

Signal Selection for Estimation and Identification in Networks of Dynamic Systems: A Graphical Model Approach

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Abstract—Network systems have become a ubiquitous modeling tool in many areas of science where nodes in a graph represent distributed processes and edges between nodes represent a form of dynamic coupling. When a network topology is already known (or partially known), two associated goals are, to derive estimators for nodes of the network, which cannot be directly observed or are impractical to measure; and to quantitatively identify the dynamic relations between nodes. We address both problems in the challenging scenario where only some outputs of the network are being measured and the inputs are not accessible. The approach makes use of the notion of d -separation for the graph associated with the network. In the considered class of networks, it is shown that the proposed technique can determine or guide the choice of optimal sparse estimators. This article also derives identification methods that are applicable to cases where loops are present providing a different perspective on the problem of closed-loop identification. The notion of d -separation is a central concept in the area of probabilistic graphical models, thus an additional contribution is to create connections between control theory and machine learning techniques.

Index Terms—Graphical models, stochastic systems, system identification.

NOMENCLATURE

$\{y_i\}_{i=1}^N$	Set of N column vector processes indexed by i .
$y = (y_1^T, \dots, y_N^T)^T$	Vector of stochastic processes obtained by stacking $y_1, \dots, y_N$.
I, J	Set of indexes in $\{1, \dots, N\}$.
$\Phi_y(\mathbf{z})$	Power spectral density of the vector process y .

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$G = (V, \vec{E})$

$W_{x|y}(\mathbf{z})$ and $W_{x|y}^C(\mathbf{z})$

$W_{x,[y_i]|y}(\mathbf{z})$ and $W_{x,[y_i]|y}^C(\mathbf{z})$

Directed graph with nodes V and edges $\vec{E}$.

Noncausal and causal Wiener filter estimating the process x from the process y , respectively.

Component of the noncausal and causal Wiener filter estimating the process x from the process y associated with the subvector y_i of y , respectively.

I. INTRODUCTION

NETWORKS of dynamic systems have become a ubiquitous tool in science to describe interactions in modular systems. Applications span areas as diverse as economics (see, e.g., [1]), social systems (see, e.g., [2] and [3]), biology (see, e.g., [4] and [5]), cognitive sciences (see, e.g., [6]), and geology (see, e.g., [7]). In many applications, the system variables being observed are not the responses to known or manipulable inputs that are actively injected to probe the network, but rather are measurements acquired while the system is naturally operating and is responding to excitations that are unknown. Scenarios where inputs are neither manipulable nor measurable include systems with critical or uninterruptible functions where injecting probing signals is deemed unacceptable to system operations (such as the power grid or the transport infrastructure), or when it is too expensive or impractical to actively manipulate the nodes (such as a financial network [8] or a gene network [4]).

This article considers networks in the dynamical structure function (DSF) form, introduced in [9], and addresses two fundamental problems in the domain of this class of systems: First, estimation of the output signal (behavior) of an individual node, and second, consistent identification of the dynamic relation (transfer function) between two nodes in the network. Both problems are investigated in the presence of latent nodes with focus on methods that preclude the manipulation or observation of the inputs. This article also bridges problems of signal estimation and consistent identification to the graphical model framework.

Toward estimation of an agent's behavior in a dynamically networked system, graphical conditions are leveraged to derive simpler estimators, which make use of a reduced number of signals. One first fundamental result is a rigorous characterization of the effects of observations' inclusion or exclusion on the estimate of an agent's activity. For example, when a set of observed nodes is used to estimate another node, it stands to reason that the introduction of new observed nodes renders

some of the nodes in the original set irrelevant. The proven characterization also demonstrates that it is possible to remove some observations and make other observations irrelevant too.

In the case of identification in networks, some approaches assume the design of an appropriate input or even the knocking-out one or more agents in their identification procedure [10]. For these reason, such approaches are not applicable to systems that are not manipulable. There are also several data-driven techniques that aim at reconstructing the underlying graph of a network [11], [12] and which also obtain the transfer functions describing the network's dynamics among the nodes [9], [12]–[15]. Some of these techniques make limiting assumptions on the structures they can identify (i.e., [11] is limited to trees). Others assume that the system is not manipulable but still rely on the measurement of the forcing inputs. For example, Gonçalves and Warnick [9] find necessary and sufficient conditions to recover the network's dynamics among the nodes given the transfer function from inputs to outputs (or an estimate of it). When inputs are not measured and just modeled as stochastic, multiple transfer functions could be compatible with the observed outputs, given that the spectral factorization of their power spectral density is in general not unique [15]. Work in [13] robustifies the approach in [9] incorporating *a priori* knowledge of the network structure in an optimization problem, but still requires an estimate of the transfer function from inputs to outputs or the actual measurements of the forcing inputs. Among the approaches based on DSF models, the work in [15] is arguably the most related to this article. The authors consider a stochastic scenario, referred to as “blind identification,” where the forcing inputs are not accessible and the objective of the identification has to be determined from the output spectral density only. Under the assumption of strictly causality for the network dynamics and mutually independent forcing inputs, Hayden *et al.* [15] show that given the outputs' spectral density matrix, its spectral factorizations compatible with the DSF dynamics are a finite number. Furthermore, under the additional minimum phase condition on the network's dynamics, the spectral factor is shown to be unique (modulo a multiplication by a signed identity matrix), so that the network can be fully identified via the methods in [9]. As an important feature, Hayden *et al.* [15] require no *a priori* knowledge of the network structure. Under the same strict causality and minimum-phase conditions, similar results that do not require *a priori* knowledge of the structure are obtained in [12] using a Granger causality approach [16].

This article follows a problem formulation close to the blind identification in [15]. The considered class of networks is expressed in a DSF form and the unknown inputs are modeled as stochastic processes. Some of the assumptions in [15] are relaxed: the dynamics is not required to be strictly causal, and only some of the outputs are being measured. Thus, while [15] assumes full knowledge of the power spectral density matrix of the outputs, here, only a minor of the output spectral density matrix is known. At the same time, compared to [15], the results we present here make use of partial *a priori* knowledge about the network structure and attempt to identify a single transfer function and not the whole dynamics. An example (see Example 5) shows how the identification problem formulated in this article can also be applied to formulate blind identification problems as in [15], but considering nonmutually independent forcing inputs.

On the other hand, if the network topology is fully known, there is a wealth of closed-loop parametric identification techniques that has been extended to general networks in order to

identify individual transfer functions. These techniques include the direct method, the joint IO method, the two-stage identification [17], and the instrumental variable method [18]. Most of these approaches can address scenarios where inputs are nonmanipulable if observed internal signals in the network can be used as predictors and/or instrumental variables. They result in sufficient conditions on how to select these signals to arrive at consistent estimation of an individual transfer function [19]. However, these techniques often require some form of knowledge about the strict causality or the degree of delay embedded in some operators (especially if feedback loops are present [18]).

This article follows a related, but nonparametric, approach. The identification of a transfer function is achieved by computing a multi-input Wiener filter estimating a signal over a selected set of other signals, in such a way that the one of the entries of the Wiener filter matches the expression of the transfer function to be identified. A precise criterion is provided for the set of agents whose activities, if available, result in an unbiased and consistent estimation of the dynamics between a specific pair of agents. No knowledge about the strict causality of specific operators is required and, compared to other techniques, the results hold even when the transfer functions are zero, but this information is not available. These features allow one to seamlessly deal with latent nodes and situations where the network topology is not known exactly, by using purely graphical criteria.

The resulting conditions are graphical and constructive. They depend exclusively on the structure of the network and the location of the edge associated with the transfer function to be identified. Furthermore, this article's framework builds bridges between the area of probabilistic graphical models [20] and extends them to the area of network of dynamical systems [12], [21]. In particular, in this article, the notion of *d*-separation on graphs, widely used in the domain of probabilistic graphical models (see [20]), plays an enabling role: it is established that *d*-separation on the graphical representation implies a notion of independence for signals in networks of linear dynamical systems based on Wiener filtering. This is achieved both in the noncausal and in the more challenging causal case, broadening the framework developed in [22].

This article is structured as follows. Section II introduces preliminary concepts including the notion of *d*-separation for directed graphs that is key for the development of the main results. Section III introduces a flexible class of networks via the DSF formalism. Section IV formalizes the two problems, which are the focuses of this article. Section V describes graphical manipulations for our network models. Section VI shows that *d*-separation defined on the graph representation implies specific sparsity properties of Wiener filters on such class of networks. Section VII makes use of *d*-separation to select appropriate signals in order to obtain unbiased identification of a transfer function in the network. Finally, Section VIII concludes this article. Throughout this article, efficacy of the results are illustrated via examples.

The following notation for the definition of subvectors and submatrices will be adopted through this article.

Notation 1 (Ordered and Unordered Sets): An unordered set (a collection of elements where the listing order is not relevant) is denoted using curly brackets. An ordered set (a collection of elements where the listing order is relevant) is denoted using round brackets.

We extend the notions of element and subset of an unordered set to ordered sets in the natural way, so that we can write $3 \in (1, 2, 3, 4, 5)$ and $(1, 2, 4) \subseteq (5, 4, 3, 2, 1)$.

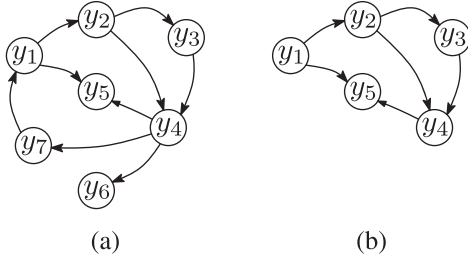


Fig. 1. (a) Directed graph and (b) its restriction to the nodes y_1, y_2, y_3, y_4 , and y_5 .

For vectors and matrices, we use an index subset notation.

Notation 2 (Index Subset Notation for Vectors): Let $v := (v_1^T, \dots, v_n^T)^T$ be a vector defined by n subvectors, $v_1, \dots, v_n$, and let $I := (i_1, \dots, i_{n_I})$ be an ordered set of integers in $\{1, \dots, n\}$. We denote by $v_I := (v_{i_1}^T, \dots, v_{i_{n_I}}^T)^T$ the vector obtained by considering the subvectors in v indexed by $(i_1, \dots, i_{n_I})$.

Notation 3 (Index Subset Notation for Matrices): Let Q be a matrix with a $n \times n$ block structure and let $I := (i_1, \dots, i_{n_I})$ and $J := (j_1, \dots, j_{n_J})$ be two ordered sets of integers in $\{1, \dots, n\}$. We denote by Q_{JI} the submatrix of Q obtained by considering the row blocks indexed by $(j_1, \dots, j_{n_J})$ and the column blocks indexed by $(i_1, \dots, i_{n_I})$.

II. PRELIMINARIES

In this section, we recall some basic concept of graph theory, including the notion of d -separation, and define the class of linear dynamic influence models, derived from the notion of DSF introduced in [9].

A. Basic Notions of Graph Theory

In this section, we want to recall some fundamental notions of graph theory, which are functional to the subsequent development. We define directed graphs and their restrictions with respect to subsets of nodes (see also [23]).

Definition 1 (Directed Graphs and Restrictions): A directed graph G is a pair $(V, \vec{E})$ where V is a set of vertices (or nodes) and $\vec{E}$ is a set of edges (or arcs), which are ordered pairs of elements of V . The restriction of G to the node set $V' \subseteq V$ is the graph $G' = (V', \vec{E}')$ where $\vec{E}' = \{(y_i, y_j) \in \vec{E} \mid y_i \in V' \text{ and } y_j \in V'\}$. ■

Fig. 1(a) shows a directed graph and Fig. 1(b) shows its restriction to a subset of its nodes.

Self-loops are edges that start and terminate on the same node.

Definition 2 (Self-Loops): A directed graph $G = (V, \vec{E})$ has a self-loop on node $y_i \in V$ if $(y_i, y_i) \in \vec{E}$.

Chains and paths are fundamental concepts in graphs.

Definition 3 (Paths and Chains): Consider a directed graph $G = (V, \vec{E})$ with vertices $y_1, \dots, y_n$. A *chain* starting from y_i and ending in y_j is an ordered sequence of distinct edges in $\vec{E}$

$$((y_{\pi_1}, y_{\pi_2}), (y_{\pi_2}, y_{\pi_3}), \dots, (y_{\pi_{\ell-1}}, y_{\pi_\ell}))$$

where $y_i = y_{\pi_1}$, $y_j = y_{\pi_\ell}$, and $(y_{\pi_p}, y_{\pi_{p+1}}) \in \vec{E}$ for all $p = 1, \dots, \ell - 1$. A *path* between two vertices y_i and y_j is an ordered

sequence of distinct ordered pairs of nodes

$$((y_{\pi_1}, y_{\pi_2}), (y_{\pi_2}, y_{\pi_3}), \dots, (y_{\pi_{\ell-1}}, y_{\pi_\ell}))$$

where $y_i = y_{\pi_1}$, $y_j = y_{\pi_\ell}$, and either $(y_{\pi_p}, y_{\pi_{p+1}}) \in \vec{E}$ or $(y_{\pi_{p+1}}, y_{\pi_p}) \in \vec{E}$ for all $p = 1, \dots, \ell - 1$. We refer to the nodes $y_{\pi_2}, \dots, y_{\pi_{\ell-1}}$ as internal nodes of a chain or path. ■

As it follows from the definition, chains are a special case of paths. All paths (and consequently also all chains) can be suggestively denoted by separating the nodes in the sequence $\{y_{\pi_p}\}_{p=1}^\ell$ with the arrow symbol $\rightarrow$ if $(y_{\pi_{p-1}}, y_{\pi_p}) \in \vec{E}$ or the symbol $\leftarrow$ if $(y_{\pi_p}, y_{\pi_{p-1}}) \in \vec{E}$. For example, in Fig. 1(a), the path $y_1 \rightarrow y_2 \rightarrow y_4 \rightarrow y_6$ is also a chain, whereas $y_1 \rightarrow y_5 \leftarrow y_4 \leftarrow y_3$ is a path, but not a chain. Here, we also say that there is a chain from vertex y_1 to vertex y_6 in the graph.

From the concept of chain, we can derive the notions of ancestry and descendance, which are common in the theory of Bayesian networks (see also [20]).

Definition 4 (Parents, Children, Ancestors, Descendants):

Consider a directed graph $G = (V, \vec{E})$. A vertex y_i is a *parent* of a vertex y_j if there is a directed edge from y_i to y_j . In such a case, y_j is a *child* of y_i . Also, y_i is an *ancestor* of y_j if $y_i = y_j$ or if there is a chain from y_i to y_j . In such a case, y_j is a *descendant* of y_i . Given a set $y_I \subseteq V$, we define the following sets:

$$pa_G(y_I) := \{w \in V \mid \exists y \in y_I : w \text{ is a parent of } y\}$$

$$ch_G(y_I) := \{w \in V \mid \exists y \in y_I : w \text{ is a child of } y\}$$

$$an_G(y_I) := \{w \in V \mid \exists y \in y_I : w \text{ is an ancestor of } y\}$$

$$de_G(y_I) := \{w \in V \mid \exists y \in y_I : w \text{ is a descendant of } y\}.$$

On a path, we define forks and colliders (see again [20]).

Definition 5 (Forks and Colliders): A path involving the nodes $y_{\pi_1}, \dots, y_{\pi_\ell}$ has a *fork* at y_{π_p} , for $1 < p < \ell$, if $y_{\pi_{p-1}}$ and $y_{\pi_{p+1}}$ are both children of y_{π_p} (that is $y_{\pi_{p-1}} \leftarrow y_{\pi_p} \rightarrow y_{\pi_{p+1}}$ appears in the path). A path has an *inverted fork* (or a *collider*) at y_{π_p} if $y_{\pi_{p-1}}$ and $y_{\pi_{p+1}}$ are both parents of y_{π_p} (that is $y_{\pi_{p-1}} \rightarrow y_{\pi_p} \leftarrow y_{\pi_{p+1}}$ appears in the path).

The following definition introduces a notion of separation on subsets of vertices in a directed graph [20].

Definition 6 (d -Separation and d -Connection): Consider a directed graph $G = (V, \vec{E})$ and three mutually disjoint sets of vertices $y_I, y_Z, y_J \subseteq V$. The set y_Z is said to d -separate y_I and y_J if for every $y_i \in y_I$ and $y_j \in y_J$, all paths between y_i and y_j meet at least one of the following conditions.

- 1) The path contains a node $y_z \in y_Z$ that is not a collider.
- 2) The path contains a collider y_k such that neither y_k nor its descendants belong to y_Z .

If y_Z d -separates y_I and y_J in the graph G , we write $dsep_G(y_I, y_Z, y_J)$. Otherwise, we write $\neg dsep_G(y_I, y_Z, y_J)$ and say that y_Z d -connects y_I and y_J .

Since the notion of d -separation plays a central role in the developments of this article, we illustrate it with some examples.

Example 1 (Examples of d -Separation): In the graph G of Fig. 1(a), we have that $dsep_G(y_3, \{y_4, y_6\})$. Also, we have that $\neg dsep_G(y_4, \{y_2, y_5, y_7\}, y_1)$ because y_5 is a collider on the path $y_4 \rightarrow y_5 \leftarrow y_1$. The smallest set that d -separates y_1 and y_4 is $\{y_2, y_7\}$, but it also holds that $dsep_G(y_4, \{y_2, y_3, y_7\}, y_1)$, $dsep_G(y_4, \{y_2, y_6, y_7\}, y_1)$, and $dsep_G(y_4, \{y_2, y_3, y_6, y_7\}, y_1)$. We have that $\neg dsep_G(y_2,$

$\{y_1, y_6\}, y_3)$ because y_6 is a descendant of y_4 , which is a collider in the path $y_2 \rightarrow y_4 \leftarrow y_3$.

B. Wiener Filter and Wiener Separation

We introduce a class of linear operators to transform sets of rationally related random processes.

Definition 7: The set $\mathcal{F}$ is defined as the set of real-rational transfer functions that are analytic and invertible on the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$. Given a transfer function $P(z) \in \mathcal{F}$, it can be uniquely represented in the time domain by a bi-infinite sequence p_k (the impulse response of $P(z)$) satisfying

$$P(z) = \sum_{k=-\infty}^{\infty} p_k z^{-k} \quad (1)$$

for all $|z| = 1$. If $k < 0$ implies $p_k = 0$, then we say that the transfer function is causal. We define the space of causal transfer functions as $\mathcal{F}^+$. If $k \leq 0$ implies $p_k = 0$, then we say that the transfer function is strictly causal.

We also define the following notation.

Definition 8: We define $y = P(z)u$ as the process obtained by computing the convolution of u with the impulse response of $P(z)$, namely $y(t) = \sum_{k=-\infty}^{\infty} p_{t-k}u(k)$ for all $t \in \mathbb{Z}$.

For example, $\frac{1}{z}u$ denotes the process u delayed by one time step. For a set of processes $S = \{y_1, \dots, y_n\}$, we denote by $P(z)S$ the set $\{P(z)y_1, \dots, P(z)y_n\}$. For example, $\frac{1}{z}S$ denotes the set of all the processes in S delayed by one time step. The defined linear operators can be used to define the following spaces of stochastic processes.

Definition 9 (Transfer Function Spaces and Spans): For a finite number of wide-sense stationary signals $y_1, \dots, y_n$, the rational transfer function span (tfspan) and the rational causal transfer span (ctfspan) are defined as

$$\begin{aligned} \text{tfspan}\{y_1, \dots, y_n\} &:= \left\{ q = \sum_{i=1}^n P_i(z)y_i \mid P_i(z) \in \mathcal{F} \right\} \\ \text{ctfspan}\{y_1, \dots, y_n\} &:= \left\{ q = \sum_{i=1}^n P_i(z)y_i \mid P_i(z) \in \mathcal{F}^+ \right\}. \end{aligned}$$

where the variable z is related to a discrete time-shift operator.

We provide a specific formulation for the Wiener filter that covers both noncausal and causal estimators.

Proposition 10 (Wiener Filtering for Processes With Rational Power Spectral Density): Let x and $y_1, \dots, y_n$ be vector processes that are jointly stationary with rational power cross-spectral densities and rational auto-spectral densities. Define $y := (y_1^T, \dots, y_n^T)^T$ and

$$\begin{aligned} X &:= \text{tfspan}\{y_1, \dots, y_n\} \\ X^C &:= \text{ctfspan}\{y_1, \dots, y_n\}. \end{aligned}$$

Consider the problems

$$\inf_{q \in X} \mathbf{E}[(x - q)^T(x - q)] \quad \text{and} \quad \inf_{q \in X^C} \mathbf{E}[(x - q)^T(x - q)]$$

where $\mathbf{E}[\cdot]$ is the expectation operator. Then, the solutions for both problems exist and are unique and given, respectively, by $\hat{x}_y = W_{x|y}(z)y$ and $\hat{x}_y^C = W_{x|y}^C(z)y$ where, if $\Phi_y(e^{i\omega}) \succ 0$, for $\omega \in [-\pi, \pi]$, the transfer functions $W_{x|y}(z)$ and $W_{x|y}^C(z)$ are unique and referred to as Wiener filter and causal Wiener filter,

respectively. Moreover, $\hat{x}_y$ is the only element in X such that for any $q \in X$

$$\mathbf{E}[(x - \hat{x}_y)q] = 0 \quad (2)$$

and $\hat{x}_y^C$ is the only element in X^C such that for any $q \in X^C$

$$\mathbf{E}[(x - \hat{x}_y^C)q] = 0. \quad (3)$$

Proof: The result is a direct consequence of the standard Wiener filter theory. Rational power spectral densities guarantee the existence of a spectral factorization (see [24]). ■

Definition 11: Let x and $y = (y_1^T \dots y_n^T)^T$ be jointly stationary processes. Let $W_{x|y}(z)y$ and $W_{x|y}^C(z)y$ be, respectively, the noncausal and causal Wiener estimates of x from y . We denote the component of the Wiener filter operating on y_i as $W_{x,[y_i]|y}$, namely

$$\begin{aligned} W_{x|y} &= (W_{x,[y_1]|y}, \dots, W_{x,[y_n]|y}) \\ W_{x|y}^C &= (W_{x,[y_1]|y}^C, \dots, W_{x,[y_n]|y}^C). \end{aligned}$$

As an extension, given an ordered subset y_I of components of y , we use the notation $W_{x,[y_I]|y}$ (or $W_{x,[y_I]|y}^C$) to denote the vector of components of the noncausal Wiener filter (or causal Wiener filter) associated with the vector of signals y_I in the estimation of x .

Definition 12 (Wiener Separation): Let y_I , y_J , and y_Z be disjoint subsets in a set of jointly stationary random processes. We say that y_I is (noncausally) Wiener-separated from y_J by y_Z if the Wiener estimate of y_J from y_I and the Wiener estimate of y_J from $y_I \cup y_Z$ are identical, namely

$$W_{y_J|(y_I, y_Z)}(z) \begin{pmatrix} y_I \\ y_Z \end{pmatrix} = W_{y_J|y_Z}(z) y_Z.$$

We denote this relation as $wsep(y_I, y_Z, y_J)$. Equivalently, we say that $wsep(y_I, y_Z, y_J)$ when all components of the Wiener filter associated with y_I vanish, namely

$$W_{y_J,[y_I]|(y_I, y_Z)} = 0.$$

Analogously, we use the notation $cwsep(y_I, y_Z, y_J)$ to denote that $W_{y_J,[y_I]|(y_I, y_Z)}^C(z) = 0$.

III. LINEAR DYNAMIC INFLUENCE MODELS AND THEIR GRAPHICAL REPRESENTATIONS

In the last decade, several network models have emerged, such as DSF models [9], linear dynamic graphs (LDGs) [12], [25], and the dynamic interconnection proposed in [26]. These models share very similar characteristics, the main one being that they can be interpreted via graphs in a way analogous to signal flow graphs [27]. While in their original formulations, these models were not exactly equivalent, subsequent generalizations have made more and more similar (i.e., DSF was originally considering only strictly causal transfer functions, whereas LDGs was considering nonstrictly causal transfer functions from the beginning). In this article, we consider a specific class of dynamic networks defined by relations among wide-sense stationary stochastic processes following the DSF approach proposed in [9], even though these results can be seamlessly applied to the network models in [12] or [26].

A DSF is a representation of a network in the form

$$y = Q(z)y + P(z)u$$

where y is a vector of outputs associated with the nodes of the network and u is the vector of inputs forcing it. The pair $(Q(\mathbf{z}), P(\mathbf{z}))$ defines the relation in the model: the matrix Q represents how the nodes are interconnected and affect each other, whereas the matrix P represents how the inputs are influencing the nodes. In DSF representation, the number of forcing inputs and the number of output nodes can be different. Also, DSF considers, in general, signals that can be either stochastic and deterministic.

In this article, we specifically consider scenarios where the forcing signals in u are not observable and can be modeled as wide-sense-stationary stochastic processes. The forcing inputs are assumed as mutually independent and act separately on each node of the network leading to a block diagonal structure for $P(\mathbf{z})$, which is assumed stable. In the context of this article, since the forcing inputs are independent and not observable, there is no loss of generality, by defining $e = P(\mathbf{z})u$, and consider e as the vector of inputs directly exciting the network. In other words, the dynamics in $P(\mathbf{z})$ can be incorporated in the power spectral density of e , $\Phi_e(\mathbf{z}) = P(\mathbf{z})\Phi_u(\mathbf{z})P^*(\mathbf{z})$, which has the same diagonal structure of $\Phi_u(\mathbf{z})$. This motivates the following definition.

Definition 13 (Linear Dynamic Influence Models):

Consider a network following the DSF dynamics $y = Q(\mathbf{z})y + P(\mathbf{z})u$, as described in [9] where

- 1) $y = (y_1^T, \dots, y_n^T)^T$ are the n output nodes;
- 2) $u = (u_1^T, \dots, u_n^T)^T$ are the n wide-sense stationary inputs;
- 3) $Q(\mathbf{z}) \in \mathcal{F}^{n \times n}$ and $P(\mathbf{z}) \in (\mathcal{F}^+)^{n \times n}$ are stable $(n \times n)$ -block matrices;
- 4) $\Phi_u(\mathbf{z}) = \mathcal{I}$ and $P(\mathbf{z})$ is block diagonal.

Define $e = P(\mathbf{z})u$, so that $\Phi_e(\mathbf{z})$ is block diagonal and the dynamics can be written as $y = Q(\mathbf{z})y + e$. We say that the DSF network is an LDIM and we denote it with the pair $(Q(\mathbf{z}), e)$.

We say that the LDIM is causal if all entries of $Q(\mathbf{z})$ are causal. We say that the LDIM is well-posed (causally well-posed) if $(\mathcal{I} - Q_{II}(\mathbf{z}))$ is invertible (causally invertible) for all $I \subseteq \{1, \dots, n\}$ with no poles on the unit circle. The well-posedness (causal well-posedness) of an LDIM guarantees that y is wide-sense stationary.

Observe that the concept of causality is often a requirement for standard well-posedness (see [28]). Note that the dynamic matrix $Q(\mathbf{z})$ determining an LDIM is assumed not necessarily causal (unless the LDIM is causal). The reason behind the distinct notions of well-posedness and causal well-posedness in Definition 13 is to allow the application of this article's results, potentially, to other domains where transfer functions are not necessarily causal (i.e., video or image processing). For a more extensive discussion on the well-posedness of DSF structures, we refer to [29].

Definition 14 (Graphical Representations of an LDIM and Its Associated Perfect Directed Graph (PDG)): Given an LDIM $(Q(\mathbf{z}), e)$ with output processes $y = (y_1^T | \dots | y_n^T)^T$ and a directed graph $G = (V, \vec{E})$ where $V = \{y_1, \dots, y_n\}$, we say that G is a graphical representation of the LDIM if $(y_i, y_j) \notin \vec{E}$ implies $Q_{ji}(\mathbf{z}) = 0$. If it also holds that $Q_{ji}(\mathbf{z}) = 0$ implies $(y_i, y_j) \notin \vec{E}$, then we say that G is the (unique) PDG associated with the LDIM. The PDG can also be seen as the restriction of the signal structure (see [9]) of the LDIM restricted to only the output nodes.

Abusing the nomenclature, we will sometimes refer to nodes, edges, paths, and chains of an LDIM even though, formally, we should refer to them as nodes, edges, paths, and chains of its graphical representation or its PDG.

IV. PROBLEM FORMULATIONS: IRRELEVANCE OF MEASUREMENTS AND IDENTIFICATION OF A TRANSFER FUNCTION UNDER GRAPH CONSTRAINTS

Given an LDIM, we consider two basic problems. We refer to the first problem as irrelevance of a measurement.

Problem 1 (Irrelevance of a Node in an Estimate): Given an LDIM with graphical representation G , determine if the least square estimate of a node y_j from the signals y_S and y_i is equivalent to the estimate of y_j from y_S only.

In other terms, Problem 1 boils down to determining that the signal y_i is irrelevant for the estimate of y_j when y_S is available. Observe that Problem 1 is formulated assuming only knowledge of the graphical representation, thus, interestingly, the solution is going to be based purely on graphical conditions and holds for all LDIMs with the same structure.

The second basic problem is about the identification of a transfer function under graph constraints via noninvasive observations.

Problem 2 (Individual Module Identification in an LDIM):

Consider an LDIM $\mathcal{G}$ with dynamics $y = Q(\mathbf{z})y + e$ and unknown $Q(\mathbf{z})$, where $y = (y_1, \dots, y_n)$ and $e = (e_1, \dots, e_n)$ are the vector of output signals and forcing signals, respectively. Under what conditions can the transfer function $Q_{ji}(\mathbf{z})$ be estimated from the observation of a prespecified subset of the signals $y_1, \dots, y_n$ only?

In this article, we present a solution to this problem, which considers only graphical conditions on a known graphical representation G of the LDIM $\mathcal{G}$.

A related problem is described in [15], where no *a priori* knowledge of the network is provided, but the transfer matrix $Q(\mathbf{z})$ is assumed strictly causal and all the node signals are observed (or equivalently, the full matrix $\Phi_y(\mathbf{z})$ is known). In [15], the problem is tackled by showing that only a finite number of factorizations of $\Phi_y(\mathbf{z}) = G(\mathbf{z})G^*(\mathbf{z})$ are compatible with the assumptions made on the network dynamics. Furthermore, modulo a multiplication by a signed identity matrix, the factor $G(\mathbf{z})$ is unique in the case of a minimum phase $(\mathcal{I} - Q(\mathbf{z}))$. After computing such a spectral factor, the results in [9] are applied to recover the network.

In our scenario, the spectral factorizations compatible with the data (partial measurement of y) and the network dynamics (nonnecessarily strictly causal $Q(\mathbf{z})$) are in general uncountably many. It is the partial *a priori* knowledge of the network structure that will enable the recovery of the target transfer function $Q_{ji}(\mathbf{z})$. Importantly, we consider situations where only a subset of the output nodes in y is being measured. This will play a fundamental role to suitably apply our results to deal with confounding variables, by assuming that the confounding variable is an unmeasured node.

V. GRAPHICAL MANIPULATIONS

Different techniques can be used to manipulate LDIMs to obtain a new network of dynamic systems. For example, the mechanisms of immersion used in [19] and described in [30] can

be applied to an LDIM, but, in general, they do not lead to the definition of another LDIM because of the potential correlation occurring between the unknown external signals. Instead, we will introduce two operations to manipulate an LDIM and obtain another LDIM: marginalization and self-loop removal.

The marginalization of a node in an LDIM is the definition of another LDIM with the same output processes, but such that the incoming links in the marginalized node are removed.

Lemma 15 (Node Marginalization Lemma): Consider an LDIM $\mathcal{G} = (Q(\mathbf{z}), e)$ with output processes $(y_1, \dots, y_n)$. Let $(V, \vec{E})$ be a graphical representation of the LDIM $\mathcal{G}$. Let $y_J \subset V$ and $y_I = V \setminus y_J$. Define two vector processes

$$y' := \begin{pmatrix} y_J \\ e_I \end{pmatrix} \quad \text{and} \quad e' := \begin{pmatrix} e_J \\ e_I \end{pmatrix} \quad (4)$$

and the transfer matrices

$$Q'_{JJ} = Q_{JJ} + Q_{JI}(\mathcal{I} - Q_{II})^{-1}Q_{IJ} \quad (5)$$

$$Q'_{JI} = Q_{JI}(\mathcal{I} - Q_{II})^{-1} \quad (6)$$

$$Q'_{II} = 0 \quad (7)$$

$$Q'_{IJ} = 0. \quad (8)$$

The following properties hold.

- 1) The vector processes y' and e' define LDIM (the marginalization of $\mathcal{G}$ with respect to y_I), which satisfies the relation $y' = e' + Q'(\mathbf{z})y'$.
- 2) Given two nodes $y_i \in y_I$ and y_j in y_J , if there is no chain of the form $y_i \rightarrow y_{\pi_1} \rightarrow \dots \rightarrow y_{\pi_\ell} \rightarrow y_j$, with $\ell \geq 1$, where $\{y_{\pi_1}, \dots, y_{\pi_\ell}\} \subseteq y_I$, then $Q_{ji} = 0$ implies $Q'_{ji} = 0$.
- 3) Given two nodes y_{j_1} and $y_{j_2} \in y_J$, if there is no chain of the form $y_{j_1} \rightarrow y_{\pi_1} \rightarrow \dots \rightarrow y_{\pi_\ell} \rightarrow y_{j_2}$, with $\ell \geq 1$, where $\{y_{\pi_1}, \dots, y_{\pi_\ell}\} \subseteq y_I$, then $Q'_{j_2j_1} = Q_{j_2j_1}$.

Furthermore, if $(Q(\mathbf{z}), e)$ is causal and causally well-posed, then $(Q'(\mathbf{z}), e')$ is causal and causally well-posed.

Proof: Note that

$$\begin{aligned} y_I &= e_I + Q_{IJ}y_J + Q_{II}y_I \\ &= (\mathcal{I} - Q_{II})^{-1}e_I + (\mathcal{I} - Q_{II})^{-1}Q_{IJ}y_J. \end{aligned}$$

Thus, substituting y_I in $y_J = Q_{JJ}y_J + Q_{JI}y_I + e_J$, it follows:

$$\begin{aligned} y'_J &:= y_J = e_J + Q_{JI}(\mathcal{I} - Q_{II})^{-1}e_I \\ &\quad + [Q_{JJ} + Q_{JI}(\mathcal{I} - Q_{II})^{-1}Q_{IJ}]y_J \end{aligned} \quad (9)$$

$$= e_J + Q'_{JI}y'_I + Q'_{JJ}y'_J. \quad (10)$$

Moreover

$$y'_I := e'_I = e_I + Q'_{II}y'_I + Q'_{IJ}y'_J.$$

Thus, 1) is proven. Now, let $T(\mathbf{z}) := (\mathcal{I} - Q_{II}(\mathbf{z}))$ where $T(\mathbf{z})$ is an invertible $N \times N$ matrix. From Cayley–Hamilton theorem [31], it follows that there exist coefficients $a_0(\mathbf{z}), \dots, a_{N-1}(\mathbf{z})$ such that

$$T^N + a_{N-1}(\mathbf{z})T^{N-1} + \dots + (-1)^N \det(T)I = 0$$

where $a_0(\mathbf{z}) := (-1)^{N-1} \det(T) \neq 0$. Multiplying the aforementioned identity by T^{-1} , it follows:

$$T^{-1} = \sum_{k=0}^{N-1} c_k T^k$$

for some coefficients c_k , which in turn admits an expansion of the form $T^{-1} = \sum_{k=0}^{N-1} d_k Q_{II}^k$. Thus, there exist coefficients d_k such that $(\mathcal{I} - Q_{II})^{-1} = \sum_{k=0}^{N-1} d_k Q_{II}^k$. Consider $y_i \in I$ and $y_j \in J$. Note that the entry (π_1, i) of the matrix Q_{II} is 0 if there is no link from y_i to y_{π_1} . Also, note that the entry (π_2, i) of the matrix $(Q_{II})^2$ can be written as

$$[(Q_{II})^2]_{\pi_2 i} = \sum_{\pi_1 \in I} [Q_{II}]_{\pi_2 \pi_1} [Q_{II}]_{\pi_1 i}$$

and thus, if there is no chain of the form $y_i \rightarrow y_{\pi_1} \rightarrow y_{\pi_2}$ for any $y_{\pi_1} \in y_I$, then $[Q_{II}]_{\pi_2 \pi_1} [Q_{II}]_{\pi_1 i} = 0$ for all $\pi_1 \in I$ implying $[(Q_{II})^2]_{\pi_2 i} = 0$. Similarly, the entry (π_3, i) of the matrix $(Q_{II})^3$ can be written as

$$[(Q_{II})^3]_{\pi_3 i} = \sum_{\pi_2 \in I} \sum_{\pi_1 \in I} [Q_{II}]_{\pi_3 \pi_2} [Q_{II}]_{\pi_2 \pi_1} [Q_{II}]_{\pi_1 i}$$

and thus, if there is no chain of the form $y_i \rightarrow y_{\pi_1} \rightarrow y_{\pi_2} \rightarrow y_{\pi_3}$ with $\{y_{\pi_1}, y_{\pi_2}, y_{\pi_3}\} \subseteq y_I$, then we have $[Q_{II}]_{\pi_3 \pi_2} [Q_{II}]_{\pi_2 \pi_1} [Q_{II}]_{\pi_1 i} = 0$ for all $\pi_1, \pi_2 \in I$ implying $[(Q_{II})^3]_{\pi_3 i} = 0$. Iterating this argument, it can be shown that if there is no chain of the form $y_i \rightarrow y_{\pi_1} \rightarrow \dots \rightarrow y_{\pi_\ell}$ with $\{y_{\pi_1}, \dots, y_{\pi_{\ell-1}}\} \subseteq y_I$ for $\ell \geq 1$, then $[(Q_{II})^\ell]_{\pi_\ell i} = 0$. Suppose there is no chain of the form $y_i \rightarrow y_{\pi_1} \rightarrow \dots \rightarrow y_{\pi_\ell} \rightarrow y_j$ with $\{y_{\pi_1}, \dots, y_{\pi_\ell}\} \subseteq y_I$. Then

$$\begin{aligned} [Q'_{JI}]_{ji} &= [Q_{JI}(\mathcal{I} - Q_{II})^{-1}]_{ji} = \sum_{k \in I} Q_{jk} [(I - Q_{II})^{-1}]_{ki} \\ &= \sum_{k \in I} \sum_{m=0}^{N-1} Q_{jk} d_m [Q_{II}^m]_{ki} = \sum_{k \in I} Q_{jk} d_0 [Q_{II}^0]_{ki} = Q_{ji} d_0 \end{aligned}$$

where the last two equalities follow, respectively, from the fact that there is no chain of the form $y_i \rightarrow y_{\pi_1} \rightarrow \dots \rightarrow y_{\pi_\ell} \rightarrow y_j$ with $\{y_{\pi_1}, \dots, y_{\pi_\ell}\} \subseteq y_I$ and the fact that $[Q_{II}^0]_{ki} = \mathcal{I}_{ki} = 0$ for $k \neq i$. (This proves 2).

For $y_{j_1} \in y_J$ and $y_{j_2} \in y_J$, following a similar argument, we show that

$$\begin{aligned} [Q'_{JJ}]_{j_2j_1} &= [Q_{JI}(\mathcal{I} - Q_{II})^{-1}Q_{IJ} + Q_{JJ}]_{j_2j_1} \\ &= Q_{j_2j_1} + \sum_{k \in I} \sum_{h \in I} [Q_{JI}]_{j_2k} [(I - Q_{II})^{-1}]_{kh} [Q_{IJ}]_{hj_1} \\ &= Q_{j_2j_1} + \sum_{k \in I} \sum_{h \in I} \sum_{m=0}^{N-1} d_m [Q_{JI}]_{j_2k} [Q_{II}^m]_{kh} [Q_{IJ}]_{hj_1} \\ &= Q_{j_2j_1} + \sum_{k \in I} \sum_{h \in I} d_0 [Q_{JI}]_{j_2k} \mathcal{I}_{kh} [Q_{IJ}]_{hj_1} \\ &= Q_{j_2j_1} + \sum_{k \in I} [Q_{JI}]_{j_2k} [Q_{IJ}]_{kj_1} = Q_{j_2j_1} \end{aligned}$$

where the last two equalities follow, respectively, from the fact that there is no chain of the form $y_i \rightarrow y_{\pi_1} \rightarrow \dots \rightarrow y_{\pi_\ell} \rightarrow y_j$ with $\{y_{\pi_1}, \dots, y_{\pi_\ell}\} \subseteq y_I$ and the fact that $[Q_{II}^0]_{kh} = \mathcal{I}_{kh} = 0$ for $k \neq h$. ■

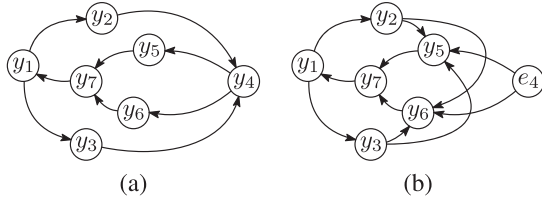


Fig. 2. (a) Graphical representation of an LDIM and (b) the graphical representation resulting from the marginalization of y_4 . For simplicity, it is assumed that in this example, the procedure has not generated any self-loops.

In Fig. 2, we report the graphical representation of an LDIM resulting from the marginalization of one of its nodes.

The marginalization procedure can introduce nonzero elements in the diagonal blocks of the transfer matrix $Q'(\mathbf{z})$, even if the diagonal blocks of Q are zero. These nonzero elements in the diagonal blocks of $Q'(\mathbf{z})$, however, can be removed using the lemma described below.

Lemma 16 (Self-Loop Removal): Consider a set of processes related to each other via

$$y = e + Q(\mathbf{z})y$$

where $y = (y_1^T \dots y_n^T)$, $e = (e_1^T \dots e_n^T)$. Consider Q to have a $n \times n$ block structure commensurate with dimensions of y_i , $i = 1, \dots, n$. Assume that block transfer matrix $Q_{kk}(\mathbf{z}) \neq 0$ for all $k \in K \subset \{1, 2, \dots, n\}$, and $Q_{kk}(\mathbf{z}) = 0$ for all $k \notin K$. Define the block diagonal transfer matrix $D(\mathbf{z})$ such that $D_{kk}(\mathbf{z}) = (I - Q_{kk})^{-1}$ for all k . A new set of processes y' following the relation:

$$y' = e' + Q'(\mathbf{z})y$$

can be defined with the following properties:

- 1) $y' = y$;
- 2) $e' = D(\mathbf{z})e$ (thus, $e'_j = e_j$ if $j \notin K$ and $\Phi_{e'} = D(\mathbf{z})\Phi_e D^*(\mathbf{z})$);
- 3) Q' , imposed with the same block structure of Q , has zero entries in its diagonal blocks;
- 4) $Q'_{ki} = D_{kk}Q_{ki}$ for $i \neq k$ (thus, $Q'_{ki} \neq 0$ if $Q_{ki} \neq 0$ for $i \neq j$ and $Q'_{ji} = Q_{ji}$ if $j \notin K$ and for all i).

Furthermore, if $(Q(\mathbf{z}), e)$ is an LDIM, then $(Q'(\mathbf{z}), e')$ is an LDIM and if $(Q(\mathbf{z}), e)$ is a causal and causally well-posed LDIM, then $(Q'(\mathbf{z}), e')$ is a causal and causally well-posed LDIM.

Proof: For every $k \in K$ observe that

$$y_k = e_k + Q_{kk}y_k + \sum_{i \neq k} Q_{ki}y_i.$$

This implies that

$$y_k = (I - Q_{kk})^{-1}e_k + \sum_{i \neq k} (I - Q_{kk})^{-1}Q_{ki}y_i.$$

Define $y' := y$. Also, for $k \in K$, define

$$\begin{aligned} e'_k &:= (I - Q_{kk})^{-1}e_k \\ Q'_{ki} &:= (I - Q_{kk})^{-1}Q_{ki} \text{ for all } i \neq k \\ Q'_{kk} &:= 0 \end{aligned}$$

and for $k \notin K$, define

$$\begin{aligned} e'_k &:= e_k \\ Q'_{ki} &= Q_{ki} \text{ for all } i. \end{aligned}$$

All the properties in the theorem statement can be verified by inspection. $\blacksquare$

Remark 17: Note that in Lemma 16, we have that

$$Q' = DQ - \text{diag}(D_{11}Q_{11}, \dots, D_{nn}Q_{nn})$$

where n is the block dimension of the transfer matrix Q .

The self-loop removal lemma can be used to remove all self-loops after marginalizing an LDIM in order to obtain a new LDIM with no self-loops.

Remark 18: The marginalization of an LDIM, with the subsequent removal of all self-loops, produces a new LDIM with no self-loops and with the property that nodes y_I have no incoming links and have outgoing links only to the nodes in $ch_G(V \setminus e_I)$.

VI. RELATION BETWEEN d -SEPARATION AND WIENER SEPARATION IN LDIMS

The main goal of this section is to show that if the graph G is a graphical representation of the LDIM $\mathcal{G} = (Q(\mathbf{z}), e)$ then $dsep_G(y_I, y_Z, y_J)$ implies $wsep(y_I, y_Z, y_J)$. Furthermore, if the LDIM is causal, we also have that $dsep_G(y_I, y_Z, y_J)$ implies $cwsep(y_I, y_Z, y_J)$ for the LDIM. Preliminary results, but only limited to the noncausal scenario, were shown in [22] following a derivation similar to the one provided in [32] for graphical models of Gaussian random variables. Such a derivation cannot be repeated for the causal scenario, since it would require an analytical expression for a matrix spectral factorization. Here, we provide a more general derivation that can tackle both scenarios at once. Such a novel derivation still borrows three lemmas from [22] and [32], which are of general utility since they can be used to obtain simpler d -separation relations in graphs.

When three disjoint sets y_I, y_J , and y_Z are a partition of the node set V of a graph, the following lemma provides a characterization of d -separation.

Lemma 19: Consider a directed graph $G = (V, \vec{E})$. Suppose y_I, y_J , and y_Z are three disjoint subsets of V such that $V = y_I \cup y_J \cup y_Z$; thus, y_I, y_J , and y_Z partition the set V . Then, y_Z d -separates y_I and y_J (i.e., $dsep_G(y_I, y_Z, y_J)$) if and only if there are no paths of the following form:

- 1) $y_i \rightarrow y_j$;
- 2) $y_j \rightarrow y_i$;
- 3) $y_i \rightarrow y_z \leftarrow y_j$

where $y_i \in y_I, y_j \in y_J$, and $y_z \in y_Z$.

Proof: See [22, Lem. 5] for the proof. $\blacksquare$

This lemma, instead, provides a way to verify d -separation in a graph G using a graph G' that is a restriction of G .

Lemma 20: Let $G = (V, \vec{E})$ be a directed graph. Let y_I, y_J and y_Z be disjoint subsets of V . Let $V' = an_G(y_I \cup y_J \cup y_Z)$ and let $G' = (V', \vec{E}')$ be the restriction of G to V' . Then

$$dsep_G(y_I, y_Z, y_J) \Leftrightarrow dsep_{G'}(y_I, y_Z, y_J).$$

Proof: This lemma is the same as [32, Cor. 1]. We refer to [32] for its proof. $\blacksquare$

The following lemma allows one to check if $dsep_G(y_I, y_Z, y_J)$ holds by testing $dsep_G(y_I, y_Z, y_J)$ where

y_I, y_Z, y_J partition the set of all ancestors of y_I, y_Z, y_J and $y_I \subseteq y_I'$ and $y_J \subseteq y_J'$.

Lemma 21: Given a directed graph $G = (V, \vec{E})$ and three disjoint sets $y_I, y_J, y_Z \subset V$. Let $y_U := an(y_I, y_J, y_Z)$. Let y_J' be the set of all nodes in y_U that are d -separated from the set y_I by y_Z

$$y_J' := \{y_j \in y_U | dsep_G(y_I, y_Z, y_j)\}.$$

Let the set y_I' be defined as complement of $\{y_J' \cup y_Z\}$ in y_U

$$y_I' := y_U \setminus \{y_J' \cup y_Z\}.$$

Then

$$dsep_G(y_I, y_Z, y_J) \Leftrightarrow dsep_G(y_I', y_Z, y_J'). \quad (11)$$

Proof: This lemma is the same as [32, Prop. 2]. We refer to [32] for its proof. ■

In a graph $G = (V, \vec{E})$, the Markov blanket of a set y_I is the set given by the nodes in $V \setminus y_I$ that are parents, children or parents of children of nodes in y_I .

Definition 22 (Markov Blanket): Given a graph G and a set y_I , define the Markov Blanket of y_I as

$$MB_G(y_I) := pa_G(y_I) \cup ch_G(y_I) \cup pa_G(ch_G(y_I)) \setminus y_I.$$

Theorem 23 (Markov Blanket Separation): Consider an LDIM $\mathcal{G}$ with set of nodes V and no self-loop in y_I . For every $y_Z \supseteq MB_G(y_I)$, we have that $wsep(y_I, y_Z, V \setminus (y_Z \cup y_I))$. Furthermore, if $\mathcal{G}$ is causal and causally well-posed, we also have $cwsep(y_I, y_Z, V \setminus (y_Z \cup y_I))$.

Proof: If y_I has a single element, this result is equivalent to [12, Ths. 27 and 34] for the noncausal and causal case, respectively. In order to show Wiener separation for a vector process y_I , it is enough to show Wiener separation for every element in y_I . We show this via some manipulations. Define $y_{\bar{I}} := y_I \setminus \{y_i\}$ and marginalize $y_{\bar{I}}$. Let G'_i be the graphical representation of the graph after the marginalization of $y_{\bar{I}}$. We want to show that $MB_{G'_i}(y_i) \subseteq MB_G(y_I)$. Trivially, all children of and parents of y_i in G'_i are, respectively, children and parents of y_I of G . The marginalization of $y_{\bar{I}}$, though can create some edges in G'_i that are not in G . There is an edge from y_k to y_ℓ in G'_i that is not in G if and only if there is a path in G from y_k to y_ℓ that involves only nodes in $y_{\bar{I}}$. Thus, y_k needs to be a parent of y_I in G and y_ℓ needs to be a child of y_I . Necessarily, then, a coparent of y_i in G'_i is a coparent of y_i in G or a parent of y_I in G . This implies that $MB_{G'_i}(y_i) \subseteq MB_G(y_I)$. Then, since $MB_{G'_i}(y_i) \subseteq y_Z$ for all $y_i \in y_I$, we have, from [12, Th. 27], that $wsep(y_i, y_Z, V \setminus (y_Z \cup y_I))$ for all $y_i \in y_I$. Identical step prove the statement $cwsep(y_i, y_Z, V \setminus (y_Z \cup y_I))$ in the causal scenario, where the fact that $\mathcal{G}$ is a causal and causally well-posed guarantees that $\mathcal{G}'_y$ is causal and causally well-posed so that [12, Th. 34] can be applied. ■

Lemmas 19–21, along with Theorem 23 allow us to introduce the first main contribution of this article. The following theorem establishes a strong connection between the concept of d -separation defined on the graph and the notion of Wiener separation that is instead associated with LDIM.

Theorem 24: In a LDIM $\mathcal{G}$ with graphical representation $G = (V, \vec{E})$, let y_I, y_Z, y_J be disjoint subsets of V . Then

$$dsep_G(y_I, y_Z, y_J) \Rightarrow wsep(y_I, y_Z, y_J).$$

Furthermore, if the LDIM is causal and causally well-posed, then $dsep_G(y_I, y_Z, y_J) \Rightarrow cwsep(y_I, y_Z, y_J)$.

Proof: Let y_I, y_Z , and y_J be three disjoint subsets such that $dsep_G(y_I, y_Z, y_J)$. Define $V' = an_G(y_I \cup y_Z \cup y_J)$ and observe that $y_I \cup y_Z \cup y_J \subseteq V'$. Since the transfer functions from $y_{V \setminus V'}$ to $y_{V'}$ are identically zero, we have that $y_{V'} = Q_{V'V'}(z)y_{V'} + e_{V'}$ allowing us to define an LDIM $\mathcal{G}'$ over the nodes $y_{V'}$. Observe that the restriction G' of G to $V' = an_G(y_I \cup y_Z \cup y_J)$ is a graphical representation for $\mathcal{G}'$. From Lemma 20, we have that $dsep_{G'}(y_I, y_Z, y_J)$. Defining $y_I' \subseteq y_I$ and $y_J' \subseteq y_J$ as in Lemma 21, we get $dsep_{G'}(y_I', y_Z, y_J')$. Observe that y_I', y_J' , and y_Z are a partition of V' , thus, from Lemma 19, we have that $MB_{G'}(y_I') \subseteq y_Z$. Also, we can write the dynamics of $\mathcal{G}'$ as

$$\begin{pmatrix} y_I' \\ y_Z \\ y_J' \end{pmatrix} = \begin{pmatrix} Q_{I'I'} & Q_{I'Z} & 0 \\ Q_{ZI'} & Q_{ZZ} & Q_{ZJ'} \\ 0 & Q_{J'Z} & Q_{J'J'} \end{pmatrix} \begin{pmatrix} y_I' \\ y_Z \\ y_J' \end{pmatrix} + \begin{pmatrix} e_{I'} \\ e_Z \\ e_{J'} \end{pmatrix}. \quad (12)$$

By applying the self-loop removal Lemma, this can be rewritten as

$$\begin{pmatrix} y_I' \\ y_Z \\ y_J' \end{pmatrix} = \begin{pmatrix} 0 & \tilde{Q}_{I'Z} & 0 \\ \tilde{Q}_{ZI'} & 0 & \tilde{Q}_{ZJ'} \\ 0 & \tilde{Q}_{J'Z} & 0 \end{pmatrix} \begin{pmatrix} y_I' \\ y_Z \\ y_J' \end{pmatrix} + \begin{pmatrix} e_{I'} \\ e_Z \\ e_{J'} \end{pmatrix} \quad (13)$$

so that we obtain an LDIM $\mathcal{G}''$ with the same output signals as in $\mathcal{G}'$. In the graphical representation G'' of $\mathcal{G}''$, we still have that $MB_{G''}(y_I') \subseteq y_Z$: no edge connects directly a node in y_I' and a node in y_J' ; if a node in y_I' and a node in y_J' shared a child in G'' then there would be from Lemma 16 a node in y_I' and a node in y_J' sharing a child in G' and this is ruled out by the fact that $dsep_{G'}(y_I, y_Z, y_J)$. From Theorem 23 applied to the LDIM $\mathcal{G}'$, we have that $wsep(y_I', MB_{G''}(y_I'), V' \setminus (MB_{G''}(y_I') \cup y_I'))$. Thus, the Wiener filter estimating y_I' from $V' \setminus y_I'$ has zero components for all signals in $V' \setminus (MB_{G''}(y_I') \cup y_I')$. Since $MB_{G'}(y_I') \subseteq y_Z$, we also have the Wiener filter estimating y_I' from $V' \setminus y_I'$ that has zero components for all signals in $V' \setminus (y_Z \cup y_I')$ implying $wsep(y_I', y_Z, V' \setminus (y_Z \cup y_I'))$. Since $y_J \subseteq y_J' = V' \setminus (y_Z \cup y_I')$, we also have $wsep(y_I', y_Z, y_J)$. Finally, since $y_I \subseteq y_I'$, we also have that $wsep(y_I, y_Z, y_J)$. The proof for the causal case follows the same steps, where the fact that the LDIM $\mathcal{G}$ is causal and causally well-posed guarantees via Lemmas 15 and 16 that the LDIMs $\mathcal{G}'$ and $\mathcal{G}''$ are causal and causally well-posed. ■

The importance of Theorem 24 can be summarized as follows: when estimating a certain node from the observations of a set of other nodes, a graphical test (d -separation) allows one to determine if some of the observations are irrelevant for the estimate and, thus, can be disregarded. Example 2 illustrates the power of Theorem 24 by showing how it is possible to answer questions on sensor locations without solving any optimization problem but relying entirely on the graphical properties of the network topology.

Example 2: Consider an LDIM with a graphical representation given by the graph G shown in Fig. 3. The objective is to estimate y_1 . As a first scenario assume that the nodes y_A, y_B, y_C , and y_D are being measured. Let us assume that y_6 is added to the set of observed signals. Using Theorem 24, we conclude that addition of y_6 to the set of observed nodes $\{y_A, y_B, y_C, y_D\}$ makes y_A irrelevant for the estimate of y_1 , since

$$dsep_G(y_1, \{y_6, y_B, y_C, y_D\}, y_A).$$

Thus, having both y_A and y_6 being measured in addition to y_B, y_C , and y_D being measured is wasteful in estimating y_1 . This

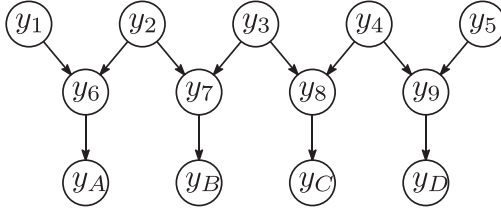


Fig. 3. Graphical representation of the LDIM of Example 2. The objective is to minimize the least square estimation error on y_1 using only three of the four observed nodes y_A , y_B , y_C , and y_D . By the iterative application of Theorem 24, it is shown that irrespective of the power spectral densities of the signals, because of the system structure, the triplet y_A , y_B , and y_C always minimizes the least square error in the estimate.

is quite intuitive, since y_6 is on the path from y_1 to node y_6 . Similarly, the further addition of y_7 to the set of observed nodes makes its child y_B irrelevant as well. Indeed, we have

$$dsep_G(y_1, \{y_6, y_7, y_C, y_D\}, \{y_A, y_B\}).$$

In general, if we use the nodes $y_6, y_7, y_8, y_9, y_A, y_B, y_C, y_D$ for estimating y_1 , Theorem 24 allows us to conclude that y_A, y_B, y_C, y_D are not needed, because

$$dsep_G(y_1, \{y_6, y_7, y_8, y_9\}, \{y_A, y_B, y_C, y_D\}).$$

A less intuitive consequence of Theorem 24 is the following. Let us go back to the situation where the observed nodes are again y_A, y_B, y_C , and y_D , but now, we have a sparsity requirement: we can only use three of those signals for the estimation of y_1 . Observe that even though y_B, y_C , and y_D are independent of y_1 , their information, in combination with y_A , can help improve the estimate of y_1 that would be obtained using only y_A . Indeed, it can be verified that, in general, the Wiener filter estimating y_1 from y_A, y_B, y_C , and y_D has four nonzero entries. However, the goal is to find a triplet of observed signals optimally estimating y_1 in the least square sense. We start with a straightforward observation: y_A necessarily belongs to an optimal triplet since any triplet containing y_A performs at least as well as $\{y_B, y_C, y_D\}$, which are all independent of y_1 . This can also be seen via Theorem 24, since

$$dsep_G(y_1, \emptyset, \{y_B, y_C, y_D\}).$$

Having established that y_A belongs to an optimal triplet for the estimation of y_1 , we find now that $\{y_A, y_B\}$ is necessarily contained in an optimal triplet. Indeed, any triplet containing $\{y_A, y_B\}$ performs as well as any other triplet containing $\{y_A\}$ when estimating y_1 , since

$$dsep_G(y_1, y_A, \{y_C, y_D\})$$

implies that the estimate using only $\{y_A\}$ performs as well as the estimate using $\{y_A, y_C, y_D\}$. Given that $\{y_A, y_B\}$ is contained in an optimal triplet for the estimation of y_1 , we eventually find that $\{y_A, y_B, y_C\}$ is necessarily optimal, since

$$dsep_G(y_1, \{y_A, y_B\}, y_D)$$

implies that $\{y_A, y_B\}$ performs as well as $\{y_A, y_B, y_D\}$. Thus, irrespective of the actual transfer functions and power spectral densities of the network, $\{y_A, y_B, y_C\}$ is always an optimal

triplet for the estimate of y_1 . Observe how only graphical properties lead to this conclusion without the need to solve any involved optimization problem with sparsity constraints.

VII. IDENTIFICATION OF INDIVIDUAL TRANSFER FUNCTIONS IN AN LDIM

This section presents the contributions of this article about the identification of individual transfer functions in an LDIM.

The single-door criterion is a powerful tool developed for the identification of parameters in structural equation models [33]. Here, we provide a similar criterion for the identification of transfer functions in an LDIM.

Lemma 25: Let $G = (V, \bar{E})$ be a graphical representation of the LDIM $\mathcal{G} = (Q(\mathbf{z}), e)$ with output $y = (y_1^T, \dots, y_n^T)$, where G is allowed to have self-loops. Let $\bar{G}$ be the graph obtained by removing the link (y_i, y_j) from the graph G . For some $y_Z \subset y$, let $y_U := an_G(y_i, y_j, y_Z)$. Define the following sets of processes:

$$y_J := \{y_k \in y_U \mid dsep_{\bar{G}}(y_k, y_Z, y_i)\}$$

$$y_{\bar{J}} := y_J \setminus \{y_j\}$$

$$y_K := \{y_k \in y_Z \mid \text{there is an edge } y_j \rightarrow y_k$$

$$\text{or there is a chain } y_j \rightarrow \dots \rightarrow y_k$$

$$\text{with internal nodes all in } y_J\}.$$

Assume $\Phi_{(y_i, y_j, y_Z)}(\mathbf{z}) > 0$ for all $|\mathbf{z}| = 1$ and the following assumptions:

- A1) y_i and y_j are d -separated by y_Z in the graph $\bar{G}$;
- A2) in the restriction of G to y_J , y_j is involved in no loops;
- A3) there is no link of the form $y_k \rightarrow y_j$ for $y_k \in y_K$.

Under the aforementioned assumptions, if y_K is empty, we have

$$Q_{ji} = W_{y_j, [y_i] | (y_i, y_Z)}. \quad (14)$$

Instead, if y_K is not empty, the following holds:

$$Q_{ji} = \left[\mathcal{I} - W_{y_j, [y_K] | (y_i, y_Z)} Q_{Kj}^{(f)} \right]^{-1} W_{y_j, [y_i] | (y_i, y_Z)} \quad (15)$$

where

$$Q_{Kj}^{(f)} = (\mathcal{I} - (Q_{KK} + Q_{K\bar{J}}(\mathcal{I} - Q_{\bar{J}\bar{J}})^{-1} Q_{\bar{J}K}))^{-1} \cdot (Q_{Kj} + Q_{K\bar{J}}(\mathcal{I} - Q_{\bar{J}\bar{J}})^{-1} Q_{\bar{J}j}). \quad (16)$$

Here, $W_{y_j, [y_i] | (y_i, y_Z)}$ and $W_{y_j, [y_K] | (y_i, y_Z)}$ are the component corresponding to y_i and y_K , respectively, in the Wiener filter that estimates the process y_j based on processes y_i and y_Z .

Proof: See the Appendix. ■

Equation (15) provides an immediate expression for the identification of $Q_{ji}(\mathbf{z})$ when $Q_{Kj}^{(f)}(\mathbf{z})$ is known or can be directly estimated using the power spectral densities of the observed signals. This cannot be achieved in the general case. In the following, we will provide two results describing situations where (15) can be used for the identification of $Q_{ji}(\mathbf{z})$. In Theorem 26, we will consider additional conditions guaranteeing that $Q_{Kj}^{(f)}(\mathbf{z})$ is identically zero. In Theorem 27, we will show when we can compute an estimate of $Q_{Kj}^{(f)}(\mathbf{z})$ that depends on the power spectral densities of the observed signals and Q_{ji} reducing (15) to an implicit equation in Q_{ji} .

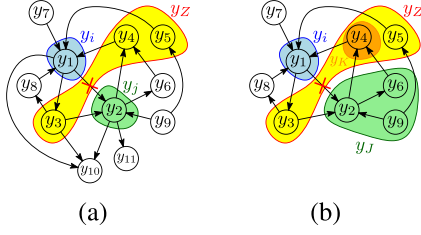


Fig. 4. (a) Application of Lemma 25 with $y_i = y_1$, $y_j = y_2$, and $y_Z = \{y_3, y_4, y_5\}$ in Example 3. (b) Restriction to $an_G(\{y_i, y_j\} \cup y_Z)$ and sets y_J and y_K . Observe that if the “door” (edge) from y_i and y_j is removed, the two nodes are d -separated by y_Z .

Before introducing Theorems 26 and 27, we illustrate the application of Lemma 25 with an example assuming $Q_{Kj}^{(f)}(\mathbf{z})$ is already known.

Example 3: Consider an LDIM with graph representation as in Fig. 4(a) where we have chosen $y_i = y_1$ and $y_j = y_2$. Let also $y_Z = \{y_3, y_4, y_5\}$. Observe that Assumption A1 is verified since $dsep(y_1, \{y_3, y_4, y_5\}, y_2)$ in the graph $\bar{G}$ obtained by removing the edge $y_1 \rightarrow y_2$ (marked with a cross). The restriction of the LDIM to $y_U = an_G(\{y_i, y_j\} \cup y_Z)$ is shown in Fig. 4(b), where it is also shown the set y_J of all the nodes in y_U that are d -separated from y_i by y_Z in $\bar{G}$. y_J is given by $\{y_2, y_6, y_9\}$. Assumption A2 is then trivially verified. The only node in y_K is y_4 , because it is the only node in y_Z for which there is a chain starting from y_j with all internal nodes in y_J . Thus, Assumption A3 is verified as well. The application of the Lemma 25 yields

$$Q_{21} = (1 - W_{y_2[y_4]|\{y_1, y_4\}} Q_{41}^{(f)})^{-1} W_{y_2[y_1]|\{y_1, y_4\}}$$

where the expression of $Q_{41}^{(f)}$ is given by (16).

Now, if the set y_Z contains no descendants of y_i , we have that y_K is empty, and thus, Q_{ji} matches the expression of $W_{y_j[y_i]|\{y_i, y_Z\}}$, leading to the following important consequence.

Theorem 26 (Single-Door Criterion for LDIMs): Let $G = (V, \bar{E})$ be a graphical representation of the LDIM $\mathcal{G} = (Q(\mathbf{z}), e)$ with output $y = (y_1^T, \dots, y_n^T)$, where G has no self-loop in y_j . Let $\bar{G}$ be the graph obtained by removing the link (y_i, y_j) from the graph G . Assume $\Phi_{(y_i, y_j, y_Z)}(\mathbf{z}) > 0$ for all $|\mathbf{z}| = 1$ and the following assumptions:

- A1) y_i and y_j are d -separated by y_Z in the graph $\bar{G}$;
- A4) y_Z has no descendants of y_j .

Then, $Q_{ji} = W_{y_j[y_i]|\{y_i, y_Z\}}$.

Proof: The absence of loops in y_j and Assumption A4 implies Assumptions A2 and A3 in Lemma 25 and the fact that $y_K = \emptyset$, giving immediately the assertion. ■

A criterion similar to Theorem 26, but valid for the identification of simple proportional gains and not general transfer functions, is known in the area of structural equation models as the single door criterion [33]. The main intuition behind Theorem 26 is the following: if there is only a “single door” (edge) from y_i to y_j that prevents these two nodes from being “separated” (in the sense of d -separation) by a set y_Z that does not contain any descendants of y_j , then the Wiener filter estimating y_j from $y_i \cup y_Z$ has component associated with y_i equal to the transfer function Q_{ji} .

The identification of transfer functions in presence of confounding variables is a topic of active investigation (see [34]) and Theorem 26 can leverage the notion of d -separation toward such a goal, as the following example shows.

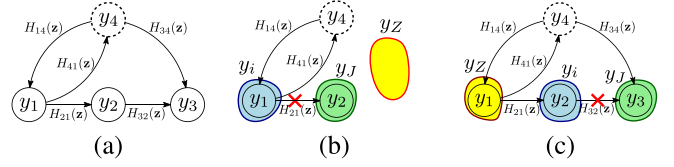


Fig. 5. (a) Network with a confounding variable y_4 . If the confounding variable was not present, the identification of the transfer functions $Q_{21}(\mathbf{z})$ and $Q_{32}(\mathbf{z})$ could be obtained as $Q_{21}(\mathbf{z}) = \Phi_{y_2 y_1} \Phi_{y_1 y_1}^{-1}$ and $Q_{32}(\mathbf{z}) = \Phi_{y_3 y_2} \Phi_{y_2 y_2}^{-1}$. The presence of the confounding variable makes these relations not correct in the general case. The application of the single door criterion (see Theorem 26) provides a solution to this identification problem that is robust with respect to the presence/absence of the y_4 . (b) Application of the single door criterion for the identification of $Q_{21}(\mathbf{z})$. (c) Application of the single door criterion for the identification of $Q_{32}(\mathbf{z})$.

Example 4 (A Network With a Confounder and No Observable External Variable): Consider the graph G of Fig. 5(a) representing an LDIM following the dynamics $y = e + Q(\mathbf{z})y$ where the only potentially nonzero entries of $Q(\mathbf{z})$ are Q_{21} , Q_{32} , Q_{14} , Q_{41} , and Q_{34} . Also assume that only y_1 , y_2 , and y_3 are observed, but not y_4 . The goal is to identify the transfer functions $Q_{21}(\mathbf{z})$ and $Q_{32}(\mathbf{z})$. Let $G_{1 \rightarrow 2}$ be the graph obtained by removing the edge $1 \rightarrow 2$ from G . Observe that $dsep_{G_{1 \rightarrow 2}}(y_1, \emptyset, y_2)$. By applying Theorem 26, with $y_Z = \emptyset$, as shown in Fig. 5(b), we get

$$Q_{21} = W_{y_2[y_1]|\{y_1\}} = \Phi_{y_2 y_1} \Phi_{y_1 y_1}^{-1}. \quad (17)$$

This is not surprising because y_2 is influenced only by node y_1 and (17) is a trivial consequence of Wiener–Khinchin theorem [24]. A more difficult task is to identify $Q_{32}(\mathbf{z})$ because of the presence of the confounding process y_4 that is not observed. Indeed the relation

$$Q_{32} = \Phi_{y_3 y_2} \Phi_{y_2 y_2}^{-1} \quad (18)$$

could be obtained from Wiener–Khinchin theorem only if $Q_{34} = 0$ or $\Phi_{y_4} = 0$. Instead, Theorem 26 provides a way to identify $Q_{32}(\mathbf{z})$. Let $G_{2 \rightarrow 3}$ be the graph obtained by removing the edge $2 \rightarrow 3$ from G . Observe that $dsep_G(y_2, y_1, y_3)$. By applying Theorem 26 with $y_Z = \{y_1\}$, as shown in Fig. 5(b), we obtain

$$Q_{32} = W_{y_3[y_2]|\{y_2, y_1\}} = (\Phi_{y_3 y_2} \quad \Phi_{y_3 y_1}) \begin{pmatrix} \Phi_{y_2 y_2} & \Phi_{y_2 y_1} \\ \Phi_{y_1 y_2} & \Phi_{y_1 y_1} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (19)$$

Thus, from (19), we have that the identification of $Q_{32}(\mathbf{z})$ can be obtained from the knowledge of the power spectral densities of y_1 , y_2 , and y_3 only. Notice that, irrespective of node y_4 and how it influences y_1 and y_3 , (19) always provides an unbiased identification for $Q_{32}(\mathbf{z})$. In particular, y_4 might exist or not in the network and still (19) would be correct. In other words, (19) is modification of (18) that is robust with respect to the presence of y_4 or the link $y_4 \rightarrow y_3$. Thus, the identification approach of this article provides a novel form of robustness against uncertainties in the network structure.

We also consider an example similar to Example 4 to show that the assumption that the unknown forcing inputs e are mutually independent can be relaxed in the framework developed in this article. This can be done by exploiting the fact that not all outputs y need to be measure to apply the single-door criterion.

Example 5: Consider a network in the DSF form $y = Q(\mathbf{z})y + P(\mathbf{z})u$ with a nonstrictly causal Q and a nondiagonal P

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ Q_{21} & 0 & 0 \\ 0 & Q_{32} & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + \begin{pmatrix} P_{11} & 0 & P_{13} \\ 0 & P_{22} & 0 \\ 0 & 0 & P_{33} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

where $\Phi_u(\mathbf{z})$ is diagonal. The objective is to determine Q_{32} and Q_{21} by knowing

$$\Phi_{(y_1, y_2, y_3)} = \begin{pmatrix} 2 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{pmatrix}; \quad \Phi_u = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$Q = \begin{pmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ 0 & * & 0 \end{pmatrix}; \quad P = \begin{pmatrix} * & 0 & * \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}$$

where $*$ denotes an entry potentially different from zero. Observe that this DSF model does not meet the assumptions in [15], since Q is not strictly causal and P is not diagonal. The results in [9] can provide Q and P from the knowledge of $G(\mathbf{z}) = (I - Q(\mathbf{z}))^{-1}P(\mathbf{z})$, by solving the matrix equality

$$G = QG + P = (Q, P) \begin{pmatrix} G \\ I \end{pmatrix} \Leftrightarrow G^T = (G^T, I) \begin{pmatrix} Q^T \\ P^T \end{pmatrix}. \quad (20)$$

Such an equation admits a solution if three elements in each row of the matrix (Q, P) are already known (DSF informativity conditions in [9]). Observe that four elements in each row of (Q, P) are known, thus the problem of computing Q and P from the knowledge G is overdetermined. On the other hand, $G(\mathbf{z})$ is not known directly, but it is known that $G(\mathbf{z})G(\mathbf{z})^* = \Phi_{(y_1, y_2, y_3)}$. From the sparsity patterns of Q and P , we have the sparsity pattern of G

$$G = (I - Q)^{-1}P = \begin{pmatrix} * & 0 & * \\ * & * & * \\ * & * & * \end{pmatrix}.$$

Multiple $G(\mathbf{z})$ satisfy such a factorization with such a sparsity pattern, for example

$$G_2 = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ \sqrt{2} & 1 & 0 \\ \frac{3}{\sqrt{2}} & 1 & \frac{1}{\sqrt{2}} \end{pmatrix}; \quad G_3 = \begin{pmatrix} -\frac{2}{\sqrt{3}} & 0 & \frac{\sqrt{2}}{\sqrt{3}} \\ \frac{1-2\sqrt{2}}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1+\sqrt{2}}{\sqrt{3}} \\ \frac{\sqrt{2}-7}{2\sqrt{3}} & \frac{1+\sqrt{2}}{2} & \frac{1+\sqrt{2}}{3} \end{pmatrix}.$$

Thus, finding the correct factor G from $G(\mathbf{z})G(\mathbf{z})^* = \Phi_{(y_1, y_2, y_3)}$ is an underdetermined problem. If we try to plug G_2 or G_3 into (20), we find no (Q, P) with the known sparsity pattern [we remind that (20) was overdetermined]. The fundamental challenge is that the factorization $\Phi_{(y_1, y_2, y_3)} = GG^*$ admits an infinite number of solutions and it is a quadratic problem in the entries of G that in turn needs to satisfy $G = (I - Q)P^{-1}$ with specific sparsity patterns for Q and P . The infinite solutions, then, need to be checked against (20) to see if they are admissible. Problems of this kind are difficult to tackle using current DSF tools because they require, in general, to solve nonlinear matrix equations.

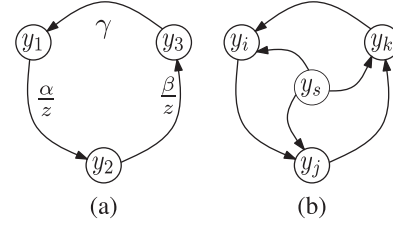


Fig. 6. (a) Loop network of Examples 6 and 7. (b) Network configuration of Theorem 27.

On the other hand, such a network is not an LDIM, since $e = P(\mathbf{z})u$ does not have a diagonal PSD. The problematic input is u_3 , which influences both y_1 and y_3 . However, by introducing a new output node y_4 , we can rewrite the system, so that it is an LDIM

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} + \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix}.$$

In this form, the system is recognized as an LDIM of the same form of the LDIM in Example 4. The application of the single-door criterion gives

$$Q_{21}(z) = \Phi_{y_2 y_1}(z) / \Phi_{y_1 y_1}(z) = \frac{2}{2} = 1$$

$$Q_{32}(z) = (\Phi_{y_3 y_1}, \Phi_{y_3 y_2}) \begin{pmatrix} \Phi_{y_1 y_1} & \Phi_{y_1 y_2} \\ \Phi_{y_2 y_1} & \Phi_{y_2 y_2} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= (3, 4) \begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1.$$

Observe that the single-door criterion has not used the fact that $\Phi_u = I$, but only that it is diagonal.

We have shown that the single-door criterion allows us to identify a single transfer function $Q_{ji}(\mathbf{z})$ in an LDIM with no self-loop at y_j without necessarily observing all the signals of the network. The main assumption is that, after removing the edge $y_i \rightarrow y_i$, it must be possible to d -separate y_i and y_j in the graph of the LDIM using no descendants of y_j . This assumption is not very limiting if the edge $y_i \rightarrow y_j$ is not involved in a directed loop. Indeed, the standard formulation of the single-door criterion has been developed in the context of directed acyclic graphs where this assumption is always verified. However, if the edge $y_i \rightarrow y_j$ is involved in a directed loop, it is not possible to d -separate y_i and y_j making no use of descendants of y_j . The following example shows how to use Lemma 25 instead of Theorem 26 when the node y_j is involved in a loop if other transfer functions of the network are known.

Example 6 (Closed Loop Identification With Knowledge of a Transfer Function): Consider again the loop network in Fig. 6(a) and this time assume that it is only known that the network only involves causal transfer functions. The objective is to identify the transfer function Q_{21} relying on the power spectral densities of the output processes of the LDIM. After removing the link $y_1 \rightarrow y_2$, it is not possible to d -separate nodes y_1 and y_2 without using node y_3 , which is a descendant of y_2 . Thus, the single-door criterion (see Theorem 26) cannot be used. However, if it is known that $Q_{32}(\mathbf{z}) = \frac{\beta}{z}$, we can apply Lemma 25 where $y_i = y_1$, $y_j = y_2$, $y_Z = y_K = \{y_3\}$,

and $Q_{Kj}^{(f)} = Q_{32}(\mathbf{z}) = \frac{\beta}{\mathbf{z}}$. For the application of Lemma 25 to compute Q_{21} , we need the quantities $W_{y_2, [y_1] | (y_1, y_3)}$ and $W_{y_2, [y_3] | (y_1, y_3)}$, which can be computed from the knowledge of the power and cross-power spectral densities of the signals y_1 , y_2 , and y_3 . Several methods exist to estimate the power and cross-power spectral densities from data. We illustrate the application of Lemma 25 using the analytical expressions of $W_{y_2, [y_1] | (y_1, y_3)}$ and $W_{y_2, [y_3] | (y_1, y_3)}$, which are

$$W_{y_2, [y_1] | (y_1, y_3)} = \frac{\alpha \Phi_{e_3} \mathbf{z}}{\Phi_{e_3} \mathbf{z}^2 + \beta^2 \Phi_{e_2}} \quad \text{and} \quad (21)$$

$$W_{y_2, [y_3] | (y_1, y_3)} = \frac{\beta \Phi_{e_2} \mathbf{z}}{\Phi_{e_3} \mathbf{z}^2 + \beta^2 \Phi_{e_2}}. \quad (22)$$

From Lemma 25, we find

$$\begin{aligned} Q_{21} &= (1 - W_{y_2, [y_3] | (y_1, y_3)} Q_{32}) W_{y_2, [y_1] | (y_1, y_3)} \\ &= \left(1 - \frac{\beta \Phi_{e_2} \mathbf{z}}{\Phi_{e_3} \mathbf{z}^2 + \beta^2 \Phi_{e_2}} \frac{\beta}{\mathbf{z}}\right) \frac{\alpha \Phi_{e_3} \mathbf{z}}{\Phi_{e_3} \mathbf{z}^2 + \beta^2 \Phi_{e_2}} = \frac{\alpha}{\mathbf{z}} \end{aligned}$$

and, thus, the transfer function Q_{21} is recovered correctly. Notice how the identification results without measuring the input signals of the network e_1 , e_2 , and e_3 , but just its outputs. Also, no assumption is needed about the color of the noises.

A. Generalized Single-Door Criterion for Closed-Loop Identification

Lemma 25 can overcome the limitations of the standard single-door criterion to deal with closed-loop identification problems even when other transfer functions in the network are not known and potentially null (to model partial knowledge of the network). To exemplify the versatility of Lemma 25, we derive a novel methodology for the identification of a transfer function involved in a closed loop in the potential presence of a measurable common process affecting all the nodes of the loop. A wealth of other, generally parametric, techniques exist to identify individual transfer functions in networks [17]–[19]. These techniques can be applied to situations where loops are present, but require the presence of at least a strictly causal transfer function in each loop and some form of knowledge about which of the transfer functions in the loop is strictly causal [17], [18], or cannot deal in general with uncertainties in the network topology because edges in the graph cannot represent null transfer functions [19]. Our method has the advantage of not requiring such knowledge and of being applicable to scenarios where edges are only potentially present. Furthermore, our method, being nonparametric, does not need an explicit noise model and the solution of nonconvex optimization problems. However, such a technique requires the solution of an implicit relation involving the transfer function $Q_{ji}(\mathbf{z})$.

Theorem 27 (Revolving Door Criterion): Consider an LDIM $\mathcal{G} = (Q(\mathbf{z}), e)$ with output (y_i, y_j, y_k, y_s) and graphical representation $G = (V, \vec{E})$ shown in Fig. 6(b). Then, the following implicit relation involving the transfer function $Q_{ji}(\mathbf{z})$ holds:

$$\begin{aligned} Q_{ji} &= \left\{ \mathcal{I} - W_{j, [y_k] | i, y_k, s} W_{y_k, [j] | J, i, s} - W_{j, [y_k] | i, y_k, s} W_{y_k, [i] | j, i, s} \right. \\ &\quad \cdot (\mathcal{I} - W_{i, [j] | y_k, j, s} Q_{ji} - W_{i, [y_k] | y_k, j, s} W_{y_k, [i] | j, i, s})^{-1} \\ &\quad \left. \cdot W_{i, [y_k] | y_k, j, s} W_{y_k, [j] | j, i, s} \right\}^{-1} W_{j, [i] | i, y_k, s}. \end{aligned} \quad (23)$$

Proof: Consider the graph $G_{i \rightarrow j}^-$ resulting from removing the link from $y_i \rightarrow y_j$ from graph G . In the graph $G_{i \rightarrow j}^-$, the paths connecting i and j are: $y_j \rightarrow y_k \rightarrow y_i$, $y_j \rightarrow y_k \leftarrow y_s \rightarrow y_i$, $y_j \leftarrow y_s \rightarrow y_k \rightarrow y_i$, and $y_j \leftarrow y_s \rightarrow y_i$, which are all paths that are d -separated by the set $\{y_s, y_k\}$. Moreover, the set $J := \{y_m | d_{\text{sep}}(y_m, \{y_s, y_k\}, y_i)\} = \{y_j\}$. Note that y_j is not in any loop in G restricted to $y_J \cup y_k = y_j \cup y_k$. Thus, all the conditions of Lemma 25 are satisfied and thus

$$Q_{ji} = \left[\mathcal{I} - W_{y_j, [y_k] | (y_i, y_k, y_s)} Q_{kj}^{(f)} \right]^{-1} W_{y_j, [y_i] | (y_i, y_k, y_s)} \quad (24)$$

where

$$\begin{aligned} Q_{kj}^{(f)} &= (\mathcal{I} - (Q_{kk} + Q_{k\bar{J}}(\mathcal{I} - Q_{\bar{J}\bar{J}})^{-1} Q_{\bar{J}k}))^{-1} \\ &\quad \cdot (Q_{kj} + Q_{k\bar{J}}(\mathcal{I} - Q_{\bar{J}\bar{J}})^{-1} Q_{\bar{J}j}). \end{aligned}$$

Note that $\bar{J} = \emptyset$ and there is no self-loop at k . Thus, $Q_{kj}^{(f)} = Q_{kj}$. It follows in a similar manner that

$$Q_{kj} = [\mathcal{I} - W_{y_k, [y_i] | (y_j, y_i, y_s)} Q_{ik}]^{-1} W_{y_k, [y_j] | (y_j, y_i, y_s)} \quad (25)$$

and

$$Q_{ik} = [\mathcal{I} - W_{y_i, [y_j] | (y_k, y_j, y_s)} Q_{ji}]^{-1} W_{y_i, [y_k] | (y_k, y_j, y_s)}. \quad (26)$$

By replacing (26) in (25) and the resulting expression into (24), we obtain the assertion using the matrix inversion lemma. ■

Remark: Theorem 27 represents only an instance of the application of Lemma 25. There are many results that can be derived on identification of transfer functions involved in loops. For instance, Theorem 27 can easily be extended to involve a closed-loop with multiple intermediate nodes between y_i and y_j and not just y_k ; the resulting analog of relation (23) will still be quadratic in Q_{21} . Furthermore, it is straightforward to generalize the results to include nonancestors of the variables y_i , y_j , y_k , and y_s as there is no loss of generality, in the conclusions of Lemma 25, by restricting the graph to the set of ancestors $U = \text{an}\{y_i, y_j, y_k, y_s\}$.

Remark: Theorem 27 provides an implicit relation for the transfer function Q_{ji} that needs to be solved. This relation is quadratic in the variable Q_{ji} providing, in general, two solutions. The following example shows that, at least in some situations, imposing the additional condition that Q_{ji} is a causal transfer function helps to determine Q_{ji} unequivocally.

Remark: The presence of the observable external signal y_s is not required, since, contrary to other identification methods, Theorem 27 can seamlessly deal with null transfer functions, as shown in the following example.

Example 7 (Closed-Loop Identification With No Knowledge of a Transfer Function): Consider again the loop network in Fig. 6 where all nodes are measured and the dynamics is known to be causal. The objective is to identify the transfer function Q_{21} . Note that the conditions for the application of Theorem 27 hold. The following can be determined from power spectral densities:

$$W_{y_3, [y_2] | (y_2, y_1)} = \frac{\beta \Phi_{e_1}}{\Phi_{e_3} \mathbf{z} \gamma^2 + \Phi_{e_1} \mathbf{z}};$$

$$W_{y_3, [y_1] | (y_2, y_1)} = \frac{\Phi_{e_3} \gamma}{\Phi_{e_3} \gamma^2 + \Phi_{e_1}};$$

$$W_{y_1, [y_3] | (y_3, y_2)} = \frac{\Phi_{e_2} \mathbf{z}^2 \gamma}{\Phi_{e_2} \mathbf{z}^2 + \alpha^2 \Phi_{e_1}};$$

$$W_{y_1, [y_2] | (y_3, y_2)} = \frac{\alpha \Phi_{e_1} \mathbf{z}}{\Phi_{e_2} \mathbf{z}^2 + \alpha^2 \Phi_{e_1}}.$$

and can thus be determined from measured time-series data. Equation (23) reduces to (27) described by

$$Q_{21} = [\alpha \mathbf{z} (\alpha \Phi_{e_3} Q_{21} \mathbf{z} \gamma^2 - \alpha^2 \Phi_{e_3} \gamma^2 - \Phi_{e_2} \mathbf{z}^2 + \alpha \Phi_{e_1} Q_{21} \mathbf{z} - \alpha^2 \Phi_{e_1})] / [\alpha \Phi_{e_3} Q_{21} \mathbf{z}^3 \gamma^2 - \alpha^2 \Phi_{e_3} \mathbf{z}^2 \gamma^2 + \alpha \beta^2 \Phi_{e_2} Q_{21} \mathbf{z} \gamma^2 - \alpha^2 \beta^2 \Phi_{e_2} \gamma^2 - \Phi_{e_2} \mathbf{z}^4 + \alpha \Phi_{e_1} Q_{21} \mathbf{z}^3 - \alpha^2 \Phi_{e_1} \mathbf{z}^2]. \quad (27)$$

Relation (27) is quadratic in $Q_{21}(z)$ and can be solved to yield two solutions described by

$$Q_{21} = \frac{\alpha}{\mathbf{z}}; \quad Q_{21} = \frac{\alpha^2 \Phi_{e_3} \mathbf{z} \gamma^2 + \Phi_{e_2} \mathbf{z}^3 + \alpha^2 \Phi_{e_1} \mathbf{z}}{(\alpha \Phi_{e_3} \mathbf{z}^2 + \alpha \beta^2 \Phi_{e_2}) \gamma^2 + \alpha \Phi_{e_1} \mathbf{z}^2}.$$

The second solution can be discarded since it is not causal.

VIII. CONCLUSION

This article builds bridges between tools from probabilistic graphical models and systems theory. As a first contribution, this article provides graphical criteria, based on d -separation, to determine the relevance of any set of nodes on the estimation of a specific node's activity. As a second contribution, this article lays a framework for the identification of a transfer function between a specified pair of network nodes in a network. Assuming a partially known topology of linear interactions between agents, the effectiveness of the methods rests on the ease of selecting a set of additional nodes whose activities, if used in a linear prediction, provide as a by-product a consistent estimate of the transfer between the specified pair of nodes. Again, the set of nodes to be measured for identifying a specified transfer function is achieved using graphical criteria based on d -separation. The results are applicable to topologies that include feedback loops, providing novel perspective on closed-loop identification techniques as well. Furthermore, this article provides insights into notions of estimation and identification that are robust with respect against uncertainties on the network topology.

APPENDIX

A. Proof of Lemma 25

The proof proceeds by manipulating the original LDIM to obtain an explicit expression for $W_{y_j|(y_i, y_z)}$. First, marginalize all nodes that are not ancestors of y_i , y_j , and y_z .

1) Marginalizing $y_U^c := y \setminus y_U$: Note that the set of vertices $y = y_U \cup y_U^c$. As y_U is an ancestor set, there can be no parents to any element of y_U in y_U^c . Thus, directed links from y_U^c to y_U are not possible, as depicted in Fig. 7(a). Note, from Lemma 15, that, as there are no links from y_U^c to y_U in the original LDIM $\mathcal{G}$, it is not possible to have a chain from y_U to y_U with all intermediate links in y_U^c . Then, the transfer matrix from the set y_U to the set y_U after marginalizing y_U^c remains unaltered and is given by $Q_{y_U y_U}$ (see property 3 of Lemma 15). Thus, after marginalizing the set y_U^c , the LDIM in Fig. 7(b) results.

The marginalization of the LDIM $\mathcal{G}$ with respect to y_U^c is complete. We further analyze the structure of the LDIM after marginalizing y_U^c . Let

$$y_I := y_U \setminus \{y_J \cup y_Z\}; \quad y_{\bar{I}} := y_I \setminus \{y_i\}.$$

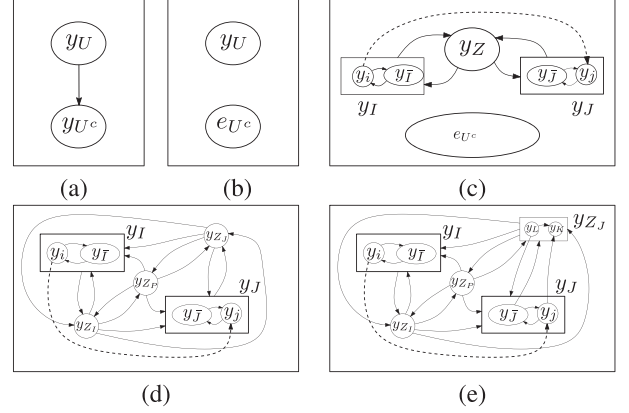


Fig. 7. Steps in the proof of Lemma 25 (a) partition of the nodes into y_U and y_U^c ; (b) marginalization of y_U^c ; (c) partition of y_U into y_I , y_J and y_Z ; (d) partition of y_Z into y_{Z_I} , y_{Z_J} and y_{Z_P} ; (e) partition of y_{Z_J} into y_{Z_K} and y_{Z_L} .

As $dsep_{\bar{G}}(y_i, y_Z, y_j)$ holds, it follows from Theorem 21 that $dsep_{\bar{G}}(y_I, y_Z, y_J)$ holds. Observe that $y_U = y_I \cup y_Z \cup y_J$ and that y_I , y_J , and y_Z are pairwise disjoint. Thus, y_I , y_J , and y_Z partition U . As in the graph $\bar{G}$, we have $dsep_{\bar{G}}(y_I, y_Z, y_J)$, links of the forms $y_k \rightarrow y_l$, $y_l \rightarrow y_k$, and $y_l \rightarrow y_c \leftarrow y_k$ for any $y_k \in y_I$, $y_l \in y_J$, and $y_c \in y_Z$ are not allowed in $\bar{G}$ (see Lemma 19). Thus, Fig. 7(c) follows where the set y_U is partitioned by y_I , y_J , and y_Z , and there are no direct links between y_I and y_J in $\bar{G}$. In Fig. 7(c), all solid links belong to $\bar{G}$ and the link $y_i \rightarrow y_j$ shown as a dashed link is added to recover the graphical structure of all links.

Define sets

$$\begin{aligned} y_{Z_J} &:= ch_{\bar{G}}(y_J) \cap y_Z \\ y_{Z_I} &:= ch_{\bar{G}}(y_I) \cap y_Z \\ y_{Z_P} &:= y_Z \setminus (y_{Z_J} \cup y_{Z_I}). \end{aligned}$$

Consider a link $y_\ell \rightarrow y_c$ where $y_\ell \in y_I$ and $y_c \in y_{Z_J}$ are present. This implies that there exists $y_p \in y_J$, which is a parent of y_c . Thus, the path $y_\ell \rightarrow y_c \leftarrow y_p$ exists with $y_\ell \in y_I$, $y_c \in y_{Z_J} \subset y_Z$, and $y_p \in y_J$. This is a contradiction as y_I , y_Z , and y_J partition y_U and $dsep_{\bar{G}}(y_I, y_Z, y_J)$ holds (see Lemma 19). Thus, there is no link from y_I to y_{Z_J} in $\bar{G}$. Moreover, from the definition of y_{Z_P} , there can be no link from y_I to y_{Z_P} in $\bar{G}$. For similar reasons, there can be no links from y_J to y_{Z_I} or to y_{Z_P} in $\bar{G}$. Furthermore, note that as $y_K = de(y_j) \cap y_{Z_I}$, it follows that $y_K \subset y_{Z_J}$. Let $y_L := y_{Z_J} \setminus y_K$. Suppose there is a link from $y_k \in y_K$ to $y_l \in y_L$. Then, $y_l \in y_J$ and $y_l \in de(y_j)$ implying $y_l \in y_K$. This is a contradiction as y_L and y_K are disjoint. Thus, there can be no link from y_K to y_L . For similar reasons, there can be no link from y_j to y_L .

Also, from Assumption A3, there are no links of the form $y_k \rightarrow y_j$ for $y_k \in y_K$. Moreover, y_K and y_L form partition of y_{Z_J} . Restricting attention to y_U and introducing the link between y_i and y_j (the dashed line), Fig. 7(d) follows. Also, note that none of the transfer functions are altered; thus the transfer matrix mapping y_U to y_U remains Q_{UU} . Thus, the sparsity pattern depicted by Fig. 7(d) along with the LDIM of Fig. 8.

$$\begin{pmatrix} y_i \\ y_{\bar{I}} \\ y_{Z_P} \\ y_{Z_I} \\ y_K \\ y_L \\ y_j \\ y_{\bar{J}} \end{pmatrix} = \begin{pmatrix} Q_{ii} & Q_{i\bar{I}} & Q_{iZ_P} & Q_{iZ_I} & Q_{iK} & Q_{iL} & 0 & 0 \\ Q_{\bar{I}i} & Q_{\bar{I}\bar{I}} & Q_{\bar{I}Z_P} & Q_{\bar{I}Z_I} & Q_{\bar{I}K} & Q_{\bar{I}L} & 0 & 0 \\ 0 & 0 & Q_{Z_P Z_P} & Q_{Z_P Z_I} & Q_{Z_P K} & Q_{Z_P L} & 0 & 0 \\ Q_{Z_I i} & Q_{Z_I \bar{I}} & Q_{Z_I Z_P} & Q_{Z_I Z_I} & Q_{Z_I K} & Q_{Z_I L} & 0 & 0 \\ 0 & 0 & Q_{K Z_P} & Q_{K Z_I} & Q_{KK} & Q_{KL} & Q_{Kj} & Q_{K\bar{J}} \\ 0 & 0 & Q_{L Z_P} & Q_{L Z_I} & 0 & Q_{LL} & 0 & Q_{L\bar{J}} \\ Q_{ji} & 0 & Q_{jZ_P} & Q_{jZ_I} & 0 & Q_{jL} & 0 & Q_{j\bar{J}} \\ 0 & 0 & Q_{\bar{J}Z_P} & Q_{\bar{J}Z_I} & Q_{\bar{J}K} & Q_{\bar{J}L} & Q_{\bar{J}j} & Q_{\bar{J}\bar{J}} \end{pmatrix} \begin{pmatrix} y_i \\ y_{\bar{I}} \\ y_{Z_P} \\ y_{Z_I} \\ y_K \\ y_L \\ y_j \\ y_{\bar{J}} \end{pmatrix} + \begin{pmatrix} e_i \\ e_{\bar{I}} \\ e_{Z_P} \\ e_{Z_I} \\ e_K \\ e_L \\ e_j \\ e_{\bar{J}} \end{pmatrix}.$$

Fig. 8. LDIM obtained after reaching the step represented in Fig. 7(d).

$$\begin{pmatrix} y_i \\ y_{\bar{I}} \\ y_{Z_P} \\ y_{Z_I} \\ y_K \\ y_L \\ y_j \\ e_{\bar{J}} \end{pmatrix} = \begin{pmatrix} Q_{ii} & Q_{i\bar{I}} & Q_{iZ_P} & Q_{iZ_I} & Q_{iK} & Q_{iL} & 0 & 0 \\ Q_{\bar{I}i} & Q_{\bar{I}\bar{I}} & Q_{\bar{I}Z_P} & Q_{\bar{I}Z_I} & Q_{\bar{I}K} & Q_{\bar{I}L} & 0 & 0 \\ 0 & 0 & Q_{Z_P Z_P} & Q_{Z_P Z_I} & Q_{Z_P K} & Q_{Z_P L} & 0 & 0 \\ Q_{Z_I i} & Q_{Z_I \bar{I}} & Q_{Z_I Z_P} & Q_{Z_I Z_I} & Q_{Z_I K} & Q_{Z_I L} & 0 & 0 \\ 0 & 0 & Q_{K Z_P} + Q_{KJ}Q_{\bar{J}Z_P}^{(e)} & Q_{K Z_I} + Q_{KJ}Q_{\bar{J}Z_I}^{(e)} & Q_{KK} + Q_{KJ}Q_{\bar{J}K}^{(e)} & Q_{KL} + Q_{KJ}Q_{\bar{J}L}^{(e)} & Q_{Kj} + Q_{KJ}Q_{\bar{J}j}^{(e)} & Q_{K\bar{J}}^{(e)} \\ 0 & 0 & Q_{L Z_P} + Q_{LJ}Q_{\bar{J}Z_P}^{(e)} & Q_{L Z_I} + Q_{LJ}Q_{\bar{J}Z_I}^{(e)} & 0 & Q_{LL} + Q_{LJ}Q_{\bar{J}L}^{(e)} & 0 & Q_{L\bar{J}}^{(e)} \\ Q_{ji} & 0 & Q_{jZ_P} + Q_{j\bar{J}}Q_{\bar{J}Z_P}^{(e)} & Q_{jZ_I} + Q_{j\bar{J}}Q_{\bar{J}Z_I}^{(e)} & 0 & Q_{jL} + Q_{j\bar{J}}Q_{\bar{J}L}^{(e)} & 0 & Q_{j\bar{J}}^{(e)} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_i \\ y_{\bar{I}} \\ y_{Z_P} \\ y_{Z_I} \\ y_K \\ y_L \\ y_j \\ e_{\bar{J}} \end{pmatrix} + \begin{pmatrix} e_i \\ e_{\bar{I}} \\ e_{Z_P} \\ e_{Z_I} \\ e_K \\ e_L \\ e_j \\ e_{\bar{J}} \end{pmatrix}$$

Fig. 9. LDIM after the marginalization of $y_{\bar{J}}$.

2) Marginalizing Set $y_{\bar{J}}$: Consider marginalization with respect to $y_{\bar{J}}$, given the LDIM in Fig. 7(d). Consider any two elements r and s in $U \setminus (y_{\bar{J}} \cup y_j \cup y_{Z_J}) = y_i \cup y_{\bar{I}} \cup y_{Z_P} \cup y_{Z_I}$. There is no chain of the form $r \rightarrow y_{\pi_1} \rightarrow y_{\pi_2} \rightarrow \dots \rightarrow y_{\pi_t} \rightarrow s$ with $y_{\pi_m} \in y_{\bar{J}}$ for $m = 1, \dots, t$. From property 3) of Lemma 15, it follows that the transfer matrix between $y_i \cup y_{\bar{I}} \cup y_{Z_P} \cup y_{Z_I}$ and itself remain unchanged in the LDIM resulting after marginalizing $y_{\bar{J}}$. Also, there is a no chain of the form $y_{\pi_1} \rightarrow \dots \rightarrow y_{\pi_\ell} \rightarrow s$ with $y_{\pi_m} \in y_{\bar{J}}$ for any $s \in y_i \cup y_{\bar{I}} \cup y_{Z_P} \cup y_{Z_I}$. It follows from property 2) of Lemma 15 that the transfer matrix from $y_{\bar{J}}$ to any $s \in y_i \cup y_{\bar{I}} \cup y_{Z_P} \cup y_{Z_I}$ remains unchanged and, thus, remains zero. Moreover, for any $s \in y_i \cup y_{\bar{I}} \cup y_{Z_P} \cup y_{Z_I}$, there is no chain of the form $y_j \rightarrow y_{\pi_1} \rightarrow \dots \rightarrow y_{\pi_\ell} \rightarrow s$ with $y_{\pi_m} \in y_{\bar{J}}$. Thus, all transfer matrices mapping y_j into y_s remain unchanged. Similarly, there are no chains contained in $y_{\bar{J}}$ that connect y_i to y_{Z_J} , $y_{\bar{I}}$ to y_{Z_J} , y_i to y_j , and $y_{\bar{I}}$ to y_j and, thus, all corresponding transfer functions matrices remain unchanged.

Assumption A2 precludes a chain connecting y_j to y_j with all intermediate nodes in $y_{\bar{J}}$; thus the map from j to j remains unchanged and is zero (from Assumption A2, the original LDIM does not have a self-loop at j and this $Q_{jj} = 0$). Also, from Assumption A3, there is no chain connecting any element of y_K to y_j , which is contained in $\bar{J}$. Thus, the transfer matrix from y_K to y_j remains at zero after marginalizing $y_{\bar{J}}$.

After marginalizing $y_{\bar{J}}$, the LDIM is described by the equation in Fig. 9.

where

$$\begin{aligned} Q_{\bar{J}m}^{(e)} &= (\mathcal{I} - Q_{\bar{J}\bar{J}})^{-1} Q_{\bar{J}m} \\ Q_{m\bar{J}}^{(e)} &= Q_{m\bar{J}} (\mathcal{I} - Q_{\bar{J}\bar{J}})^{-1}. \end{aligned} \quad (28)$$

This completes the marginalization with respect to the set $y_{\bar{J}}$.

Removing self-loops at y_K and y_L and subsuming the effect of $e_{\bar{J}}$ into corresponding noise terms, we have the relation described in Fig. 10 where the following holds.

- 1) $Q_{K\alpha}^{(f)} = [\mathcal{I} - (Q_{KK} + Q_{KJ}Q_{\bar{J}K}^{(e)})]^{-1} [Q_{K\alpha} + Q_{KJ}Q_{\bar{J}\alpha}^{(e)}]$ for $\alpha \in \{Z_P, Z_I, L, j\}$, $Q_{K\bar{J}}^{(f)} = [\mathcal{I} - (Q_{KK} + Q_{KJ}Q_{\bar{J}K}^{(e)})]^{-1} Q_{K\bar{J}}^{(e)}$, with $e_K^{(f)} = [\mathcal{I} - (Q_{KK} + Q_{KJ}Q_{\bar{J}K}^{(e)})]^{-1} e_K$ and $\epsilon_K = e_K^{(f)} + Q_{KJ}Q_{\bar{J}j}^{(e)}$.
- 2) $Q_{L\alpha}^{(f)} = [\mathcal{I} - (Q_{LL} + Q_{LJ}Q_{\bar{J}L}^{(e)})]^{-1} [Q_{L\alpha} + Q_{LJ}Q_{\bar{J}\alpha}^{(e)}]$ for $\alpha \in \{Z_P, Z_I\}$, $Q_{L\bar{J}}^{(f)} = [\mathcal{I} - (Q_{LL} + Q_{LJ}Q_{\bar{J}L}^{(e)})]^{-1} Q_{L\bar{J}}^{(e)}$, with $e_L^{(f)} = [\mathcal{I} - (Q_{LL} + Q_{LJ}Q_{\bar{J}L}^{(e)})]^{-1} e_L$, and $\epsilon_L = e_L^{(f)} + Q_{LJ}Q_{\bar{J}j}^{(e)}$.
- 3) $Q_{j\alpha}^{(f)} = Q_{j\alpha} + Q_{j\bar{J}}Q_{\bar{J}\alpha}^{(e)}$ for $\alpha \in \{Z_P, Z_I, L\}$ and $Q_{j\bar{J}}^{(f)} = Q_{j\bar{J}}^{(e)}$, and $\epsilon_j = e_j + Q_{j\bar{J}}Q_{\bar{J}j}^{(e)}$.

Note that (29) is not in the LDIM form because the noise processes are no longer independent. The relation of ϵ_j from the last row of (29) can be substituted in the row corresponding y_K to eliminate ϵ_j from the row to obtain the relation in Fig. 11.

Let us compute the Wiener estimate $\hat{y}_j$ of y_j using the processes $\{y_i\} \cup y_Z$. We have that

$$\begin{aligned} \hat{y}_j &:= \arg \min_{q \in Q} \|y_j - q\| \\ \text{s.t. } Q &= \text{tfspan}\{y_i, y_Z\}. \end{aligned}$$

Since $y_{\bar{I}}$ is d -separated from y_j by $y_Z \cup \{y_i\}$ [see Fig. 7(d)], it follows from Theorem 24 that $y_Z \cup \{y_i\}$ Wiener separates $y_{\bar{I}}$ and y_j , we have that

$$\begin{aligned} \hat{y}_j &= \arg \min_{q \in Q} \|y_j - q\| \\ \text{s.t. } Q &= \text{tfspan}\{y_i, y_{\bar{I}}, y_Z\}. \end{aligned}$$

$$\begin{pmatrix} y_i \\ y_{\bar{I}} \\ y_{Z_P} \\ y_{Z_I} \\ y_K \\ y_L \\ y_j \end{pmatrix} = \begin{pmatrix} Q_{ii} & Q_{i\bar{I}} & Q_{iZ_P} & Q_{iZ_I} & Q_{iK} & Q_{iL} & 0 \\ Q_{\bar{I},i} & Q_{\bar{I}\bar{I}} & Q_{\bar{I}Z_P} & Q_{\bar{I}Z_I} & Q_{\bar{I}K} & Q_{\bar{I}L} & 0 \\ 0 & 0 & Q_{Z_P Z_P} & Q_{Z_P Z_I} & Q_{Z_P K} & Q_{Z_P L} & 0 \\ Q_{Z_I i} & Q_{Z_I \bar{I}} & Q_{Z_I Z_P} & Q_{Z_I Z_I} & Q_{Z_I K} & Q_{Z_I L} & 0 \\ 0 & 0 & Q_{K Z_P}^{(f)} & Q_{K Z_I}^{(f)} & 0 & Q_{KL}^{(f)} & Q_{Kj}^{(f)} \\ 0 & 0 & Q_{L Z_P}^{(f)} & Q_{L Z_I}^{(f)} & 0 & 0 & 0 \\ Q_{ji} & 0 & Q_{jZ_P}^{(f)} & Q_{jZ_I}^{(f)} & 0 & Q_{jL}^{(f)} & 0 \end{pmatrix} \begin{pmatrix} y_i \\ y_{\bar{I}} \\ y_{Z_P} \\ y_{Z_I} \\ y_K \\ y_L \\ y_j \end{pmatrix} + \begin{pmatrix} e_i \\ e_{\bar{I}} \\ e_{Z_P} \\ e_{Z_I} \\ \epsilon_K - Q_{Kj}^{(f)} \epsilon_j \\ \epsilon_L \\ \epsilon_j \end{pmatrix}. \quad (29)$$

Fig. 10. Relation obtained after removing self-loops at y_K and y_L and subsuming the effect of $e_{\bar{J}}$ into corresponding noise terms. This relation does not represent an LDIM because the noise vector does not have a block diagonal power spectral density.

$$\begin{pmatrix} y_i \\ y_{\bar{I}} \\ y_{Z_P} \\ y_{Z_I} \\ y_K \\ y_L \\ y_j \end{pmatrix} = \begin{pmatrix} Q_{ii} & Q_{i\bar{I}} & Q_{iZ_P} & Q_{iZ_I} & Q_{iK} & Q_{iL} & 0 \\ Q_{\bar{I},i} & Q_{\bar{I}\bar{I}} & Q_{\bar{I}Z_P} & Q_{\bar{I}Z_I} & Q_{\bar{I}K} & Q_{\bar{I}L} & 0 \\ 0 & 0 & Q_{Z_P Z_P} & Q_{Z_P Z_I} & Q_{Z_P K} & Q_{Z_P L} & 0 \\ Q_{Z_I i} & Q_{Z_I \bar{I}} & Q_{Z_I Z_P} & Q_{Z_I Z_I} & Q_{Z_I K} & Q_{Z_I L} & 0 \\ Q_{Kj}^{(f)} Q_{ji} & 0 & Q_{K Z_P}^{(f)} + Q_{Kj}^{(f)} Q_{jZ_P}^{(f)} & Q_{K Z_I}^{(f)} + Q_{Kj}^{(f)} Q_{jZ_I}^{(f)} & 0 & Q_{KL}^{(f)} + Q_{Kj}^{(f)} Q_{jL}^{(f)} & 0 \\ 0 & 0 & Q_{L Z_P}^{(f)} & Q_{L Z_I}^{(f)} & 0 & 0 & 0 \\ Q_{ji} & 0 & Q_{jZ_P}^{(f)} & Q_{jZ_I}^{(f)} & 0 & Q_{jL}^{(f)} & 0 \end{pmatrix} \begin{pmatrix} y_i \\ y_{\bar{I}} \\ y_{Z_P} \\ y_{Z_I} \\ y_K \\ y_L \\ y_j \end{pmatrix} + \begin{pmatrix} e_i \\ e_{\bar{I}} \\ e_{Z_P} \\ e_{Z_I} \\ \epsilon_K \\ \epsilon_L \\ \epsilon_j \end{pmatrix}. \quad (30)$$

Fig. 11. Relation obtained after substituting y_j from the last row of (29) in the row corresponding y_K .

Using (30), we have

$$\begin{aligned} \hat{y}_j &= Q_{ji} y_i + Q_{jZ_P}^{(f)} y_{Z_P} + Q_{jZ_I}^{(f)} y_{Z_I} \\ &\quad + Q_{jL}^{(f)} y_L + \arg \min_{q \in Q} \|\epsilon_j - q\| \\ \text{s.t. } Q &= \text{tfspan}\{y_i, y_{\bar{I}}, y_{Z_P}, y_{Z_I}, y_L, y_K\}. \end{aligned} \quad (31)$$

From (30), we have that $\text{tfspan}\{y_i, y_{\bar{I}}, y_{Z_P}, y_{Z_I}, y_L, y_K\} = \text{tfspan}\{e_i, e_{\bar{I}}, e_{Z_P}, e_{Z_I}, \epsilon_K, \epsilon_L\}$. We also note that processes $\{e_i, e_{\bar{I}}, e_{Z_P}, e_{Z_I}\}$ are independent of $\epsilon_j = e_j + Q_{j\bar{J}}^{(f)} e_{\bar{J}}$. Thus, (31) leads to

$$\begin{aligned} \hat{y}_j &= Q_{ji} y_i + Q_{jZ_P}^{(f)} y_{Z_P} + Q_{jZ_I}^{(f)} y_{Z_I} + Q_{jL}^{(f)} y_L \\ &\quad + \arg \min_{q \in Q} \|\epsilon_j - q\| \\ \text{s.t. } Q &= \text{tfspan}\{\epsilon_K, \epsilon_L\}. \end{aligned}$$

Thus

$$\begin{aligned} \hat{y}_j &= Q_{ji} y_i + Q_{jZ_P}^{(f)} y_{Z_P} + Q_{jZ_I}^{(f)} y_{Z_I} + Q_{jL}^{(f)} y_L \\ &\quad + W_{\epsilon_j, [\epsilon_K]}(\epsilon_K, \epsilon_L) \epsilon_K + W_{\epsilon_j, [\epsilon_L]}(\epsilon_K, \epsilon_L) \epsilon_L. \end{aligned} \quad (32)$$

Note from (30) that

$$\begin{aligned} \epsilon_K &= y_K - Q_{Kj}^{(f)} Q_{ji} y_i - [Q_{KZ_P}^{(f)} + Q_{Kj}^{(f)} Q_{jZ_P}^{(f)}] y_{Z_P} \\ &\quad + [Q_{KZ_I}^{(f)} + Q_{Kj}^{(f)} Q_{jZ_I}^{(f)}] y_{Z_I} - [Q_{KL}^{(f)} + Q_{Kj}^{(f)} Q_{jL}^{(f)}] y_L \\ \epsilon_L &= y_L - Q_{LZ_P}^{(f)} y_{Z_P} - Q_{LZ_I}^{(f)} y_{Z_I}. \end{aligned} \quad (33)$$

Substituting, (33) in (32), it follows that

$$\begin{aligned} \hat{y}_j &= [\mathcal{I} - W_{\epsilon_j, [\epsilon_K]}(\epsilon_K, \epsilon_L) Q_{Kj}^{(f)}] Q_{ji} y_i \\ &\quad + [W_{\epsilon_j, [\epsilon_K]}(\epsilon_K, \epsilon_L)] y_K \\ &\quad + [Q_{jZ_P}^{(f)} - W_{\epsilon_j, [\epsilon_K]}(\epsilon_K, \epsilon_L) (Q_{KZ_P}^{(f)} + Q_{Kj}^{(f)} Q_{jZ_P}^{(f)}) \\ &\quad - W_{\epsilon_j, [\epsilon_L]}(\epsilon_K, \epsilon_L) Q_{LZ_P}^{(f)}] y_{Z_P} \end{aligned}$$

$$\begin{aligned} &+ [Q_{jZ_I}^{(f)} - W_{\epsilon_j, [\epsilon_L]}(\epsilon_K, \epsilon_L) (Q_{KZ_I}^{(f)} + Q_{Kj}^{(f)} Q_{jZ_I}^{(f)}) \\ &\quad - W_{\epsilon_j, [\epsilon_L]}(\epsilon_K, \epsilon_L) Q_{LZ_I}^{(f)}] y_{Z_I} \\ &+ [Q_{jL}^{(f)} - W_{\epsilon_j, [\epsilon_K]}(\epsilon_K, \epsilon_L) (Q_{KL}^{(f)} + Q_{Kj}^{(f)} Q_{jL}^{(f)}) \\ &\quad + W_{\epsilon_j, [\epsilon_L]}(\epsilon_K, \epsilon_L)] y_L. \end{aligned} \quad (34)$$

Note that if $v = (e_i^T \ e_{\bar{I}}^T \ e_{Z_P}^T \ e_{Z_I}^T \ e_K^T \ e_L^T)^T$, then the power spectral density matrix $\Phi_{vv}(e^{j\omega})$ is assumed positive definite for every $\omega \in R$. From the uniqueness of the Wiener filter of Proposition 10, we get

$$\begin{aligned} W_{y_j, [y_K]}(y_i, y_{Z_P}, y_{\bar{I}}) &= W_{\epsilon_j, [\epsilon_K]}(\epsilon_K, \epsilon_L) \text{ and} \\ W_{y_j, [y_i]}(y_i, y_{Z_P}, y_{\bar{I}}) &= [\mathcal{I} - W_{\epsilon_j, [\epsilon_K]}(\epsilon_K, \epsilon_L) Q_{Kj}^{(f)}] Q_{ji}. \end{aligned}$$

Thus, $Q_{ji} = [\mathcal{I} - W_{\epsilon_j, [\epsilon_K]}(\epsilon_K, \epsilon_L) Q_{Kj}^{(f)}]^{-1} W_{y_j, [y_i]}(y_i, y_{Z_P}, y_{\bar{I}})$ leads to $Q_{ji} = \mathcal{I} - W_{y_j, [y_K]}(y_i, y_{Z_P}, y_{\bar{I}}) Q_{Kj}^{(f)}$ because of $W_{y_j, [y_i]}(y_i, y_{Z_P}, y_{\bar{I}}) = W_{y_j, [y_i, y_{Z_P}]}$ and eventually to $Q_{ji} = [\mathcal{I} - W_{y_j, [y_K]}(y_i, y_{Z_P}) Q_{Kj}^{(f)}]^{-1} W_{y_j, [y_i]}(y_i, y_{Z_P})$ where we have established that

$$Q_{ji} = [\mathcal{I} - W_{y_j, [y_K]}(y_i, y_{Z_P}) Q_{Kj}^{(f)}]^{-1} W_{y_j, [y_i]}(y_i, y_{Z_P})$$

where

$$Q_{Kj}^{(f)} = [\mathcal{I} - (Q_{KK} + Q_{K\bar{J}} Q_{\bar{J}K}^{(e)})]^{-1} [Q_{Kj} + Q_{K\bar{J}} Q_{\bar{J}j}^{(e)}]$$

with

$$Q_{\bar{J}K}^{(e)} = (\mathcal{I} - Q_{\bar{J}\bar{J}})^{-1} Q_{\bar{J}K}; \text{ and } Q_{\bar{J}j}^{(e)} = (\mathcal{I} - Q_{\bar{J}\bar{J}})^{-1} Q_{\bar{J}j}.$$

This completes the proof. The case for $y_K = \emptyset$ is analogous.

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