.69})$$

Combining (B.64), (B.68), and (B.69), we have

$$\begin{aligned} \mathbb{P}[\hat{\tau}_{\beta,i} \leq \hat{\tau}_{\hat{\beta},i}^+, \hat{\tau}_{\hat{\beta},i}^+ \leq T_f] &\leq \mathbb{P}[\tau_\beta^Y \leq T_f] \\ &= \mathbb{P}[\sup_{0 \leq t \leq T_f} Y_d(t) \geq \beta], \end{aligned} \quad (\text{B.70})$$

where the last equality follows from the definition of τ_β^Y . Since, $Y_d(h)$ is a Wiener process with constant drift, using Girsanov's theorem, we can show that,

$$\mathbb{P}[\sup_{0 \leq t \leq T_f} Y_d(t) \geq \beta] \leq e^{-\frac{2\bar{B}c_p T_f^{-\frac{1}{2}+\epsilon}(\beta-\hat{\beta})}{\sigma_d^2}}.$$

Therefore, from (B.70), it follows that

$$\mathbb{P}[\hat{\tau}_{\beta,i} \leq \hat{\tau}_{\hat{\beta},i}^+, \hat{\tau}_{\hat{\beta},i}^+ \leq T_f] \leq e^{-\frac{2\bar{B}c_p T_f^{-\frac{1}{2}+\epsilon}(\beta-\hat{\beta})}{\sigma_d^2}}.$$

Similarly,

$$\begin{aligned} &\mathbb{P}[\hat{\tau}_{\beta,i} \leq T_f, \hat{\tau}_{\hat{\beta},i}^+ > T_f] \\ &= \mathbb{P}[\hat{\tau}_{\beta,i} \leq T_f | \hat{\tau}_{\hat{\beta},i}^+ > T_f] \mathbb{P}[\hat{\tau}_{\hat{\beta},i}^+ > T_f] \\ &= \mathbb{P}[\sup_{\hat{\tau}_{\hat{\beta},i} \leq t \leq T_f} X_d^{\pi'}(t) \geq \beta | \hat{\tau}_{\hat{\beta},i}^+ > T_f] \mathbb{P}[\hat{\tau}_{\hat{\beta},i}^+ > T_f] \\ &= \mathbb{P}[\sup_{0 \leq h \leq T_f - \hat{\tau}_{\hat{\beta},i}} X_d^{\pi'}(h + \hat{\tau}_{\hat{\beta},i}) \geq \beta | \hat{\tau}_{\hat{\beta},i}^+ > T_f] \mathbb{P}[\hat{\tau}_{\hat{\beta},i}^+ > T_f]. \end{aligned}$$

Note that here, for $\omega \in \{\omega : \hat{\tau}_{\hat{\beta},i}^+(\omega) > T_f\}$, we have $X_d^{\pi'}(h + \hat{\tau}_{\hat{\beta},i}, \omega) \geq 0$ for all

$h \in [0, T_f - \hat{\tau}_{\hat{\beta}, i}]$. Hence, following a similar procedure as before, we can show that

$$\mathbb{P}[\hat{\tau}_{\beta, i} \leq T_f, \hat{\tau}_{\hat{\beta}, i}^+ > T_f] \leq e^{-\frac{2\bar{B}c_p T_f^{-\frac{1}{2}+\epsilon}(\beta-\hat{\beta})}{\sigma_d^2}}.$$

This completes the proof.

B.12 Proof of Lemma 4

Note that, to hit the level β , the process $X_d(t)$ must hit the intermediate level $\hat{\beta}$ first, since $\beta > \hat{\beta}$ and $X_d(t)$ is continuous. Therefore, by definition of epoch, the event $\{\hat{\tau}_\beta \leq T_f\}$ must happen within an epoch. Let the number of epochs occurring within the interval $[0, T_f]$ be denoted by the random variable K . Here, an upper bound on $\mathbb{P}[\hat{\tau}_\beta \leq T_f]$ is given as follows:

$$\begin{aligned} \mathbb{P}[\hat{\tau}_\beta \leq T_f] &\leq \mathbb{P}[\{\cup_{i=0}^{K-1} \{\hat{\tau}_{\beta, i} \leq \hat{\tau}_{\hat{\beta}, i}^+, \hat{\tau}_{\hat{\beta}, i}^+ \leq T_f\}\} \cup \{\hat{\tau}_{\beta, K} \leq T_f, \hat{\tau}_{\hat{\beta}, K}^+ > T_f\}] \\ &\leq \mathbb{P}[\cup_{i=0}^{K-1} \{\hat{\tau}_{\beta, i} \leq \hat{\tau}_{\hat{\beta}, i}^+, \hat{\tau}_{\hat{\beta}, i}^+ \leq T_f\}] + \mathbb{P}[\hat{\tau}_{\beta, K} \leq T_f, \hat{\tau}_{\hat{\beta}, K}^+ > T_f] \\ &= \mathbb{P}[\cup_{i=0}^{K-1} G_i] + \mathbb{P}[\tilde{G}_K], \end{aligned} \tag{B.71}$$

where $G_i := \{\hat{\tau}_{\beta, i} \leq \hat{\tau}_{\hat{\beta}, i}^+, \hat{\tau}_{\hat{\beta}, i}^+ \leq T_f\}$, and $\tilde{G}_K := \{\hat{\tau}_{\beta, K} \leq T_f, \hat{\tau}_{\hat{\beta}, K}^+ > T_f\}$. Let $H := \{K > K_{max}\}$ and $H^c := \{K \leq K_{max}\}$. From corollary 1 and lemma 3, we have

$$\begin{aligned} \mathbb{P}[G_i] &\leq e^{-\frac{2\bar{B}c_p T_f^{-\frac{1}{2}+\epsilon}(\beta-\hat{\beta})}{\sigma_d^2}}, \quad \mathbb{P}[\tilde{G}_K] \leq e^{-\frac{2\bar{B}c_p T_f^{-\frac{1}{2}+\epsilon}(\beta-\hat{\beta})}{\sigma_d^2}}, \\ \mathbb{P}[H] &\leq K_{max} e^{-\left(\frac{\hat{\beta}^2}{2\sigma_d^2 T} - \frac{2\bar{B}\hat{\beta}c_p T_f^{-\frac{1}{2}+\epsilon}}{\sigma_d^2}\right)}. \end{aligned} \tag{B.72}$$

Further,

$$\begin{aligned}
& \mathbb{P}[\cup_{i=0}^{K-1} G_i] \\
&= \mathbb{P}[\{\cup_{i=0}^{K-1} G_i\}, H^c] + \mathbb{P}[\{\cup_{i=0}^{K-1} G_i\}, H] \\
&\leq \mathbb{P}[\{\cup_{i=0}^{K-1} G_i\}, H^c] + \mathbb{P}[H] \\
&\leq \mathbb{P}[\{\cup_{i=0}^{K-1} G_i\} | H^c] \mathbb{P}[H^c] + \mathbb{P}[H] \\
&\leq \mathbb{P}[\{\cup_{i=0}^{K_{max}-1} G_i\} | H^c] \mathbb{P}[H^c] + \mathbb{P}[H] \\
&= \mathbb{P}[\{\cup_{i=0}^{K_{max}-1} G_i\}, H^c] + \mathbb{P}[H] \\
&\leq \mathbb{P}[\cup_{i=0}^{K_{max}-1} G_i] + \mathbb{P}[H] \\
&\leq \sum_{i=0}^{K_{max}-1} \mathbb{P}[G_i] + \mathbb{P}[H] \\
&\leq K_{max} e^{-\frac{2\overline{B}c_p T_f^{-\frac{1}{2}+\epsilon}}{\sigma_d^2} (\beta-\hat{\beta})} + K_{max} e^{-\left(\frac{\hat{\beta}^2}{2\sigma_d^2 T} - \frac{2\overline{B}\hat{\beta}c_p T_f^{-\frac{1}{2}+\epsilon}}{\sigma_d^2}\right)},
\end{aligned}$$

where the last inequality follows from (B.72). Therefore, from (B.71), it follows that

$$\begin{aligned}
& \mathbb{P}[\hat{\tau}_\beta \leq T_f] \\
&\leq K_{max} \left[e^{-\left(\frac{\hat{\beta}^2}{2\sigma_d^2 T} - \frac{2\overline{B}\hat{\beta}c_p T_f^{-\frac{1}{2}+\epsilon}}{\sigma_d^2}\right)} \right] + (K_{max} + 1) \left[e^{-\frac{2\overline{B}c_p T_f^{-\frac{1}{2}+\epsilon} (\beta-\hat{\beta})}{\sigma_d^2}} \right]. \tag{B.73}
\end{aligned}$$

With $\hat{\beta} = T_f^\gamma$, $\beta = T_f^{(\frac{1}{2}-\psi)}$, and $T = T_\gamma = \frac{1}{4\overline{B}c_p} T_f^{2\gamma-(\epsilon-\psi)}$, it follows that

$$\begin{aligned}
& \mathbb{P}[\hat{\tau}_\beta \leq T_f] \\
&\leq K_{max,\gamma} \left[e^{-\frac{2\overline{B}c_p}{\sigma_d^2} (T_f^{\epsilon-\psi} - T_f^{-\frac{1}{2}+\gamma+\epsilon})} \right] + (K_{max,\gamma} + 1) \left[e^{-\frac{2\overline{B}c_p}{\sigma_d^2} (T_f^{\epsilon-\psi} - T_f^{-\frac{1}{2}+\gamma+\epsilon})} \right] \\
&= (2K_{max,\gamma} + 1) e^{-\frac{2\overline{B}c_p}{\sigma_d^2} (T_f^{\epsilon-\psi} - T_f^{-\frac{1}{2}+\gamma+\epsilon})}, \tag{B.74}
\end{aligned}$$

where $K_{max,\gamma} = \frac{T_f}{T_\gamma} = 4\overline{B}c_p T_f^{1-2\gamma+(\epsilon-\psi)}$. If $\psi < \epsilon < \frac{1}{2} - \gamma$, we have, $1 - 2\gamma + \epsilon - \psi > 0$, $\epsilon - \psi > 0$, and $-\frac{1}{2} + \gamma + \epsilon < 0$. Therefore, it follows that $K_{max,\gamma} = 4\overline{B}c_p T_f^{1-2\gamma+(\epsilon-\psi)}$ polynomially increases with T_f , whereas $e^{-\frac{2\overline{B}c_p}{\sigma_d^2} (T_f^{\epsilon-\psi} - T_f^{-\frac{1}{2}+\gamma+\epsilon})}$ exponentially decreases

to zero as T_f increases. Thus, we can show that,

$$\begin{aligned}
& \lim_{T_f \rightarrow \infty} (2K_{max, \gamma} + 1) e^{-\frac{2\bar{B}c_p}{\sigma_d^2} (T_f^{\epsilon-\psi} - T_f^{-\frac{1}{2}+\gamma+\epsilon})} \\
&= \lim_{T_f \rightarrow \infty} (8\bar{B}c_p T_f^{1-2\gamma+(\epsilon-\psi)} + 1) e^{-\frac{2\bar{B}c_p}{\sigma_d^2} (T_f^{\epsilon-\psi} - T_f^{-\frac{1}{2}+\gamma+\epsilon})} \\
&= 0.
\end{aligned}$$

Thus, from (B.74), we have

$$\lim_{T_f \rightarrow \infty} \mathbb{P}[\hat{\tau}_\beta \leq T_f] = \lim_{T_f \rightarrow \infty} \mathbb{P}[\sup_{0 \leq t \leq T_f} X_d^{\pi'}(t) \geq \beta] = 0.$$

This completes the proof.

B.13 Proof of Theorem 7

Note that, with $N = \tilde{N}_\infty = c_\infty \sqrt{T_f}$ and $\beta = T_f^{(\frac{1}{2}-\psi)} < \tilde{N}_\infty$, from proposition 2, we have the following:

$$\begin{aligned}
& \mathbb{P}[A_1 \cup A_2 \cup A_3 \cup A_4] \\
& \leq \mathbb{P}[X_{c, T_f}^{\pi', sup} \geq 2\tilde{N}_\infty - \beta] + \mathbb{P}[X_{c, T_f}^{\pi', inf} \leq \beta] \\
& + \mathbb{P}[\sup_{0 \leq t \leq T_f} |X_d^{\pi'}(t)| \geq \beta].
\end{aligned} \tag{B.75}$$

Here, we analyze the quantity $\mathbb{P}[X_{c, T_f}^{\pi', sup} \geq 2\tilde{N}_\infty - \beta]$. Recall that, with $N = \tilde{N}_\infty$, we have

$$X_c^{\pi'}(t) = \tilde{N}_\infty + \sigma(W_1(t) + W_2(t)).$$

Since, $(W_1(t) + W_2(t)) \sim \sqrt{2}W_c(t)$ ([70]), where $W_c(t)$ is a standard Wiener process, $X_c^{\pi'}(t)$ can be constructed from $X_c^{\pi'}(t) = \tilde{N}_\infty + \sigma_c W_c(t)$, where $\sigma_c = \sqrt{2}\sigma$. Considering

the initial p.u. energy level, $\alpha = \frac{1}{2}$, we have the following:

$$\begin{aligned}
& \mathbb{P}[X_{c,T_f}^{\pi',sup} \geq 2\tilde{N}_\infty - \beta] \\
&= \mathbb{P}[\sup_{0 \leq t \leq T_f} (\tilde{N}_\infty + \sigma_c W_c(t)) \geq 2\tilde{N}_\infty - \beta] \\
&= \mathbb{P}[\sup_{0 \leq t \leq T_f} (\sigma_c W_c(t)) \geq \tilde{N}_\infty - \beta] \\
&= \mathbb{P}[\sup_{0 \leq t \leq T_f} (W_c(t)) \geq \frac{\tilde{N}_\infty - \beta}{\sigma_c}].
\end{aligned}$$

Note that $W_c(t)$ is a standard Brownian motion. Denoting $\Phi(\cdot)$ as the cumulative distribution function of a standard normal random variable, we can show that ([70]):

$$\begin{aligned}
& \mathbb{P}[\sup_{0 \leq t \leq T_f} (W_c(t)) \geq \frac{\tilde{N}_\infty - \beta}{\sigma_c}] \\
&= 2(1 - \Phi(\frac{\tilde{N}_\infty - \beta}{\sigma_c \sqrt{T_f}})) \\
&\leq e^{-\frac{(\tilde{N}_\infty - \beta)^2}{2\sigma_c^2 T_f}},
\end{aligned}$$

where the last inequality follows from (B.31) (see Appendix B.5). Thus, it follows that.

$$\mathbb{P}[X_{c,T_f}^{\pi',sup} \geq 2\tilde{N}_\infty - \beta] \leq e^{-\frac{(\tilde{N}_\infty - \beta)^2}{2\sigma_c^2 T_f}}. \quad (\text{B.76})$$

Similarly, we can show that,

$$\mathbb{P}[X_{c,T_f}^{\pi',inf} \leq \beta] \leq e^{-\frac{(\tilde{N}_\infty - \beta)^2}{2\sigma_c^2 T_f}}. \quad (\text{B.77})$$

Thus, from (B.75), (B.76), and (B.77), we have the following:

$$\begin{aligned}
& \mathbb{P}[A_1 \cup A_2 \cup A_3 \cup A_4] \\
&\leq 2e^{-\frac{(\tilde{N}_\infty - \beta)^2}{2\sigma_c^2 T_f}} + \mathbb{P}[\sup_{0 \leq t \leq T_f} |X_d^{\pi'}(t)| \geq \beta].
\end{aligned} \quad (\text{B.78})$$

With $\beta = T_f^{(\frac{1}{2}-\psi)}$, we further have,

$$\begin{aligned}
& \lim_{T_f \rightarrow \infty} \left(2e^{-\frac{(\tilde{N}_\infty - T_f^{(\frac{1}{2}-\psi)})^2}{2\sigma_c^2 T_f}} + \mathbb{P}[\sup_{0 \leq t \leq T_f} |X_d^{\pi'}(t)| \geq T_f^{(\frac{1}{2}-\psi)}] \right) \\
&= \lim_{T_f \rightarrow \infty} \left(2e^{-\frac{(\tilde{N}_\infty - T_f^{(\frac{1}{2}-\psi)})^2}{2\sigma_c^2 T_f}} \right) \\
&+ \lim_{T_f \rightarrow \infty} \mathbb{P}[\sup_{0 \leq t \leq T_f} |X_d^{\pi'}(t)| \geq T_f^{(\frac{1}{2}-\psi)}] \\
&= \lim_{T_f \rightarrow \infty} \left(2e^{-\frac{(\tilde{N}_\infty - T_f^{(\frac{1}{2}-\psi)})^2}{2\sigma_c^2 T_f}} \right), \tag{B.79}
\end{aligned}$$

where the last step follows from (3.41). Note that, $\tilde{N}_\infty = c_\infty \sqrt{T_f} = 2\sigma \sqrt{\ln(2/\delta)} \sqrt{T_f}$. Further, since $\sigma_c = \sqrt{2}\sigma$, we have.

$$\begin{aligned}
& \lim_{T_f \rightarrow \infty} \left(2e^{-\frac{(\tilde{N}_\infty - T_f^{(\frac{1}{2}-\psi)})^2}{2\sigma_c^2 T_f}} \right) \\
&= \lim_{T_f \rightarrow \infty} \left(2e^{-\left(\sqrt{\ln(2/\delta)} - \frac{1}{2\sigma} T_f^{(-\psi)}\right)^2} \right) \\
&= 2e^{-\ln(2/\delta)} = \delta. \tag{B.80}
\end{aligned}$$

Hence, from (B.78), (B.79), and (B.80), we have,

$$\lim_{T_f \rightarrow \infty} \mathbb{P}[A_1 \cup A_2 \cup A_3 \cup A_4] \leq \delta.$$

This completes the proof.

B.14 Proof of Lemma 5

Note that with $N = K\tilde{N}_\infty$, from (3.28), we have

$$X_c^{\pi'}(t) = K\tilde{N}_\infty + \sigma_c W_c(t).$$

Recall from (3.22) that, $\tilde{N}_\infty = 2\sigma\sqrt{\ln(2/\delta)}\sqrt{T_f}$. Thus, with $K = \sqrt{1 + \frac{\ln(1/(1-\Delta))}{\ln(2/\delta)}} + \frac{\Delta T_f^{-\psi} + c_\beta T_f^{-\frac{1}{2}}}{2\sigma\sqrt{\ln(2/\delta)}}$ and $\beta_{T_f} = \Delta T_f^{\frac{1}{2}-\psi} + c_\beta$, we have,

$$K\tilde{N}_\infty - \beta_{T_f} = 2\sigma\sqrt{\ln\left(\frac{2}{(1-\Delta)\delta}\right)}\sqrt{T_f}. \quad (\text{B.81})$$

Further, note that, $K\tilde{N}_\infty - \beta_{T_f} > 0$. Here, with $N = K\tilde{N}_\infty$, we have the following:

$$\begin{aligned} & \mathbb{P}[X_{c,T_f}^{\pi',sup} \geq 2N - \beta_{T_f}] \\ &= \mathbb{P}[X_{c,T_f}^{\pi',sup} \geq 2K\tilde{N}_\infty - \beta_{T_f}] \\ &= \mathbb{P}[\sup_{0 \leq t \leq T_f} (K\tilde{N}_\infty + \sigma_c W_c(t)) \geq 2K\tilde{N}_\infty - \beta_{T_f}] \\ &= \mathbb{P}[\sup_{0 \leq t \leq T_f} (\sigma_c W_c(t)) \geq K\tilde{N}_\infty - \beta_{T_f}] \\ &= \mathbb{P}[\sup_{0 \leq t \leq T_f} (W_c(t)) \geq \frac{K\tilde{N}_\infty - \beta_{T_f}}{\sigma_c}]. \end{aligned}$$

Note that $W_c(t)$ is a standard Brownian motion. Denoting $\Phi(\cdot)$ as the cumulative distribution function of a standard normal random variable, we can show that ([70]):

$$\begin{aligned} & \mathbb{P}[\sup_{0 \leq t \leq T_f} (W_c(t)) \geq \frac{K\tilde{N}_\infty - \beta_{T_f}}{\sigma_c}] \\ &= 2\left(1 - \Phi\left(\frac{K\tilde{N}_\infty - \beta_{T_f}}{\sigma_c\sqrt{T_f}}\right)\right) \\ &\leq e^{-\frac{(K\tilde{N}_\infty - \beta_{T_f})^2}{2\sigma_c^2 T_f}}, \end{aligned}$$

where the last inequality follows from (B.31) (see Appendix B.5). Thus, it follows that, with $N = K\tilde{N}_\infty$,

$$\mathbb{P}[X_{c,T_f}^{\pi',sup} \geq 2K\tilde{N}_\infty - \beta_{T_f}] \leq e^{-\frac{(K\tilde{N}_\infty - \beta_{T_f})^2}{2\sigma_c^2 T_f}}. \quad (\text{B.82})$$

Similarly, we can show that,

$$\mathbb{P}[X_{c,T_f}^{\pi',inf} \leq \beta_{T_f}] \leq e^{-\frac{(K\tilde{N}_\infty - \beta_{T_f})^2}{2\sigma_c^2 T_f}}. \quad (\text{B.83})$$

Hence, from (B.82), and (B.83), we have

$$\begin{aligned} & \mathbb{P}[X_{c,T_f}^{\pi',sup} \geq 2K\tilde{N}_\infty - \beta_{T_f}] + \mathbb{P}[X_{c,T_f}^{\pi',inf} \leq \beta_{T_f}] \\ & \leq 2e^{-\frac{(K\tilde{N}_\infty - \beta_{T_f})^2}{2\sigma_c^2 T_f}}. \end{aligned} \quad (\text{B.84})$$

Since $\sigma_c = \sqrt{2}\sigma$, from (B.81) it follows that

$$2e^{-\frac{(K\tilde{N}_\infty - \beta_{T_f})^2}{2\sigma_c^2 T_f}} = 2e^{-\ln\left(\frac{2}{(1-\Delta)\delta}\right)} = (1-\Delta)\delta.$$

Hence, from (B.84), we have the desired result.

B.15 Proof of Lemma 6

We define $T := \frac{\Delta}{4\bar{B}c_p} T_f^{2\gamma - (\epsilon - \psi)}$, $K_{max} := \frac{T_f}{T} = \frac{4\bar{B}c_p}{\Delta} T_f^{1 + (\epsilon - \psi) - 2\gamma}$, and $\hat{\beta} := \Delta T_f^\gamma$. Since, $T_f^\gamma < T_f^{(\frac{1}{2} - \psi)}$ and $\Delta \in (0, 1)$, we have $\hat{\beta} = \Delta T_f^\gamma < \Delta T_f^{(\frac{1}{2} - \psi)}$. Thus with $\beta := \Delta T_f^{(\frac{1}{2} - \psi)}$, from (3.39), we have

$$\begin{aligned} & \mathbb{P}[\sup_{0 \leq t \leq T_f} X_d^{\pi'}(t) \geq \beta] \\ & \leq (2K_{max} + 1)e^{-\frac{2\bar{B}c_p\Delta}{\sigma_d^2} T_f^\epsilon (T_f^{-\psi} - T_f^{-\frac{1}{2} + \gamma})} \\ & = \left(\frac{8\bar{B}c_p}{\Delta} T_f^{1 + (\epsilon - \psi) - 2\gamma} + 1\right)e^{-\frac{2\bar{B}c_p\Delta}{\sigma_d^2} [T_f^{-\psi} - T_f^{-\frac{1}{2} + \gamma}] T_f^\epsilon}. \end{aligned}$$

Since,

$$(8\bar{B}c_p T_f^{1 + (\epsilon - \psi) - 2\gamma} + \Delta)e^{-\frac{2\bar{B}c_p\Delta [T_f^{-\psi} - T_f^{-\frac{1}{2} + \gamma}] T_f^\epsilon}{\sigma_d^2}} = \frac{\Delta\delta}{2},$$

it follows that

$$\mathbb{P}[\sup_{0 \leq t \leq T_f} X_d^{\pi'}(t) \geq \beta] \leq \frac{\Delta\delta}{2}.$$

Similarly from (3.40), we have

$$\mathbb{P}[\inf_{0 \leq t \leq T_f} X_d^{\pi'}(t) \leq -\beta] \leq \frac{\Delta\delta}{2}.$$

Further, note that $\beta_{T_f} = \Delta T_f^{(\frac{1}{2}-\psi)} + c_\beta > \Delta T_f^{(\frac{1}{2}-\psi)} = \beta$, since $c_\beta > 0$. Hence, $\sup_{0 \leq t \leq T_f} X_d^{\pi'}(t) \geq \beta_{T_f} \implies \sup_{0 \leq t \leq T_f} X_d^{\pi'}(t) \geq \beta$, and also $\inf_{0 \leq t \leq T_f} X_d^{\pi'}(t) \leq -\beta_{T_f} \implies \inf_{0 \leq t \leq T_f} X_d^{\pi'}(t) \leq -\beta$. Therefore,

$$\begin{aligned} & \mathbb{P}[\sup_{0 \leq t \leq T_f} X_d^{\pi'}(t) \geq \beta_{T_f}] \\ & \leq \mathbb{P}[\sup_{0 \leq t \leq T_f} X_d^{\pi'}(t) \geq \beta] \leq \frac{\Delta\delta}{2}. \end{aligned}$$

and,

$$\begin{aligned} & \mathbb{P}[\inf_{0 \leq t \leq T_f} X_d^{\pi'}(t) \leq -\beta_{T_f}] \\ & \leq \mathbb{P}[\inf_{0 \leq t \leq T_f} X_d^{\pi'}(t) \leq -\beta] \leq \frac{\Delta\delta}{2}. \end{aligned}$$

Thus, from the inequality (3.32), it follows that

$$\begin{aligned} & \mathbb{P}[\sup_{0 \leq t \leq T_f} |X_d^{\pi'}(t)| \geq \beta_{T_f}] \\ & \leq \mathbb{P}[\sup_{0 \leq t \leq T_f} X_d^{\pi'}(t) \geq \beta_{T_f}] \\ & + \mathbb{P}[\inf_{0 \leq t \leq T_f} X_d^{\pi'}(t) \leq -\beta_{T_f}] \\ & \leq \Delta\delta. \end{aligned}$$

This completes the proof.

Appendix C

C.1 Probability distribution function of Phasor Data

Since each of the samples drawn from the distribution of $\mathbf{X}$ belongs to one of the K different topologies, we can deduce the following: for $\mathbf{x} \in \mathbb{R}^{2N}$,

$$\mathbb{P}[\mathbf{X} \leq \mathbf{x}] = \sum_{k=1}^K \mathbb{P}[\mathbf{X} \leq \mathbf{x} | \mathbf{Z} = \mathbf{e}_k] \mathbb{P}[\mathbf{Z} = \mathbf{e}_k]. \quad (\text{C.1})$$

Recall that, for $\alpha \in E_k := \{\alpha' : \mathbf{Z}(\alpha') = \mathbf{e}_k\}$, the random variable $\mathbf{X}(\alpha) \sim \mathcal{N}(\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)$. Thus we have $\mathbb{P}[\mathbf{X} \leq \mathbf{x} | \mathbf{Z} = \mathbf{e}_k] = \mathbb{P}[\{\alpha \in E_k : \mathbf{X}(\alpha) \leq \mathbf{x}\}] = F_{\mathbf{X}|\mathbf{Z}, \phi_k}(\mathbf{x} | \mathbf{Z} = \mathbf{e}_k)$, where $F_{\mathbf{X}|\mathbf{Z}, \phi_k}(\cdot)$ denotes the cumulative distribution function of a Gaussian random variable with parameters $\phi_k = (\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)$. Further recall that $\pi_k = \mathbb{P}[\mathbf{Z} = \mathbf{e}_k]$. Therefore, from (C.1), $\mathbb{P}[\mathbf{X} \leq \mathbf{x}]$ can be written as

$$\begin{aligned} \mathbb{P}[\mathbf{X} \leq \mathbf{x}] &= \sum_{k=1}^K F_{\mathbf{X}|\mathbf{Z}, \phi_k}(\mathbf{x} | \mathbf{Z} = \mathbf{e}_k) \mathbb{P}[\mathbf{Z} = \mathbf{e}_k] \\ &= \sum_{k=1}^K \pi_k F_{\mathbf{X}|\mathbf{Z}, \phi_k}(\mathbf{x} | \mathbf{Z} = \mathbf{e}_k). \end{aligned} \quad (\text{C.2})$$

Note that, since $\pi_k = \mathbb{P}[\mathbf{Z} = \mathbf{e}_k]$, we have $\pi_k \geq 0$ for all $k \in \{1, \dots, K\}$, and since there are K different topologies, we have $\sum_{k=1}^K \pi_k = \sum_{k=1}^K \mathbb{P}[\mathbf{Z} = \mathbf{e}_k] = 1$. Here, note that the distribution of $\mathbf{X}$ depends on $\boldsymbol{\Phi}_K = (\pi_1, \dots, \pi_K, \phi_1, \dots, \phi_K)$. From (C.2), we derive the PDF of $\mathbf{X}$, which can be written as follows:

$$f_{\mathbf{X}|\boldsymbol{\Phi}_K}(\mathbf{x}) = \sum_{k=1}^K \pi_k f_{\mathbf{X}|\mathbf{Z}, \phi_k}(\mathbf{x} | \mathbf{Z} = \mathbf{e}_k),$$

with $\sum_{k=1}^K \pi_k = 1$, $\pi_k \geq 0$ for all $k \in \{1, \dots, K\}$.

C.2 A brief overview of EM algorithm

Recall that, $\mathbf{Z}$ is a random vector of indicator variables taking values in $\{\mathbf{e}_1, \dots, \mathbf{e}_K\}$, where $\mathbf{e}_k \in \mathbb{R}^K, i \in \{1, \dots, K\}$ denote standard basis vectors of $\mathbb{R}^K$. Here we can derive the following: $f_{\mathbf{X}|\Phi_K}(\mathbf{x}) = \sum_{\mathbf{z} \in \{\mathbf{e}_1, \dots, \mathbf{e}_K\}} f_{\mathbf{X}, \mathbf{Z}|\Phi_K}(\mathbf{x}, \mathbf{z})$, where $f_{\mathbf{X}, \mathbf{Z}|\Phi_K}(\mathbf{x}, \mathbf{z})$ denotes the joint density of $\mathbf{X}$ and $\mathbf{Z}$. Thus the problem (4.8) can be equivalently written as

$$\begin{aligned} \hat{\Phi}_K &= \operatorname{argmax}_{\Phi_K} \ln[\prod_{n=1}^S \sum_{\mathbf{z} \in \{\mathbf{e}_1, \dots, \mathbf{e}_K\}} f_{\mathbf{X}, \mathbf{Z}|\Phi_K}(\mathbf{x}^{(n)}, \mathbf{z})] \\ \text{such that } &\sum_{i=1}^K \pi_k = 1, \pi_k \geq 0 \text{ for all } k \in \{1, \dots, K\}. \end{aligned} \quad (\text{C.3})$$

However, finding an analytical solution remains intractable, hence in the EM algorithm, we try to maximize a lower bound of the objective function iteratively [104]. Suppose, the current estimate of Φ_K is Φ_K^l at iteration l . Here we can derive the following:

$$\begin{aligned} &\ln[\prod_{n=1}^S \sum_{\mathbf{z} \in \{\mathbf{e}_1, \dots, \mathbf{e}_K\}} f_{\mathbf{X}, \mathbf{Z}|\Phi_K}(\mathbf{x}^{(n)}, \mathbf{z})] \\ &= \sum_{n=1}^S \ln[\sum_{\mathbf{z} \in \{\mathbf{e}_1, \dots, \mathbf{e}_K\}} f_{\mathbf{X}, \mathbf{Z}|\Phi_K}(\mathbf{x}^{(n)}, \mathbf{z})] \\ &= \sum_{n=1}^S \ln[\sum_{\mathbf{z} \in \{\mathbf{e}_1, \dots, \mathbf{e}_K\}} \frac{f_{\mathbf{X}, \mathbf{Z}|\Phi_K}(\mathbf{x}^{(n)}, \mathbf{z}) f_{\mathbf{Z}|\mathbf{X}, \Phi_K^l}(\mathbf{z}|\mathbf{X}=\mathbf{x}^{(n)})}{f_{\mathbf{Z}|\mathbf{X}, \Phi_K^l}(\mathbf{z}|\mathbf{X}=\mathbf{x}^{(n)})}] \\ &= \sum_{n=1}^S \ln(\mathbb{E}_{\mathbf{Z} \sim f_{\mathbf{Z}|\mathbf{X}, \Phi_K^l}} \left[\frac{f_{\mathbf{X}, \mathbf{Z}|\Phi_K}(\mathbf{x}^{(n)}, \mathbf{Z})}{f_{\mathbf{Z}|\mathbf{X}, \Phi_K^l}(\mathbf{Z}|\mathbf{X}=\mathbf{x}^{(n)})} \middle| \mathbf{X} = \mathbf{x}^{(n)} \right]) \\ &\geq \sum_{n=1}^S [\mathbb{E}_{\mathbf{Z} \sim f_{\mathbf{Z}|\mathbf{X}, \Phi_K^l}} [\ln(f_{\mathbf{X}, \mathbf{Z}|\Phi_K}(\mathbf{x}^{(n)}, \mathbf{Z})) | \mathbf{X} = \mathbf{x}^{(n)}] \\ &\quad - \mathbb{E}_{\mathbf{Z} \sim f_{\mathbf{Z}|\mathbf{X}, \Phi_K^l}} [\ln(f_{\mathbf{Z}|\mathbf{X}, \Phi_K^l}(\mathbf{Z}|\mathbf{X} = \mathbf{x}^{(n)})) | \mathbf{X} = \mathbf{x}^{(n)}]], \end{aligned} \quad (\text{C.4})$$

where the last line follows from Jensen's inequality [104]. Thus, Φ_K^{l+1} which maximizes the lower bound (C.4) can be computed as

$$\begin{aligned} \Phi_K^{l+1} &= \operatorname{argmax}_{\Phi_K} \sum_{n=1}^S \\ &[\mathbb{E}_{\mathbf{Z} \sim f_{\mathbf{Z}|\mathbf{X}, \Phi_K^l}} [\ln(f_{\mathbf{X}, \mathbf{Z}|\Phi_K}(\mathbf{x}^{(n)}, \mathbf{Z})) | \mathbf{X} = \mathbf{x}^{(n)}] \\ \text{such that } &\sum_{i=1}^K \pi_k = 1, \pi_k \geq 0 \text{ for all } k \in \{1, \dots, K\}, \end{aligned}$$

which defines the EM algorithm. Here, in the E-step, we compute the expectation, and in the M-step we maximize it. As explained in Section 4.4.1, the computation of the expectation involves computation of $h_k^{n,l} := \mathbb{P}[\mathbf{Z} = \mathbf{e}_k | \mathbf{X} = \mathbf{x}^{(n)}, \Phi_K^l]$ for all

$k \in \{1, \dots, K\}$, whereas the term $\ln[f_{\mathbf{X}, \mathbf{Z}|\Phi_K}(\mathbf{x}^{(n)}, \mathbf{Z})]$ can be analytically expressed utilizing (4.5). We compute $h_k^{n,l}$ as shown below [104]:

$$\begin{aligned} h_k^{n,l} &= \frac{f_{\mathbf{X}|\mathbf{Z}, \Phi_K^l}(\mathbf{x}^{(n)}|\mathbf{Z}=\mathbf{e}_k)\mathbb{P}[\mathbf{Z}=\mathbf{e}_k]}{\sum_{k=1}^K f_{\mathbf{X}|\mathbf{Z}, \Phi_K^l}(\mathbf{x}^{(n)}|\mathbf{Z}=\mathbf{e}_k)\mathbb{P}[\mathbf{Z}=\mathbf{e}_k]} \\ &= \frac{\pi_k^l |\boldsymbol{\Sigma}_k^l|^{-\frac{1}{2}} e^{-\frac{1}{2}(\mathbf{x}^{(n)} - \boldsymbol{\mu}_k^l)^T (\boldsymbol{\Sigma}_k^l)^{-1} (\mathbf{x}^{(n)} - \boldsymbol{\mu}_k^l)}}{\sum_{k=1}^K \pi_k^l |\boldsymbol{\Sigma}_k^l|^{-\frac{1}{2}} e^{-\frac{1}{2}(\mathbf{x}^{(n)} - \boldsymbol{\mu}_k^l)^T (\boldsymbol{\Sigma}_k^l)^{-1} (\mathbf{x}^{(n)} - \boldsymbol{\mu}_k^l)}}, \end{aligned}$$

where π_k^l , $\boldsymbol{\mu}_k^l$, and $\boldsymbol{\Sigma}_k^l$ are the estimates of the corresponding quantities at iteration l . In the M-step, utilizing the constraints on π_k and solving the maximization problem, it can be shown [104] that the updated values of π_k^{l+1} , $\boldsymbol{\mu}_k^{l+1}$ and $\boldsymbol{\Sigma}_k^{l+1}$ are given by

$$\begin{aligned} \pi_k^{l+1} &= \frac{\sum_{n=1}^S h_k^{n,l}}{S}, \quad \boldsymbol{\mu}_k^{l+1} = \frac{\sum_{n=1}^S h_k^{n,l} \mathbf{x}^{(n)}}{\sum_{n=1}^S h_k^{n,l}} \\ \boldsymbol{\Sigma}_k^{l+1} &= \frac{\sum_{n=1}^S h_k^{n,l} (\mathbf{x}^{(n)} - \boldsymbol{\mu}_k^{l+1})(\mathbf{x}^{(n)} - \boldsymbol{\mu}_k^{l+1})^T}{\sum_{n=1}^S h_k^{n,l}}. \end{aligned}$$

In the EM algorithm, we iteratively compute $h_k^{n,l}$ in E-step and update π_k^l , $\boldsymbol{\mu}_k^{l+1}$, $\boldsymbol{\Sigma}_k^{l+1}$ in the M-step until convergence.