

Distributed Finite-Time Termination for Consensus Algorithm in Switching Topologies

Govind Saraswat^{ID}, *Member, IEEE*, Vivek Khatana^{ID}, Sourav Patel^{ID}, and Murti V. Salapaka^{ID}, *Fellow, IEEE*

Abstract—In this article, we present a finite-time stopping criterion for consensus algorithms in networks with dynamic communication topology. Prior state of the art has established convergence to the consensus value; however, the asymptotic convergence of these algorithms poses a challenge in practical settings where the response from agents is required in finite time. To this end, we propose a maximum-minimum protocol that propagates the global maximum and minimum values of agent states (while running the consensus algorithm) in the network. This article focuses on establishing that the global maximum and minimum values are strictly monotonic even for a dynamic topology, and they can be used to distributively ascertain the closeness to convergence in finite time. We rigorously show that each node can have access to the global maximum and minimum by running the proposed maximum-minimum protocol to realize a finite-time stopping criterion for the otherwise asymptotic consensus algorithm. The practical utility of the algorithm is illustrated through experiments where each agent is instantiated by a NodeJS *socket.io* server.

Index Terms—Distributed consensus, multi-agent systems, network-based computing systems, switching topology.

I. INTRODUCTION

THE availability of a large number of low-cost sensors and the development of suitable network protocols have led to the development of modern-day, multi-agent systems (MAS). In many practical domains, MAS are often designed to attain coordinated objectives, such as movement coordination among a group of mobile autonomous vehicles for traffic optimization [1] or task allocation among agents (nodes), such as unmanned aerial vehicles or autonomous underwater vehicles for search and survey operations [2], [3]. Such systems are limited by their size and mobile nature, which restrict computational and

sensing resources, rendering distributed algorithms well-suited for the coordination of these MAS [4].

A number of works in the literature have proposed distributed coordination algorithms for MAS, such as those considering asynchronous sampling and nonidentical packet loss [5], nonlinear agent dynamics with position feedback [6], and with transmission constraints and noise [7], [8]. Some specific works have proposed distributed coordination algorithms for MAS achieving consensus on the average of the agents' initial state values [9] and on the privacy-preserving consensus on directed graphs [10]. Most of these algorithms consider a static/fixed interconnection topology between the agents involved in the consensus algorithms; however, many practical distributed decision-making applications involve a dynamic interconnection topology. Some examples of such scenarios are described next.

Ad hoc cognitive radio networks. In cognitive radio networks, secondary users can sense the spectrum to detect the presence of primary users. In a spectrum-sensing consensus algorithm [11], secondary users mutually transmit and receive their states according to the real-time (dynamic) network topology, regardless of whether primary users are present. The topology is created when secondary users establish communication links with their own neighbors to locally exchange information among them. The algorithm iterations are repeatedly done until all the individual states converge toward the average of the initial value of the states to make a local decision.

Control of autonomous agents. It is often necessary to coordinate a collection of autonomous agents (e.g., cars or unmanned aerial vehicles). For example, consider a fleet of self-driving cars where each car can communicate with its neighboring cars. One might wish for the cars to meet a global objective, such as maintaining a particular formation where the neighbors of a car can change in real time. A distributed decision is usually needed in such situations. All the cars can distributively agree on a direction or an average speed (or both). Such a coordination model was investigated in [2].

The authors in [12] introduced a gossip-based decentralized scheme, called push-sum protocol, to compute the average of the initial state values of nodes. The push-sum protocol has been shown to converge to the average exponentially fast. The authors of [13] proposed a ratio-consensus protocol in which agents in a fixed topological network asymptotically converge to the ratio of the sum of the initial values of the two states maintained by each agent. The algorithms in [12], [13] have a

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Govind Saraswat is with National Renewable Energy Laboratory, Golden, CO 80401 USA (e-mail: govind.saraswat@gmail.com).

Vivek Khatana, Sourav Patel, and Murti V. Salapaka are with Department of Electrical and Computer Engineering, University of Minnesota, Minneapolis, MN 55455 USA (e-mail: khata010@umn.edu; patel292@umn.edu; murtis@umn.edu).

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significant advantage in that the protocol can be realized in a truly distributed manner, without requiring any centralized dissemination of parameters, making them suitable for plug-and-play *ad hoc* networks. The article [14] extends this approach to the case where the network topology is dynamic (as required in the previous examples) under the condition that the union of the communication graphs at different time instants remains strongly connected infinitely often. The consensus algorithm in [14] provides asymptotic guarantees for agreement between the states of the agents. Many MAS with real-time applications, however, require the consensus value to be used by each agent for a subsequent task or action, and thus a finite-time distributed stopping criterion for many applications is imperative. Moreover, if the agents can distributively detect the convergence within a prespecified tolerance of the consensus value, they will avoid running the algorithm longer than necessary and save often scarce power and computational resources.

The finite-time synchronization phenomena for coupled networks have been thoroughly studied in the literature, where a controller synthesis is required to synchronize the trajectories of the states in the network within a finite time. For example, reference [15] investigated a framework that considers agents in a coupled network with Markovian topology and impulsive perturbations. The continuous-time (non-) linear perturbed system is analyzed to determine the sufficient conditions guaranteeing the existence of a finite setting time for synchronization of the agents in the network. The authors of [16], [17], [18] looked into a coupled system with nonlinear coupling strengths and nonidentical uncertain bounded external disturbances; discontinuous neural networks with nonlinear activation functions; and mixed delays in the network as well as fuzzy cellular neural networks with proportional delays, respectively. Similarly, references [19], [20] present an adaptive neural network finite-time control strategy and an optimal controller design, respectively, for nonlinear systems with uncertainties using back-stepping and power integrator technique. Article [21] considers the issue of adaptive neural decentralized tracking control for a class of output-constrained switched interconnected nonlinear systems with unknown backlash-like hysteresis control input. A nonlinear function approximator and state observer are designed using neural networks. References [15]–[21] need a stabilizing controller to be explicitly designed for each agent in the network ($(N - 1)$ controllers for an N multi-agent system) to synchronize the perturbed coupled systems within a finite time such that the states of any two nodes in the system are identical in the expected sense after the finite time.

Unlike these works, however, the network state updates considered in the current article are a discrete-time linear first-order system without any stochastic perturbations, and the objective of the first-order multi-agent system is to synchronize to the average of the initial conditions of all the agents in the network, where the network topology can change with time by updating the state information with neighboring agents and without having to design controllers at any agent in the network to enforce synchronization.

The problem of designing a finite-time consensus protocol has been considered in the literature. Articles [22], [23] designed *continuous-time* updates that can be terminated in finite time to achieve consensus among the agents. Article [22] proposed an appointed-time consensus scheme based on continuous-time updates under directed spanning trees with switching topologies for a network of agents with double integrator dynamics, whereas article [23] proposed a continuous-time average consensus protocol for switching topologies where the state consensus is achieved in finite time; however, the applicability of the continuous-time updates requires a suitably designed event-triggered communication mechanism for practical applications. The finite-time consensus problem for multiple nonidentical second-order nonlinear systems based on a *leader-follower* architecture under a *static* interconnection topology was studied in [24]. The works [25], [26], [27], [28], [29] focused on the design of a finite-time stopping criteria for *distributed* consensus algorithms with *discrete-time* updates over *static* communication topologies. The authors in [25] presented a method to achieve the consensus value in a finite number of iterations under static graph topologies. Here, each node can calculate the final consensus value using the minimal polynomial associated with the weight matrix in the state update iterations; however, to calculate the coefficients of the minimal polynomial, each node needs to run N (total number of agents) different linear iterations for at least $N + 1$ time steps. Reference [26] achieved finite-time consensus based on the network observability matrix. Here, every node needs storage and computational abilities commensurate with the requirements of matrix inversions and rank calculations, which makes it unsuitable for MAS based on computationally less capable Internet of Things systems. The authors of [27], [28], [29] designed a computationally advantageous distributed detection mechanism to ascertain the convergence of average consensus algorithms under static interconnection topologies. They established that the sequence of global maximum (and minimum) state values of the agents following a consensus algorithm is a monotonic sequence converging to the consensus when the network topology is fixed. A distributed maximum (and minimum) protocol was proposed to propagate these global maximum (and minimum) state values to achieve finite-time consensus within a prespecified tolerance margin. Having additional states corresponding to the global maximum and minimum values helps each node to detect the progress toward consensus. Each node can simultaneously detect the convergence in finite time; thus, the consensus protocol is terminated by each node at the same iteration. These works, however, do not consider switching topologies, and adopting the same framework for the network with changing topologies can lead to convergence toward erroneous states in the network.

In this article, we focus on answering the following key research issues:

- i) Can the protocols developed to detect the convergence of distributed consensus algorithms under static topologies [27], [28], [29] be used as a finite-time criteria in

the case when the communication topology of the agents is *dynamic* ?

- ii) If not, then how can we design a criterion to detect when the agents achieve (approximate) consensus to the average of their initial values?

In this sequel, we show that the approaches in [27], [28], [29] are not directly applicable when the network topology is dynamic. Here, we propose an extension of this approach to develop a distributed finite-time criterion that can be applied to networks with dynamic topology. The work in [30] introduced the concept of a node's influence on the other parts of the network for a static graph. Here, we extend that idea to a novel concept of a "time-path," which incorporates the influence of a node on the other nodes of the network with changes in time as well as the topology. We establish the existence of time-paths of finite length for all pairs of agents in the network. Leveraging the existence of time-paths, we propose a new maximum-minimum protocol to propagate the global maximum and minimum state values. The article makes the following contributions:

Statement of contribution:

- 1) This article presents an algorithm with a distributed stopping criterion for ratio consensus in the presence of directed switching topologies. We augment the ratio consensus algorithm with two additional states: the global maximum and the global minimum of the values held by the agents. We show that these values are monotonic in nature and converge to the consensus value even when the network is dynamic. The stopping criterion can be set such that if the criterion is met by an agent, then it has access to the consensus value within any prespecified tolerance margin. Here, the maximum and minimum consensus-based distributed stopping approach [27], [28] is extended to the case of time-varying topologies. To the best of the authors' knowledge, such a criterion for average consensus algorithms in switching topologies does not exist in the literature.
- 2) We establish an upper bound on the number of iterations required for a node in the network to influence all other nodes. To achieve the upper bound, we introduce a novel concept of a time-path. We prove the existence of finite-length time-paths for every pair of nodes, and we present the maximum-minimum protocol to propagate the global maximum and global minimum state values in the network.
- 3) The scheme proposed here is shown to be amenable for distributed implementation because it only requires each agent to have access to an upper bound on the number of nodes. The performance of the proposed algorithm is illustrated by experimentally realizing a 1,000-node network with dynamic topology created using a NodeJS framework. This demonstrates the scalability of the proposed protocol. Here, each node is implemented as a *socket.io* [31] server. This experimentally validates the distributed stopping criterion in the presence of switching topologies, thus providing

evidence that our algorithm is indeed applicable for real-time applications [32].

The rest of the paper is organized as follows. In Section II, the basic definitions needed for subsequent development as well as the setup for distributed averaging using ratio consensus are presented. In Section III, analytical results for the distributed finite-time termination of ratio consensus in switching topology using maximum and minimum consensus algorithms are discussed. Theoretical findings are validated with the experiments presented in Section IV, followed by conclusions in Section V.

II. DEFINITIONS AND BACKGROUND

In this section, we present basic notions of graph theory and linear algebra, which are essential for the subsequent developments as well as the relevant background results. Detailed descriptions of graph theory and linear algebra notions are available in [33] and [34], respectively.

Definition 1: (Cardinality of a set) Let A be a set. The cardinality of a set, A , denoted by $|A|$, is a measure of the number of elements of the set A .

Definition 2: (Directed graph) A directed graph (denoted as digraph), $\mathcal{G}$, is a pair, $(\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is a set of vertices or nodes, and $\mathcal{E}$ is a set of edges, which are ordered subsets of two distinct elements of $\mathcal{V}$. If an edge from $j \in \mathcal{V}$ to $i \in \mathcal{V}$ exists, then it is denoted as $(i, j) \in \mathcal{E}$.

Definition 3: (Path) In a directed graph, a directed path from node i to j exists if there is a sequence of distinct directed edges of $\mathcal{G}$ of the form $(k_1, i), (k_2, k_1), \dots, (j, k_m)$.

Definition 4: (Strongly connected graph) A directed graph is strongly connected if it has a directed path between each pair of distinct nodes i and j .

Definition 5: (Column stochastic matrix) A real $n \times n$ matrix $A = [a_{ij}]$ is called a column stochastic matrix if $1 \geq a_{ij} \geq 0$ for $1 \leq i, j \leq n$ and $\sum_i a_{ij} = 1$ for $1 \leq j \leq n$.

Definition 6: (Irreducible matrix) A $n \times n$ matrix A is reducible if we may partition $\{1, \dots, n\}$ into two nonempty subsets, E, F , such that $A_{ij} = 0$ if $i \in E, j \in F$. Matrix A is called irreducible if it is not reducible.

Definition 7: (Aperiodic matrix) Let A be a $n \times n$ matrix. Let $\mathcal{G}$ be a directed graph induced by the zero/nonzero structure of A . Then matrix A is aperiodic if $\mathcal{G}$ is strongly connected with at least one node has a self-loop.

Definition 8: (Primitive matrix) A $N \times N$ matrix A is said to be primitive if for any $i, j \in \{1, \dots, N\}$, there exist a $m \in \mathbb{N}$, such that $(A^m)(i, j) > 0$, that is, it is possible to reach any state from any other state in a finite number of hops.

Next, the key result from [35], which enables reaching the average consensus in the presence of dynamic topology, is summarized. Consider a scenario where the network topology is dynamic but with a fixed set of nodes, $\mathcal{V}$ ($|\mathcal{V}| = n$), i.e., at any given instant, k , the network is described by a digraph, $\mathcal{G}(k) = (\mathcal{V}, \mathcal{E}(k))$. Let $\mathcal{P}(k) = (p_{ij}(k))$ be the weighted adjacency matrix associated with the digraph, $\mathcal{G}(k)$. Here, $\mathcal{G}(k) \in \bar{\mathcal{G}} = \{\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_m\}$, $m \leq 2^{n^2-n}$ is the set of all possible digraphs for a given set of nodes, $\mathcal{V}$. We assume that a node

always has access to its own information, i.e., for any node, $i \in \mathcal{V}$, $(i, i) \in \mathcal{E}(k)$, for all k . Now, we present a few definitions related to dynamic topology.

Definition 9: (Union of digraphs) Given a collection of digraphs, $\{\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_m\}$ (for some $m \geq 1$), of the form $\mathcal{G}(k) = (\mathcal{V}, \mathcal{E}(k))$, $k = 1, 2, \dots, m$, the union of digraphs is defined as $\mathcal{G}_{1,2,\dots,m} = (\mathcal{V}, \cup_{k=1}^m \mathcal{E}(k))$.

Definition 10: (In-neighborhood of node i at instant k) The set of in-neighbors of node $i \in \mathcal{V}$ at instant k is denoted by $N^-(i, k) = \{j : (j, i) \in \mathcal{E}(k)\}$.

Definition 11: (Out-neighborhood of node i at instant k) The set of out-neighbors of node $i \in \mathcal{V}$ at instant k is denoted by $N^+(i, k) = \{j : (i, j) \in \mathcal{E}(k)\}$.

Definition 12: (Out-degree of node i at instant k) The out-degree of a node, i , at any instant, k , is equal to the cardinality of the set of out-neighbors of i , $|N^+(i, k)|$.

Each node, $i \in \mathcal{V}$, maintains two states at time k , denoted by $x_i(k)$ (referred to as the numerator state of node i) and $y_i(k)$ (referred to as the denominator state of node i). Node i updates its state at the $(k+1)^{th}$ iteration according to the following policy:

$$x_i(k+1) = p_{ii}(k)x_i(k) + \sum_{j \in N^-(i,k) \setminus \{i\}} p_{ij}(k)x_j(k) \quad (1)$$

$$y_i(k+1) = p_{ii}(k)y_i(k) + \sum_{j \in N^-(i,k) \setminus \{i\}} p_{ij}(k)y_j(k), \quad (2)$$

where $y_i(0) = 1$ for all $i \in \mathcal{V}$. Using (1) and (2), we get

$$\frac{x_i(k+1)}{y_i(k+1)} = \frac{p_{ii}(k)x_i(k) + \sum_{j \in N^-(i,k) \setminus \{i\}} p_{ij}(k)x_j(k)}{p_{ii}(k)y_i(k) + \sum_{j \in N^-(i,k) \setminus \{i\}} p_{ij}(k)y_j(k)}. \quad (3)$$

Throughout the rest of the paper, we consider the network with dynamic topology to satisfy the following assumptions, and the underlying communication network is assumed to be delay free.

Assumption 1: The digraphs $\mathcal{G}(k) = (\mathcal{V}, \mathcal{E}(k))$ are strongly connected for all $k \in \mathbb{N}$.

Assumption 2: $\mathcal{P}(k)$ for all k is a column-stochastic matrix.

One choice of weights that satisfies Assumption 2 is the *out-degree-based equal neighbor weights rule* [36]: Here, for all $j \in \mathcal{V}$ and $k \geq 0$:

$$p_{ij}(k) = \begin{cases} \frac{1}{1+|N_j^+(k)|}, & \text{if } i \in N_j^+(k) \\ 0, & \text{if } i \notin N_j^+(k) \end{cases} \quad (4)$$

and,

$$p_{jj}(k) = \frac{1}{1+|N_j^+(k)|} \text{ for all } k \geq 0, \quad (5)$$

where $|N_j^+(k)|$ denotes the out-degree of a node, $j \in \mathcal{V}$. Note that because the out-degree is known locally, each agent can decide the weights, $p_{ij}(k)$, in a decentralized manner, leading to a distributed synthesis. Using the equal neighbor weights rule, (4)–(5), the updates (1) and (2) are done in the following manner:

Every agent j sends the weighted information, $p_{ij}(k)x_j(k)$ and $p_{ij}(k)y_j(k)$, to every agent i in its out-neighborhood. Then, at the next time instant, agent i scales its own estimates, $x_i(k), y_i(k)$, with a weight $p_{ii}(k)$, and adds it to the information sent by all the agents in its in-neighborhood, $j \in N_i^-(k)$, to get the estimates $x_i(k+1)$ and $y_i(k+1)$. Thus, a distributed implementation of the updates (1) and (2) is realized. Note that $\mathcal{P}(k)$ synthesized using (4) and (5) satisfies the primitiveness and irreducibility properties (see [35] Lemma 2). We next present the convergence result from [35], which forms the basis of our work.

Theorem 1: Consider a sequence of digraphs of the form $\mathcal{G}(k) = (\mathcal{V}, \mathcal{E}(k))$, $k = 0, 1, 2, \dots$, satisfying Assumption 1 and Assumption 2. With the update rule (1) and (2), the ratio $\frac{x_i(k)}{y_i(k)}$ asymptotically converges to $\frac{\sum_{i=1}^n x_i(0)}{n}$ for all $i = 1, \dots, n$, that is, the ratio of the numerator and denominator states converge to the average of the initial conditions of x_i variables (referred to as the ratio consensus).

Proof: See Theorem 2 in [35] for the proof. ■

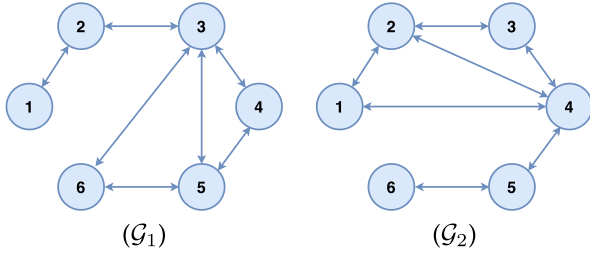
Remark 1.1: Under Assumption 2, $\mathcal{P}(k)$ is only column stochastic and not row stochastic ($\sum_{i=1}^n p_{ij}(k) \neq 1$ for all $i = 1, \dots, n$). Here, $x_i(k)$ and $y_i(k)$ converges to $\gamma_i \sum_{i=1}^n x_i(0)$ and $\gamma_i \sum_{i=1}^n y_i(0)$, respectively, where γ_i is the i^{th} entry of the vector, γ given by $\lim_{k \rightarrow \infty} \prod_{s=1}^k \mathcal{P}(s) = \gamma \mathbf{1}_n$, where $\mathbf{1}_n$ is the n -dimensional vector with all entries equal to one. By having the ratio $\frac{x_i(k)}{y_i(k)}$ and initializing $y_i(0) = 1$ for all $i \in \mathcal{V}$, each agent i 's state, $y_i(k)$, converges to $n\gamma_i$, and the ratio $\frac{x_i(k)}{y_i(k)}$ converges to the average of the initial states of x_i 's, $\frac{\sum_{i=1}^n x_i(0)}{n}$.

Remark 1.2: Note that even though Theorem 1 does not require $\mathcal{P}(k)$ to be row-stochastic, Lemma 4.2 in [37] proves that the $x_i(k)$ and $y_i(k)$ following the updates of (1) and (2) are almost surely bounded below by some positive constant. Thus, the state updates will not become arbitrarily small, and the ratio state will always be well defined. Further, due to the weak ergodicity [38] of the products of $\mathcal{P}(k)$'s, along with the bounded initializations $x_i(0)$ and $y_i(0)$, the individual states, $x_i(k)$ and $y_i(k)$, will also remain bounded.

Remark 1.3: Note that the discrete-time consensus algorithm is agnostic to the frequency with which the underlying graph topology is changing. In particular, only the network state at the time of communication is used. All the constraints on the network are imposed at time instant of communication, that is, any switching happening between the two discrete time instants is not considered.

A. Motivation

As discussed briefly in Section I, the authors of [27], [28] presented a finite-time criterion for a consensus algorithm using the global maximum and minimum of the ratio states in the case when the topology of the network is static. In this subsection, we highlight the limitation of that work when the communication topology is dynamic. It was shown in [27], [28] that within D_{max} iterations, every node has access to the global maximum and minimum, where D_{max} is an upper

Fig. 1. Both $\mathcal{G}_1$ and $\mathcal{G}_2$ have diameter 3.

bound on the graph diameter. The result from [27], [28] does not hold for a network with dynamic topology. The following counterexample highlights the case where an upper bound on the maximum diameter ($D_{max} = \{max(D(\mathcal{G})); \mathcal{G} \in \bar{\mathcal{G}}\}$, where $D(\mathcal{G})$ is the diameter of graph $\mathcal{G}$) does not provide access to the global maximum and minimum to all the nodes of the network. Let us consider a network of six nodes, where the network topology can be either of undirected graphs $\mathcal{G}_1$ or $\mathcal{G}_2$ (see Fig. 1), that is, $\bar{\mathcal{G}} = \{\mathcal{G}_1, \mathcal{G}_2\}$, with switching as described in Table I. Here, the diameter of both graphs $D = 3$.

Definition 13: (Range of a node) Let us define the range of any node i with initial time, k , and time horizon, $t > 0$ as:

$$R_i(k, t) := \bigcup_{m \in R_i(k, t-1)} \{l : l \in N^+(m, k + t - 1)\},$$

with $R_i(k, 0) = \{i\}$, $|R_i(k, 0)| = 1$, and $N^+(m, k)$ is the out-neighborhood of node m at instant k .

The range of node 1 starting from any instant k is also shown in Table I. It is clear that it takes five iterations for the range to contain all the nodes, which is more than the bound on the maximum diameter ($D_{max} = 3$). Therefore, using the criteria proposed in [27], [28] leads to two issues:

- 1) Lemma 2.2 in [28] and Theorem 4.1 in [27] (strict monotonicity of global maximum/minimum states implementing the average consensus protocol and the ratio consensus protocol, respectively) assume that the global maximum/minimum are available to each agent within D_{max} iterations, which is deemed untrue for this example (Table I).
- 2) Thus, nodes implementing Algorithm 1 of [27], [28] do not have access to the global maximum/minimum and might terminate the algorithm (when the *local* maximum and minimum are within the tolerance) before actual global convergence (when the *global* maximum and minimum are the within tolerance).

The results in Section IV will further highlight these problems in a numerical simulation.

III. DISTRIBUTED FINITE-TIME TERMINATION

In this section, the definitions and convergence of the maximum and minimum consensus algorithms are established. Subsequently, a finite-time termination criterion for the average consensus in the case of switching topology is developed based on these algorithms. Let us consider the maximum and minimum values of the ratio of the consensus protocols given

TABLE I
NETWORK TOPOLOGY AND RANGE ($R_1(k, t)$) OF NODE 1 STARTING AT INSTANT k FOR ITERATIONS $t = 1, 2, 3, 4, 5$

Iteration	Topology	Range of node 1
$t = 1$	$\mathcal{G}_1$	$R_1(k, 1) = \{1, 2\}$
$t = 2$	$\mathcal{G}_1$	$R_1(k, 1) = \{1, 2, 3\}$
$t = 3$	$\mathcal{G}_2$	$R_1(k, 1) = \{1, 2, 3, 4\}$
$t = 4$	$\mathcal{G}_2$	$R_1(k, 1) = \{1, 2, 3, 4, 5\}$
$t = 5$	$\mathcal{G}_2$	$R_1(k, 1) = \{1, 2, 3, 4, 5, 6\}$

by (1) and (2) over all nodes at any time instant k , given by:

$$M(k) := \max_{i \in \mathcal{V}} \frac{x_i(k)}{y_i(k)}, y_j(k) \neq 0, j \in \mathcal{V}, \quad (6)$$

$$m(k) := \min_{i \in \mathcal{V}} \frac{x_i(k)}{y_i(k)}, y_j(k) \neq 0, j \in \mathcal{V} \quad (7)$$

The following Lemma shows that the ratio of states at each node stays within the maximum and minimum for subsequent iterations.

Lemma 1: Consider the ratio consensus protocol comprising two updates, given by (1) and (2). Let Assumption 1 and Assumption 2 hold. Then for all time instants $k' \geq k$ and for all $i \in \mathcal{V}$:

$$m(k) \leq \frac{x_i(k')}{y_i(k')} \leq M(k).$$

Proof: We first prove the inequality for $M(k)$ using induction. By definition of $M(k)$, for $k' = k$, the proof is trivial. Suppose it is asserted that for $k' = k + l$, $l \geq 1$, $\frac{x_i(k+l)}{y_i(k+l)} \leq M(k)$ for all $i \in \mathcal{V}$. Then we have:

$$\begin{aligned} \frac{x_i(k+l+1)}{y_i(k+l+1)} &= \frac{\sum_{j \in N^-(i, k+l)} p_{ij}(k+l) x_j(k+l)}{\sum_{j \in N^-(i, k+l)} p_{ij}(k+l) y_j(k+l)} \\ &= \frac{p_{ii}(k+l) \frac{x_i(k+l)}{y_i(k+l)} + \sum_{j \in N^-(i, k+l) \setminus \{i\}} p_{ij}(k+l) \frac{x_j(k+l)}{y_j(k+l)}}{p_{ii}(k+l) + \sum_{j \in N^-(i, k+l) \setminus \{i\}} p_{ij}(k+l) \frac{y_j(k+l)}{y_i(k+l)}}. \end{aligned}$$

It follows from the inductive assumption that:

$$\begin{aligned} \frac{x_i(k+l+1)}{y_i(k+l+1)} &\leq \\ \frac{p_{ii}(k+l) M(k) + \sum_{j \in N^-(i, k+l) \setminus \{i\}} p_{ij}(k+l) M(k) \frac{y_j(k+l)}{y_i(k+l)}}{p_{ii}(k+l) + \sum_{j \in N^-(i, k+l) \setminus \{i\}} p_{ij}(k+l) \frac{y_j(k+l)}{y_i(k+l)}} \\ &= M(k). \end{aligned}$$

Therefore, $\frac{x_i(k+l+1)}{y_i(k+l+1)} \leq M(k)$ for all $i \in \mathcal{V}$. The other inequality is similar and is left to the reader. ■

Next, Lemma 2 strengthens the result of Lemma 1 to a strict inequality.

Lemma 2: Consider the ratio consensus protocol comprising two updates, given by (1) and (2), where the initial time instant is k . Let Assumption 1 and Assumption 2 hold., Let

$M(k)$ and $m(k)$ be as in (6) and (7). Let i be a node such that $\frac{x_i(k')}{y_i(k')} < M(k)$, and let j be a node such that $\frac{x_j(k')}{y_j(k')} > m(k)$ for some time instant $k' \geq k$. Then for all time instants $k'' \geq k'$, $\frac{x_i(k'')}{y_i(k'')} < M(k)$ and $\frac{x_j(k'')}{y_j(k'')} > m(k)$.

Proof: The proof is based on induction and follows similarly to the proof of Lemma 1 and is left to the reader. ■

The following definition and lemmas introduce the concept of a time-path and derive a bound on the number of iterations required for one node to access the information of any other node in the network.

Definition 14: (Time-path) In the case of the switching topology, a time-path of length l at time t from node $i \in V$ to $j \in V$ is a sequence of nodes, $k_1, k_2, \dots, k_{l-1}$, such that $(k_1, i) \in \mathcal{E}(t), (k_2, k_1) \in \mathcal{E}(t+1), \dots, (j, k_{l-1}) \in \mathcal{E}(t+l-1)$. In other words, node j has access to node i 's information after l time steps through the nodes $k_1, k_2, \dots, k_{l-1}$ starting at time t .

Lemma 3: Consider a network where Assumption 1 and Assumption 2 hold. Then for any node $i \in \mathcal{V}$, the following holds:

- 1) $R_i(k, t-1) \subset R_i(k, t)$ for all $t = 1, 2, \dots$
- 2) $R_i(k, n-1) = \mathcal{V}$.

Proof: Here, $R_i(k, t+1)$ is the union of out-neighborhoods of all the elements of $R_i(k, t)$ at instant t . Clearly, $R_i(k, t-1) \subseteq R_i(k, t)$. Indeed, $l \in R_i(k, t-1)$ implies $l \in N^+(l, k+t-1)$, which, in turn, implies, $l \in R_i(k, t)$.

Now, for $t = 1$:

$$R_i(k, 1) = \{m_1 : m_1 \in N^+(i, k)\} = N^+(i, k).$$

If $R_i(k, 1) = \mathcal{V}$, the claim is proven. Suppose $R_i(k, 1) \subset \mathcal{V}$. Note that at time instant $k+1$, the graph is strongly connected. Thus, there exists an outgoing edge between the set of nodes $R_i(k, 1)$ and $\mathcal{V} \setminus R_i(k, 1)$. That is, there exists $m_2 \in \mathcal{V} \setminus R_i(k, 1)$, such that $m_2 \in N^+(m_1, k+1)$ for some $m_1 \in R_i(k, 1)$. Thus, $m_2 \in R_i(k, 2)$. It follows that $m_2 \in R_i(k, 2) \setminus R_i(k, 1)$, which implies that:

$$|R_i(k, 2) \setminus R_i(k, 1)| \geq 1.$$

Now, as $R_i(k, 1) \subset R_i(k, 2)$, we have:

$$|R_i(k, 2)| = |R_i(k, 1)| + |R_i(k, 2) \setminus R_i(k, 1)|$$

which implies $|R_i(k, 2)| \geq |R_i(k, 1)| + 1$. Note here that $R_i(k, 1)$ is a proper subset of $R_i(k, 2)$. Following in the same manner, we get:

$$|R_i(k, t)| \geq |R_i(k, t-1)| + 1 \geq |R_i(k, t-2)| + 2 \geq \dots$$

$$\geq |R_i(k, 0)| + t,$$

and thus $|R_i(k, t)| \geq t+1$. For $t = n-1$, $|R_i(k, n-1)| \geq n$; however $R_i(k, t) \subseteq \mathcal{V}$ and thus $|R_i(k, t)| \leq n$ for all t . Thus, $|R_i(k, n-1)| = n$, implying $R_i(k, n-1) = \mathcal{V}$. Therefore, the claim is true. ■

Lemma 4: Let Assumption 1 and Assumption 2 hold. Then for any two nodes, $i, j \in \mathcal{V}$, and at any time instant k , there

exists a time-path of length s from i to j with $s \leq n-1$. In other words, j has access to information of i in s number of time steps.

Proof: Note here that $R_i(k, t)$ is the set of nodes influenced by node i 's current state within the next t time steps. Now, we show the existence of a time-path. We have $R_i(k, 0) \subset R_i(k, 1) \subset R_i(k, 2) \subset \dots \subset R_i(k, s) = \mathcal{V}$. Let s be the smallest time step, such that $j \in R_i(k, s)$. From Lemma 3, $s \leq n-1$. As $j \in R_i(k, s)$, there exists $m_{s-1} \in R_i(k, s-1)$, such that $j \in N^+(m_{s-1}, k+s-1)$. As $m_{s-1} \in R_i(k, s-1)$, it follows that there exists $m_{s-2} \in R_i(k, s-2)$, such that $m_{s-1} \in N^+(m_{s-2}, k+s-2)$. Choosing m_i 's in this manner, we get $m_{s-2} \in R_i(k, s-2), m_{s-3} \in R_i(k, s-3), \dots, m_1 \in R_i(k, 1)$, such that $m_1 \in N^+(i, k)$. Therefore, there exists a time-path $(m_1, i), (m_2, m_1), \dots, (j, m_{s-1})$ with $s \leq n-1$. ■

Using Lemma 3 and Lemma 4, we next provide a sampling of the $M(k)$ and $m(k)$, such that the resulting subsequences are strictly monotonic and converge to the average of the initial conditions of x_i variables.

Lemma 5: Consider the ratio consensus protocol comprising two updates, given by (1) and (2). Let Assumption 1 and Assumption 2 hold. Let $M(k)$ and $m(k)$ be as in (6) and (7), such that $m(k) < M(k)$, where the initial time instant is k . Then for all $k' \geq k+n'$ and for all $i \in \mathcal{V}$:

$$m(k) < \frac{x_i(k')}{y_i(k')} < M(k), \quad (8)$$

where n' is an upper bound on $n-1$.

Proof: As $m(k) < M(k)$, it follows that there exists a node $i \in \mathcal{V}$, such that $\frac{x_i(k)}{y_i(k)} < M(k)$. Let $j \in \mathcal{V}$ be an arbitrary node, then from Lemma 4, there exists a time-path of length l from node i to node j at instant k , where $l \leq n-1 \leq n'$. Let this path be denoted as $(m_1, i), (m_2, m_1), \dots, (j, m_{l-1})$. Then:

$$\begin{aligned} & \frac{x_{m_1}(k+1)}{y_{m_1}(k+1)} \\ &= \frac{p_{m_1 i}(k)x_i(k) + \sum_{u \in N^-(m_1, k) \setminus \{i\}} p_{m_1 u}(k)x_u(k)}{p_{m_1 i}(k)y_i(k) + \sum_{u \in N^-(m_1, k) \setminus \{i\}} p_{m_1 u}(k)y_u(k)} \\ &= \frac{p_{m_1 i}(k)\frac{x_i(k)}{y_i(k)} + \sum_{u \in N^-(m_1, k) \setminus \{i\}} p_{m_1 u}(k)\frac{x_u(k)}{y_u(k)}}{p_{m_1 i}(k) + \sum_{u \in N^-(m_1, k) \setminus \{i\}} p_{m_1 u}(k)\frac{y_u(k)}{y_i(k)}} \\ &< \frac{p_{m_1 i}(k)M(k) + \sum_{u \in N^-(m_1, k) \setminus \{i\}} p_{m_1 u}(k)\frac{x_u(k)}{y_i(k)}}{p_{m_1 i}(k) + \sum_{u \in N^-(m_1, k) \setminus \{i\}} p_{m_1 u}(k)\frac{y_u(k)}{y_i(k)}}. \end{aligned}$$

By definition of $M(k), \frac{x_u(k)}{y_u(k)} \leq M(k)$ for all $u \in \mathcal{V}$; thus, $x_u(k) \leq y_u(k)M(k)$. It follows that:

$$\frac{x_{m_1}(k+1)}{y_{m_1}(k+1)} <$$

$$\frac{p_{m_1 i}(k)M(k) + \sum_{u \in N^-(m_1, k) \setminus \{i\}} p_{m_1 u}(k)M(k) \frac{y_u(k)}{y_i(k)}}{p_{m_1 i}(k) + \sum_{u \in N^-(m_1, k) \setminus \{i\}} p_{m_1 u}(k) \frac{y_u(k)}{y_i(k)}} = M(k).$$

Thus, $\frac{x_{m_1}(k+1)}{y_{m_1}(k+1)} < M(k)$. Therefore, from Lemma 2, it follows that for all $k' \geq k+1$, $\frac{x_{m_1}(k')}{y_{m_1}(k')} < M(k)$. Similarly, if $k' \geq k+2$, then $\frac{x_{m_2}(k')}{y_{m_2}(k')} < M(k)$ and that for all $k' \geq k+l$, $\frac{x_j(k')}{y_j(k')} < M(k)$. Note that because $n' \geq n-1 \geq l$, it follows that $k+n' \geq k+l$. The condition $k' \geq k+n'$ is independent of the index j and where the node j was chosen arbitrarily. Thus, it can be concluded that $\frac{x_i(k')}{y_i(k')} < M(k)$ for all $k' > k+n'$ and for all $i \in \mathcal{V}$. This completes the proof for $M(k)$. The other inequality for $m(k)$ can be proven similarly and is left to the reader. ■

Remark 1.4: Note that from Lemma 5, after a finite number of iterations given by n' , all ratios of the nodal states under (1) and (2) become strictly less than the maximum value of the ratio in the network in the past and strictly greater than the minimum value of the ratio in the network in the past.

The following theorem shows that after a finite time, the maximum value of the ratio in the network decreases, and the minimum value of the ratio in the network increases.

Theorem 2: Consider the ratio consensus protocol comprising two updates given by (1) and (2). Let Assumption 1 and Assumption 2 hold, and let the initial ratio vector be:

$$r(un') := \left[\frac{x_1(un')}{y_1(un')} \frac{x_2(un')}{y_2(un')} \cdots \frac{x_N(un')}{y_N(un')} \right]$$

such that $\min r(un') < \max r(un')$, where $u = 0, 1, 2, \dots$, and n' is an upper bound on $n-1$. Then, $M((u+1)n') < M(un')$ and $m((u+1)n') > m(un')$.

Proof: Let $k = un'$, then using Lemma 5, it follows that for $k' \geq (u+1)n'$, $\frac{x_i(k')}{y_i(k')} < M(un')$ for all $i \in \mathcal{V}$. Hence, $M((u+1)n') := \max_{i \in \mathcal{V}} \frac{x_i((u+1)n')}{y_i((u+1)n')} < M(un')$. This completes the proof of the inequality involving $M(un')$. The other inequality involving $m(un')$ can be proved similarly and is left to the reader. ■

Theorem 3: Consider the ratio consensus protocol comprising two updates given by (1) and (2). Let Assumption 1 and Assumption 2 hold. Then:

$$\lim_{u \rightarrow \infty} M(un') = \frac{\sum_{j=1}^n x_j(0)}{n} \quad \text{and} \quad \lim_{u \rightarrow \infty} m(un') = \frac{\sum_{j=1}^n x_j(0)}{n},$$

where n' is an upper bound on $n-1$.

Proof: This result follows directly from Theorem 1 and Theorem 2 and is left to the reader. ■

Corollary 3.1: Consider the ratio consensus protocol comprising two updates given by (1) and (2). Let Assumption 1 and Assumption 2 hold, and let n' be an upper bound on $n-1$. Then:

$$\lim_{u \rightarrow \infty} M(un') - m(un') = 0.$$

Proof: The proof follows from Theorem 3 and is left to the reader. ■

We next present the maximum-minimum consensus protocol in the case of dynamic topology and use the preceding theorems to design a finite-time termination criterion.

A. Maximum-Minimum Consensus Protocol

The maximum and minimum consensus protocols, denoted by MXP and MNP, compute the maximum and minimum of the given initial node conditions:

$$z(0) = [z_1(0) \ z_2(0) \ \dots \ z_n(0)]^T, \\ w(0) = [w_1(0) \ w_2(0) \ \dots \ w_n(0)]^T$$

in a distributed manner, respectively. They take $z(0)$ and $w(0)$ as inputs and generate a sequence of node values based on the following update rules for node i :

$$z_i(k+1) = \max_{j \in N^-(i, k)} z_j(k), \quad (9)$$

$$w_i(k+1) = \min_{j \in N^-(i, k)} w_j(k). \quad (10)$$

Proposition 1: The MXP protocol given by (9) converges to $\max_{j \in \mathcal{V}} z_j(0)$ in finite time $k \leq n'$ for any $n' \geq n-1$.

Proof: Let m be a node with state value at $z_m(0) = \max_{j \in \mathcal{V}} z_j(0)$. Then, at $k=1$, all nodes connected to m at instant $k=0$ will have the maximum value $z_m(0)$. Then, from the definition of $R_m(0, k)$ in Lemma 4, if $m_1 \in R_m(0, 1)$ then $z_{m_1}(1) = z_m(0)$. Similarly, at $k=2$, all the nodes connected to the elements of the set $R_m(0, 1)$ at instant $k=1$ will have the maximum value, that is, if $m_2 \in R_m(0, 2)$, then $z_{m_2}(2) = z_m(0)$, and so on. At $k=n-1$, if $m_{n-1} \in R_m(0, n-1)$, then $z_{m_{n-1}}(n-1) = z_m(0)$. As $R_m(0, n-1) = \mathcal{V}$, we have for all $j \in \mathcal{V}$, $z_j(n-1) = z_m(0)$. Thus, at instant $k=n-1$, all nodes will have the maximum value. As $n' \geq n-1$, $z_j(k)$ converges to $\max_{j \in \mathcal{V}} z_j(0)$ in finite time $k \leq n'$. ■

Proposition 2: The MNP protocol given by (10) converges to $\min_{j \in \mathcal{V}} w_j(0)$ in finite time $k \leq n'$ for any $n' \geq n-1$.

Proof: Similar to the proof of Proposition 1. ■

B. Distributed Finite-Time Termination Algorithm for Ratio Consensus

Here, we propose an algorithm using the MXP-MNP protocol, which allows each node to simultaneously detect the convergence of the ratio consensus within a prespecified threshold ρ . In the proposed algorithm, the initial conditions for the MXP-MNP protocol are set as the ratio of the initial values held by the nodes.

Definition 15: (Epoch) Let n' be an upper bound on $n-1$, then the u^{th} epoch is defined as the state update iteration at the instant un' for any positive integer u .

The MXP-MNP protocol is re-initialized at every u^{th} epoch, that is, $k = un'$, where $u = 1, 2, \dots$, with $z(un') = \frac{x(un')}{y(un')}$ and $w(un') = \frac{x(un')}{y(un')}$, respectively. We define $\bar{\alpha}_i(un') := \max_{j \in \mathcal{V}} z_j(un') = M(un')$, $\underline{\alpha}_i(un') = \min_{j \in \mathcal{V}} w_j(un') = m(un')$ and

Algorithm 1: Finite-time termination of ratio consensus for switching topology (at each node $i \in \mathcal{V}$).

Input:

$x_j(0), y_j(0) = 1, j \in N^-(i, 0), \rho$ //Initial condition

Initialize:

$k := 0; z_i := x_i(0)/y_i(0); w_i := x_i(0)/y_i(0); \quad u := 1;$

Repeat:

/*ratio consensus updates of node i given by (1) and (2) */

$x_i(k+1) := \sum_{j \in N^-(i,k)} p_{ij}(k)x_j(k);$

$y_i(k+1) := \sum_{j \in N^-(i,k)} p_{ij}(k)y_j(k);$

/* global max-min updates of node i given by (9) and (10), respectively */

$z_i := \max_{j \in N^-(i,k)} z_j;$

$w_i := \min_{j \in N^-(i,k)} w_j;$

if $k = un'$ **then**

$\bar{\alpha}_i := z_i; \underline{\alpha}_i := w_i; \beta_i := \bar{\alpha}_i - \underline{\alpha}_i;$

if $\beta_i < \rho$ **then**

break //stop x_i, y_i, z_i and w_i updates

else

set $z_i = x_i(un')/y_i(un'); w_i = x_i(un')/y_i(un');$

$u = u + 1;$

else

$k = k + 1;$

$\beta_i(un') := \bar{\alpha}_i - \underline{\alpha}_i = M(un') - m(un')$. Each node compares β_i with ρ at every epoch, and if $\beta_i < \rho$, then it terminates the consensus protocol. Details of this scheme are given in Algorithm 1.

Theorem 4: Algorithm 1 simultaneously converges in finite-time at each node.

Proof: From Corollary 3.1, it follows that $M(un') - m(un') \rightarrow 0$ as $u \rightarrow \infty$. Thus, for any given $\rho > 0$, there exists an integer $t(\rho)$, such that for all $u \geq t(\rho)$, $|M(un') - m(un')| < \rho$ for all nodes in the network. As each node has access to $M(un')$ and $m(un')$, convergence is simultaneously detected by each node at the same iteration. ■

Remark 4.1: Notice that by using the maximum-minimum protocol, the global maximum and minimum values at any instant k are available to each node at instant $k + n'$. Further, the only global parameter needed for Algorithm 1 is knowledge of the number of nodes in the network. Note, however, that each node does not need to know the actual number of nodes but only some upper bound on the same. In most applications, an upper bound on the number of nodes is readily available.

This result is a nontrivial extension of [27], [28] because these considered only a static network, where the finite-time consensus is based on knowledge of the upper bound on the graph diameter. For the specific example with a six node network in Section II-A, it was observed that it takes five iterations for the range to contain all the nodes. In other words, it requires at least 5 ($= n - 1$) iterations for any node to receive information from node 1, which follows directly from Lemma 4. Thus, choosing n' as $(n - 1)$ will lead to strict monotonicity of the global maximum/minimum, as described in Theorem 2. Moreover, each node implementing Algorithm 1 will have a finite-time distributed criterion to detect global convergence of the consensus updates (2) and (1).

Remark 4.2: Here, we summarize the dependence of the proposed finite-time termination algorithm on various design parameters. The tolerance ρ has an inverse relationship with the time it takes for the algorithm to converge; the lower the tolerance, the longer it takes for the algorithm to converge. Weights p_{ij} need to be column-stochastic for the “ratio consensus” algorithm to converge. Further, weights p_{ij} also govern the rate of convergence. The termination protocol is agnostic to the weights, and it will detect convergence independent of the weights. All y_i ’s need to be initialized with 1, otherwise the “ratio consensus” algorithm will not converge to the average.

IV. EXPERIMENTAL RESULTS

Consider a network of 1000 agents with switching topology represented by a digraph chosen at random from a set of 50 strongly connected digraphs for every time instant. Three instances of these digraphs are shown in Fig. 2. Here, each agent running the consensus Algorithm 1 is implemented using a NodeJS *socket.io* server. This is the first time where a salable real-world experimental test bed is implemented for consensus algorithms in a time-varying network. Here, at each time instant, the network is based on a random topology (chosen from 50 strongly connected digraphs), with the weights using the equal neighbor weights rule (4) and (5). This guarantees that Assumption 2 is always valid. This provides an experimental environment where the information exchange between the agents happens through a communication channel (with arbitrary delays), thus proving the applicability of our protocol in a real-world setting, unlike the simulation-based validation present in most existing literature.

The initial condition of the numerator states are chosen at random from (0,1). With $\rho = 0.01$ and $n' = 1000$, Algorithm 1 results in the distributed finite-time termination of computations performed by the agents in 6000 iterations, as shown in Fig. 3. Observe that the ratios of the nodal states are close to the average of the numerator’s initial conditions (consensus value). The consensus value for this instance is 0.514. This experiment was repeated multiple times by randomly choosing digraphs for each iteration as well as the initial values of the agents. The proposed MXP-MNP protocol was able to distributively terminate the algorithm each time. As noted in Remark 4.1, the global maximum and minimum state values at k are available at $n' + k$. This is shown in Fig. 3, where shifting the MXP and MNP plots left by n' time instants, we get the MXP' and MNP' plots, which coincide with the global maximum and minimum.

Remark 4.3: This protocol uses the knowledge of only the upper bound on the number of nodes (n') and checks for convergence every n' number of iterations. As a result, after the ρ -convergence, the MXP-MNP protocol might take additional $2n'$ iterations for convergence in the worst-case scenario. Further, when computational and storage capacity is not a constraint for an agent, it can run the n' MXP-MNP protocol, which are shifted in time, with the first one starting at $k = 1$, the second one starting at $k = 2$, and so on. In this case, detection will only be delayed by n' if the agents monitor all the

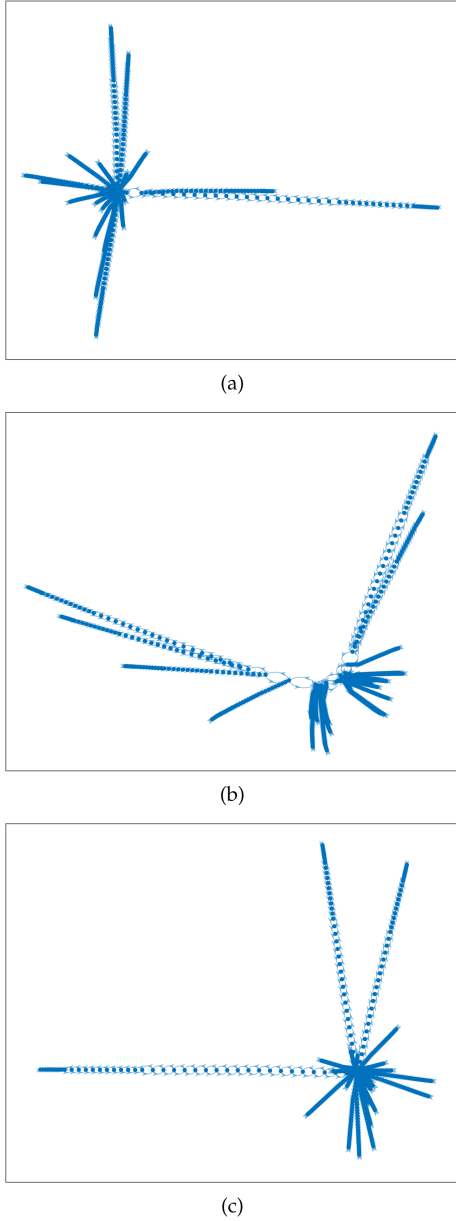


Fig. 2. Topology shown as a bidirectional graph at time instants (a) $k = 109$, (b) $k = 239$, and (c) $k = 275$.

MXP-MNP protocols. Note that without additional information on the network topology, which imposes sharing private information (for example, degree of neighbors), detection and stopping upon convergence cannot be achieved in fewer iterations. For example, a chain network takes $(n - 1)$ iterations for information to travel from one end to another, so even if an individual node has achieved convergence, it can take up to $(n - 1)$ iterations for all the other nodes in the network to know about it. Thus, in a case when the agents do not have information on the underlying communication topology, the global stopping criterion will always be delayed by at least n' .

We next present the specific example discussed in Section II-A, where it was shown analytically that some nodes will not have access to the global maximum and minimum within D_{max} iterations. We consider a 6-node network, where the topology

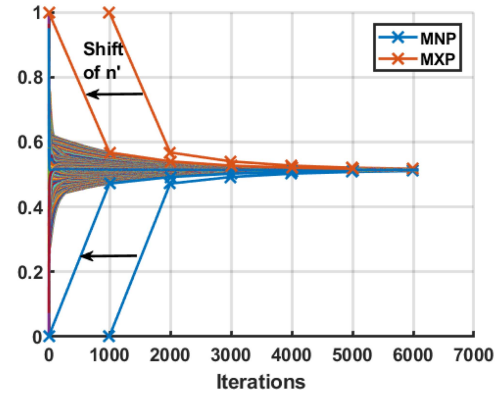


Fig. 3. Finite-time termination of ratio consensus on a 1000-node dynamic network in 6000 iterations with $\rho = 0.01$. Each curve (without markers) in the plot is the time evolution of the ratio state of 1000 nodes. MXP/MNP (when using n' as sampling) are monotonic and correctly converge. The MXP/MNP curves are shifted left to coincide with global maximum and minimum, as mentioned in Remark 4.1.

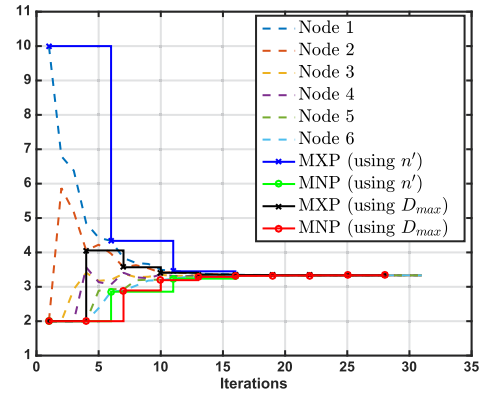


Fig. 4. Node ratio state trajectories for the ratio consensus of the 6-node network shown in Fig. 1. The MXP/MNP for sampling length of both $D_{max} = 3$ and $n' = 5$ are plotted. MXP when D_{max} is used as sampling is non-monotonic. MXP/MNP (when using n' as sampling) are monotonic and correctly converge. The MXP/MNP curves are shifted left to coincide with the global maximum and minimum.

is given by the sequence $\{\mathcal{G}_2, \mathcal{G}_2, \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_2, \mathcal{G}_1, \}$, with $\mathcal{G}_1$ and $\mathcal{G}_2$, as defined in Fig. 1. The initial state value of the nodes is chosen as $x(0) = [10, 2, 2, 2, 2, 2]$, with an average of 3.33. As mentioned earlier, each node is implemented using a NodeJS *socket.io* server. We use the sampling length of both $D_{max} = 3$ and $n' = 5$ to reset z_i and w_i in Algorithm 1, and we plot the MXP/MNP of node 5 in Fig. 4 for both cases. The MXP/MNP plots are shifted by the sampling length to coincide with the initialized values of z_i and w_i . The results clearly show that the MXP, when using D_{max} (MXP_{dia}), plot for node 5 is non-monotonic. Moreover, MXP_{dia} is unable to capture the global maximum value of the network (because some nodes have state values greater than MXP_{dia} between iterations 1 and 10). Further, the algorithm will actually stop incorrectly at the third iteration for node 5 because MXP_{dia} and MNP_{dia} have equal value. On the other hand, MXP/MNP (using n') are monotonic and converge when the states have actually converged.

We also present the evolution of the numerator ($x_i(k)$'s) and denominator states ($y_i(k)$'s) in Fig. 5(a) and (b),

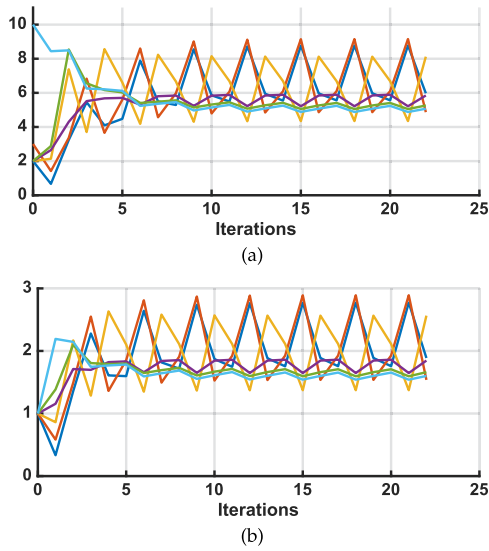


Fig. 5. Node (a) numerator and (b) denominator state trajectories for the ratio consensus of the 6-node network shown in Fig. 1. Clearly, both states for each node are bounded above and below.

respectively. Clearly, as mentioned in Remark 1.2, both the numerator and denominator are bounded above and below.

Remark 4.4: The analysis presented in this article does not consider communication delays; however, using our experimental test bed, we demonstrate that the scheme is suitable for a real-world communication network with delays. In future work, we will assume that the communication delay in the network is bounded uniformly by some τ [29], [39]. Establishing that our MXP-MNP protocol converges to the global maximum and minimum under these assumptions (switching topology with bounded communication delays) is an ongoing work.

V. CONCLUSION

In this article, we present a protocol for distributed finite-time termination of consensus algorithms in networks with dynamic topology. We introduced a novel concept of a time-path, which helps in analyzing the influence of an agent in the network on all other agents. We prove the existence of finite-length time-paths, and we establish the strict monotonic property of the global maximum and minimum of the ratio of the two state values of each node after every n' (upper bound on the number of nodes) number of iterations. A new maximum-minimum protocol for dynamic topology is presented and used to design the distributed finite-time termination algorithm. We discussed that the existing algorithms for finite-time termination based on static topology can fall short in the case of a dynamic topology for real-time applications, such as *ad hoc* cognitive radio networks and the control of autonomous agents. The effectiveness of our algorithm in these cases is demonstrated by experimentally realizing a network with dynamic topology created using a NodeJS framework. Extension of the proposed framework in the presence of noisy perturbations, bounded delay communication channels between each agent in the network, and application to the real-world control of multi-agent systems is the subject of future work.

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REFERENCES

- [1] A. Jadbabaie et al., "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Trans. Autom. Control*, vol. 48, no. 6, pp. 988–1001, Jun. 2003.
- [2] J. A. Fax and R. M. Murray, "Information flow and cooperative control of vehicle formations," *IFAC Proc. Volumes*, vol. 35, no. 1, 2002, pp. 115–120.
- [3] R. Olfati-Saber, J. A. Fax, and R. M. Murray, "Consensus and cooperation in networked multi-agent systems," *Proc. IEEE Proc. IRE*, vol. 95, no. 1, pp. 215–233, 2007.
- [4] M. Mesbahi and M. Egerstedt, *Graph Theoretic Methods in Multiagent Networks*. Princeton, NJ, United States: Princeton Univ. Press, 2010.
- [5] G. Zhao and C. Hua, "Sampled-data leaderless and leader-following consensus of multiagent systems under nonidentical packet losses," *IEEE Trans. Netw. Sci. Eng.*, vol. 9, no. 2, pp. 795–806, Mar./Apr. 2022.
- [6] G. Wang, "Consensus control in heterogeneous nonlinear multiagent systems with position feedback and switching topologies," *IEEE Trans. Netw. Sci. Eng.*, vol. 9, no. 5, pp. 3546–3557, Sep./Oct. 2022.
- [7] X. Wang and H. Su, "Robust consensus of multiagent dynamics with transmission constraints and noises," *IEEE Trans. Netw. Sci. Eng.*, vol. 9, no. 3, pp. 1540–1552, May/Jun. 2022.
- [8] J. Cai, J. Feng, J. Wang, and Y. Zhao, "Tracking consensus of multi-agent systems under switching topologies via novel smc: An event-triggered approach," *IEEE Trans. Netw. Sci. Eng.*, vol. 9, no. 4, pp. 2150–2163, Jul./Aug. 2022.
- [9] H. Gao and Y. Wang, "Algorithm-level confidentiality for average consensus on time-varying directed graphs," *IEEE Trans. Netw. Sci. Eng.*, vol. 9, no. 2, pp. 918–931, Mar./Apr. 2022.
- [10] J. Zhang, J. Lu, and J. Lou, "Privacy-preserving average consensus via finite time-varying transformation," *IEEE Trans. Netw. Sci. Eng.*, vol. 9, no. 3, pp. 1756–1764, May/Jun. 2022.
- [11] Z. Li, F. R. Yu, and M. Huang, "A distributed consensus-based cooperative spectrum-sensing scheme in cognitive radios," *IEEE Trans. Veh. Technol.*, vol. 59, no. 1, pp. 383–393, Jan. 2009.
- [12] D. Kempe, A. Dobra, and J. Gehrke, "Gossip-based computation of aggregate information," in *Proc. IEEE 44th Annu. Symp. Found. Comput. Sci.*, 2003, pp. 482–491.
- [13] A. D. Dominguez-Garcia and C. N. Hadjicostis, "Coordination and control of distributed energy resources for provision of ancillary services," in *Proc. IEEE 1st Int. Conf. Smart Grid Commun.*, 2010, pp. 537–542.
- [14] T. Charalambous and C. N. Hadjicostis, "Average consensus in the presence of dynamically changing directed topologies and time delays," in *Proc. IEEE 53rd Conf. Decis. Control*, 2014, pp. 709–714.
- [15] X. Yang and J. Lu, "Finite-time synchronization of coupled networks with markovian topology and impulsive effects," *IEEE Trans. Autom. Control*, vol. 61, no. 8, pp. 2256–2261, Aug. 2015.
- [16] X. Yang, D. W. Ho, J. Lu, and Q. Song, "Finite-time cluster synchronization of t-s fuzzy complex networks with discontinuous subsystems and random coupling delays," *IEEE Trans. Fuzzy Syst.*, vol. 23, no. 6, pp. 2302–2316, Dec. 2015.
- [17] X. Yang, Q. Song, J. Liang, and B. He, "Finite-time synchronization of coupled discontinuous neural networks with mixed delays and nonidentical perturbations," *J. Franklin Inst.*, vol. 352, no. 10, pp. 4382–4406, 2015.
- [18] R. Tang, X. Yang, and X. Wan, "Finite-time cluster synchronization for a class of fuzzy cellular neural networks via non-chattering quantized controllers," *Neural Netw.*, vol. 113, pp. 79–90, 2019.
- [19] Y. Li, K. Li, and S. Tong, "Adaptive neural network finite-time control for multi-input and multi-output nonlinear systems with positive powers of odd rational numbers," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 31, no. 7, pp. 2532–2543, Jul. 2020.

- [20] Y. Li, T. Yang, and S. Tong, "Adaptive neural networks finite-time optimal control for a class of nonlinear systems," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 31, no. 11, pp. 4451–4460, Nov. 2019.
- [21] Y. Zhao, H. Zhang, Z. Chen, H. Wang, and X. Zhao, "Adaptive neural decentralised control for switched interconnected nonlinear systems with backlash-like hysteresis and output constraints," *Int. J. Syst. Sci.*, vol. 53, no. 7, pp. 1545–1561, 2022.
- [22] Y. Liu, Y. Zhao, W. Ren, and G. Chen, "Appointed-time consensus: Accurate and practical designs," *Automatica*, vol. 89, pp. 425–429, 2018.
- [23] X. Wang, J. Li, J. Xing, R. Wang, L. Xie, and X. Zhang, "A novel finite-time average consensus protocol for multi-agent systems with switching topology," *Trans. Inst. Meas. Control*, vol. 40, no. 2, pp. 606–614, 2018.
- [24] Y. Zhao, Z. Duan, G. Wen, and G. Chen, "Distributed finite-time tracking of multiple non-identical second-order nonlinear systems with settling time estimation," *Automatica*, vol. 64, pp. 86–93, 2016.
- [25] S. Sundaram and C. N. Hadjicostis, "Finite-time distributed consensus in graphs with time-invariant topologies," in *Proc. Amer. Control Conf.*, 2007, pp. 711–716.
- [26] M. Toulouse, B. Q. Minh, and Q. T. Minh, "Invariant Properties and Bounds on a Finite Time Consensus Algorithm," in *Proc. Trans. Large-Scale Data- Knowl.-Centered Syst. XLI*, 2019, pp. 32–58.
- [27] M. Prakash, S. Talukdar, S. Attree, S. Patel, and M. V. Salapaka, "Distributed stopping criterion for ratio consensus," in *Proc. 56th Annu. Allerton Conf. Commun., Control, Comput.*, 2018, pp. 131–135.
- [28] V. Yadav and M. V. Salapaka, "Distributed protocol for determining when averaging consensus is reached," in *Proc. 45th Annu. Allerton Conf.*, 2007, pp. 715–720.
- [29] M. Prakash, S. Talukdar, S. Attree, V. Yadav, and M. V. Salapaka, "Distributed stopping criterion for consensus in the presence of delays," *IEEE Trans. Control Netw. Syst.*, vol. 7, no. 1, pp. 85–95, Mar. 2019.
- [30] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Trans. Autom. control*, vol. 49, no. 9, pp. 1520–1533, Sep. 2004.
- [31] P. Mulder and K. Breseman, *Node.js for Embedded Systems: Using Web Technologies to Build Connected Devices*. Sebastopol, CA, USA: O'Reilly Media, 2016. [Online]. Available: <https://books.google.com/books?id=uwJfgrEACAAJ>
- [32] S. Patel, S. Attree, S. Talukdar, M. Prakash, and M. V. Salapaka, "Distributed apportioning in a power network for providing demand response services," in *Proc. IEEE Int. Conf. Smart Grid Commun.*, 2017, pp. 38–44.
- [33] R. Diestel, *Graph Theory*. Berlin, Germany: Springer, 2006.
- [34] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge, UK :Cambridge University Press, 2012.
- [35] C. N. Hadjicostis and T. Charalambous, "Average consensus in the presence of delays and dynamically changing directed graph topologies," 2012, *arXiv:1210.4778*.
- [36] A. Olshevsky and J. N. Tsitsiklis, "Convergence speed in distributed consensus and averaging," *SIAM J. Control Optim.*, vol. 48, no. 1, pp. 33–55, 2009.
- [37] F. Bénézit, V. Blondel, P. Thiran, J. Tsitsiklis, and M. Vetterli, "Weighted gossip: Distributed averaging using non-doubly stochastic matrices," in *Proc. IEEE Int. Symp. Inf. Theory.*, 2010, pp. 1753–1757.
- [38] P. Brémaud, *Markov Chains: Gibbs Fields, Monte Carlo Simulation, and Queues*. Berlin, Germany : Springer, 2013.
- [39] A. Sampath and C. Tripti, "Synchronization in Distributed Systems," in *Proc. Int. Conf. Comput. and Inf. Tech.*, N. Meghanathan, D. Nagamalai, and N. Chaki, Eds. Berlin, Germany: Springer, 2012, pp. 417–424.



Govind Saraswat (Member, IEEE) received the B. Tech degree in electrical engineering from the Indian Institute of Technology, Delhi, India, in 2007, and the Ph.D. degree in electrical engineering from University of Minnesota, Twin Cities in 2014. He is currently a part of Sensing and Predictive Analytics Group with National Renewable Energy Laboratory, Golden, CO, USA, where he works on data-driven technology for energy systems planning and operation. His research interests include power system modeling and analysis, measurement-based operation and control, machine learning, and optimization.



Vivek Khatana received the B.Tech degree in electrical engineering from the Indian Institute of Technology, Roorkee, India, in 2018. He is currently working toward the Ph.D. degree with the Department of Electrical Engineering, University of Minnesota, Minneapolis, MN, USA. His research interests include distributed optimization, consensus algorithms, distributed control and stochastic calculus, and power system analysis and control.



Sourav Patel received the B.Tech. degree in instrumentation and control engineering from the National Institute of Technology, Jalandhar, India, in 2011, and the M.S. degree in electrical engineering in 2018 from the University of Minnesota, Twin-Cities, Minneapolis, MN, USA, where he is currently working toward the Ph.D. degree in electrical engineering. In 2011, he joined National Thermal Power Corporation Ltd., Kaniha, India, as a Control and Instrumentation Engineer. His research interests include control and systems theory, coordination and communication protocols for distributed energy resources towards smart grid applications.



Murli V. Salapaka (Fellow, IEEE) received the B.Tech. degree in mechanical engineering from the Indian Institute of Technology, Madras, India, in 1991, and the M.S. and Ph.D. degrees in mechanical engineering from the University of California, Santa Barbara, CA, USA, in 1993 and 1997, respectively. From 1997 to 2007, he was the Faculty Member with the Electrical and Computer Engineering Department, Iowa State University, Ames, IA, USA. He is currently the Director of Graduate Studies and the Vincentine Hermes Luh Chair Professor with the Electrical and Computer Engineering Department, University of Minnesota, Minneapolis, MN, USA. His research interests include control and network science, nanoscience and single molecule physics. Dr. Salapaka was the recipient of the NSF CAREER Award and the ISU Young Engineering Faculty Research Award for the years 1998 and 2001.