

Exact Topology Learning in a Network of Cyclostationary Processes

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Abstract—Learning the structure of a network from time-series data, in particular cyclostationary data, is of significant interest in many disciplines such as power grids, biology, finance. In this article, an algorithm is presented for reconstruction of the topology of a network of cyclostationary processes. To the best of our knowledge, this is the first work to guarantee exact recovery without any assumptions on the underlying structure. The method is based on a lifting technique by which cyclostationary processes are mapped to vector wide sense stationary processes and further on semi-definite properties of matrix Wiener filters for the said processes. We demonstrate the performance of the proposed algorithm on a Resistor-Capacitor network and present the accuracy of reconstruction for varying sample sizes.

I. INTRODUCTION

Networks are a framework used extensively for analysis of the behavior of complex systems like the brain [1], climate [2], gene regulation dynamics [3], disease spread [4], power grid [5] and others. Network representations help identify essential influence pathways in a complex system, thereby enhance the understanding of the dynamics of complex systems. A problem of interest to multiple communities like control theory, machine learning and signal processing is the inference of the influence pathways among the entities of interest from observation of the entities, which is sometimes referred as structure learning or topology learning [6], [7], [8]. For example: given a time series collection of stock prices over a time horizon, it is of interest to infer the influence paths among the collection of stocks from the observed time series of stock prices.

Fundamentally, there exist two different approaches for inference of the influence pathways among the entities from observations. The first is active learning, where a particular entity is perturbed by an external agent and its influence on the rest of the entities is examined [9]. However, the disadvantage of such an approach is that it is not always possible to actively excite a specific entity to glean the network structure. For example: it is not always possible to turn off or change generator set points to infer the structure of a power distribution network. The second approach to structure learning involves passive or non invasive approaches, where, the system is not perturbed actively and topology is inferred solely from the observations of the variables of

interest [10],[11], [12],[13]. In this article we focus on the passive approach to topology inference.

Inference of the network structure from observations assuming that the entities are a collection of random variables is studied in [10], [13]. However, such approaches are ineffective when lagged correlations are significant and the past of one entity influences the present of another entity. In such situations, the observed entities are modeled as stochastic processes or time series. In the time series setting, topology inference under the assumption of wide sense stationary time series using multivariate Wiener filtering is described in [14] and using power spectral density in [15]. These approaches are not applicable to the case of non stationary time series. In this article we utilize multivariate Wiener filtering for topology inference with consistency guarantees for non stationary time series, which are cyclostationary. Cyclostationary processes are characterized by a periodic mean and correlation function. Many phenomenon in nature exhibit periodic behaviour [16]; examples include weather patterns [17][2], regulatory processes in human body [18], [19], [20], trends in stock market [21], [22], motion of mechanical systems [23], [24], communication systems [25], [26], and planet movements [27]. In [28], the authors use multivariate Wiener filtering for topology inference in a collection of cyclostationary time series. However, the approach presented in [28] introduces possibly many spurious links and does not provide exact inference of the underlying network topology from the observations. In this article, we extend the work in [28] and present an algorithm with guarantees for *exact* reconstruction of a network structure in a collection of cyclostationary processes. Our approach does not require any structural assumptions on the underlying graph structure nor any knowledge of the system parameters. Our results are based on the properties of the Wiener filters of collection of wide sense stationary time series for inference of exact topology in a network of wide sense stationary processes [29], [30]. We validate the algorithm presented through implementation on time series of nodal states generated from an interconnected Resistor-Capacitor (RC) network with input cyclostationary process. We compare the performance of our algorithm against the algorithms presented in [28] and show the superiority of the approach presented here over prior work. To the authors' best knowledge, this is the first work on exact topology reconstruction of cyclostationary processes with provable guarantees.

The rest of the paper is organized as follows. Section II describes the learning problem of a general linear dynamical system with cyclostationary inputs. In section III, the approach for tackling the topology learning problem from cyclostationary time series data is presented. Analytic results

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as well as an implementable algorithm is presented. The performance of the algorithm is demonstrated on a simulated RC network in Section IV. Conclusions are presented in Section V.

Notation:

The symbol $:=$ denotes a definition

$\|x\|$: 2 norm of a vector x

$[x]_i$: i -th element of a vector x

$x(k)$: k^{th} sample of time domain signal x

$\times(z)$: z-transform of $x(k)$

$[A]_{j,:}$: j -th row of matrix A

$[A]_{:,i}$: i -th column of matrix A

$[A]_{i,j}$: i -th row, j -th column element of matrix A

$\mathbf{0}$: zero matrix of appropriate dimension

A^* : the conjugate transpose of matrix A

A' : transpose of a matrix A

$\|A\|_\infty$: largest row sum of matrix A

$A \succ 0$: A is a positive definite matrix

$A \prec 0$: A is a negative definite matrix

II. LINEAR DYNAMICAL SYSTEM WITH CYCLOSTATIONARY INPUTS

In this section, we briefly define and describe a networked cyclostationary process.

A. Introduction to Cyclostationary Process

A random process $x(t) \in L^2(\Omega, \mathcal{F}, \mathcal{P})$, with its mean function $m(t) := \mathbb{E}[x(t)]$ and correlation function $R_x(s, t) := \mathbb{E}[x(s)x(t)]$ is wide sense cyclostationary (WSCS) or periodically correlated with period $T > 0$ if for every $s, t \in \mathbb{Z}$ there is no value smaller than T such that $m(t) = \mathbb{E}[x(t)] = \mathbb{E}[x(t+T)] = m(t+T)$ and $R_x(s, t) = \mathbb{E}[x(s)x(t)] = \mathbb{E}[x(s+T)x(t+T)] = R_x(s+T, t+T)$. The random processes $x(t)$ and $e(t)$ are said to be **jointly wide sense cyclostationary (JWSCS)** if $x(t)$ and $e(t)$ are cyclostationary with period T and the cross correlation function $R_{x,e}(s, t)$ is periodic with period T that is $R_{x,e}(s, t) = R_{x,e}(s+T, t+T)$. An example of a cyclostationary process is a periodic signal superimposed with a **wide sense stationary (WSS)** signal. A WSS signal by itself is also a cyclostationary signal with period $T = 1$.

B. Network of Cyclostationary Processes

Consider a system with m sub-systems indexed by i with the dynamics of each sub-system x_i given by

$$\sum_{n=1}^l a_{n,i} \frac{d^n x_i}{dt^n} = \sum_{j=1, j \neq i}^m b_{ij}(x_j(t) - x_i(t)) + p_i(t), \quad (1)$$

$i \in \{1, 2, \dots, m\}$, where $x_i(t)$ is the time series data of the sub-system i ; $a_{m,i}$ and b_{ij} are system parameters and $p_i(t)$ is a forcing function acting on sub-system i , which is a zero mean WSCS of period T , uncorrelated with $\{p_j(t)\}_{i \neq j}$. Since a linear transformation of $\{p_j(k)\}_{j=1}^m$ results in $\{x_j(k)\}_{j=1}^m$, the collection of $(\{x_j(k)\}_{j=1}^m, \{p_j(k)\}_{j=1}^m)$ are JWSCS [28]. All the processes and the system parameters of (1) are real valued and b_{ij} are non-negative. Equation (1) is known as Linear Dynamical Model (LDM).

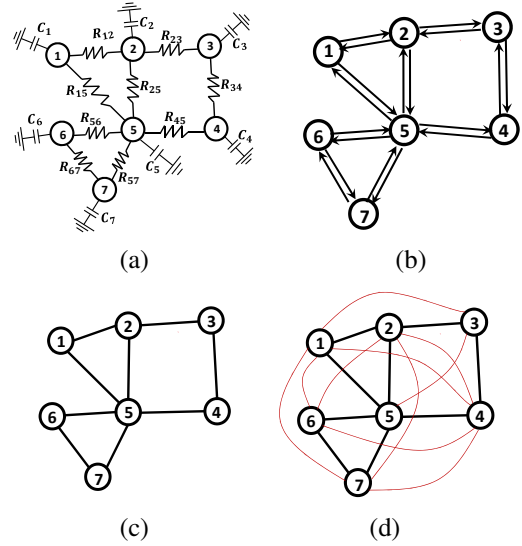


Fig. 1. (a) RC network, (b) directed graph, (c) it's topology and (d) kin topology/moral graph.

Converting (1) from time domain to z-domain by applying bilinear transform (Tustin's method [31]) on (1), we obtain

$$\begin{aligned} S_i(z)x_i(z) &= \sum_{j=1, j \neq i}^m b_{ij}x_j(z) + p_i(z), \\ \Rightarrow x_i(z) &= \sum_{j=1}^m H_{ij}(z)x_j(z) + e_i(z) \end{aligned} \quad (2)$$

where, $H_{ii}(z) = 0$, $S_i(z) = \sum_{n=1}^l a_{n,i} (\frac{2(1-z^{-1})}{\Delta t(1+z^{-1})})^n + \sum_{j=1, j \neq i}^m b_{ij}$, $H_{ij}(z) = \frac{b_{ij}}{S_i(z)}$, $e_i(z) = \frac{p_i(z)}{S_i(z)}$. $x_i(z)$, $p_i(z)$, $e_i(z)$ are the z-transforms of $x_i(k)$, $p_i(k)$ and $e_i(k)$ respectively. $e_j(k)$ is the input noise which is zero mean WSCS and uncorrelated with $\{e_i\}_{i=1, i \neq j}^m$.

The Linear Dynamic Graph (LDG) given by $\mathcal{G}(V, E)$ associated with the LDM described by (1) is obtained by setting each vertex (node) $i \in V$ to represent every subsystem i and by placing a directed edge in E from vertex i to vertex j if $b_{ji} \neq 0$. The set of children of node x_j is defined as $\mathcal{C}_G(x_j) := \{x_i | (x_j, x_i) \in E\}$, its set of parents as $\mathcal{P}_G(x_j) := \{x_i | (x_i, x_j) \in E\}$ and its set of kins as $\mathcal{K}_G(x_j) := \{x_i | x_i \neq x_j \text{ and } x_i \in \mathcal{C}_G(x_j) \cup \mathcal{P}_G(x_j) \cup \mathcal{P}_G(\mathcal{C}_G(x_j))\}$. The kin topology/moral graph of the directed graph $\mathcal{G}(V, E)$ is defined as an undirected graph $\mathcal{G}_M(V, \tilde{E})$ with a vertex set V and an edge set $\tilde{E} = \{(x_i, x_j) | x_i \in \mathcal{K}_G(x_j) \forall j\}$. The topology of the directed graph $\mathcal{G}(V, E)$ is defined as an undirected graph $\mathcal{G}_T(V, \tilde{E})$ with a vertex set V and an edge set $\tilde{E} = \{(x_i, x_j) | (x_i, x_j) \text{ or } (x_j, x_i) \in E\}$ and is denoted by $top(\mathcal{G}) = \mathcal{G}_T$. Example of topology and kin topology are shown in Fig. 1(c) and (d) respectively. In the topology $\mathcal{G}_T(V, \tilde{E})$ of the directed graph $\mathcal{G}$, a neighbor is defined as $\mathcal{N}_{\mathcal{G}_T}(x_j) := \{x_i | (x_i, x_j) \in \tilde{E}\}$ and a two hop neighbor is defined as $\mathcal{N}_{\mathcal{G}_T}(x_j, 2) := \{x_i \in V | (x_j, x_k), (x_i, x_k) \in \tilde{E} \text{ for some } x_k \in V\}$.

C. Lifting to vector wide sense stationary processes

Any cyclostationary process with a period T can be represented as multivariate, vector, wide sense stationary process through the *lifting* process [32] described next. Processes $x_i(k)$, $e_i(k)$ and $p_i(k)$ are lifted to T -variate WSS processes $X_i(k) = [x_i(k) \dots x_i(k+T-1)]'$, $E_i(k) = [e_i(k) \dots e_i(k+T-1)]'$ and $P_i(k) = [p_i(k) \dots p_i(k+T-1)]'$ respectively. The z -transform's of $X_i(k)$, $E_i(k)$ and $P_i(k)$ are denoted by $X_i(z)$, $E_i(z)$ and $P_i(z)$ respectively. The vector process $E_i(k)$ is uncorrelated with $\{E_j(k)\}_{j \neq i}$. The dynamics of $\{X_j\}_{j=1}^m$ is given by

$$\begin{aligned} X_j(z) &= \sum_{i=1}^m \mathcal{H}_{ji}(z) X_i(z) + E_j(z), \text{ where,} \\ X_j(z) &= [1 \ z \ \dots \ z^{T-1}]' x_j(z), \\ E_j(z) &= [1 \ z \ \dots \ z^{T-1}]' e_j(z), \\ \mathcal{H}_{ji}(z) &= \mathcal{I}_{T \times T} H_{ji}(z), \end{aligned} \quad (3)$$

with $\mathcal{H}_{ji}$ is a $T \times T$ transfer matrix.

The $X_i(k)$ process is said to be (weakly) stationary if $E([X_i(k)]_j) = m_i(j)$, where $[X_i(k)]_j$ is the j^{th} element of $X_i(k)$ and $R_{X_i}^{jk}(s, t) = E([X_i(s)]_j [X_i(t)]_k) = E(x_i(s+j-1)x_i(t+k-1)) = E(x_i(s-t+j)x_i(k)) = R_{X_i}^{jk}(s-t)$ for all $s, t \in \mathbb{Z}$, and $j, k \in \{1, 2, \dots, T\}$. The power spectral density matrix $\Phi_{P_i}(z), \Phi_{E_i}(z)$ are block diagonal matrices. Note that $e_i(z) = \frac{p_i(z)}{S_i(z)}$ and therefore $P_i(z) = S_i(z)E_i(z)$. Thus,

$$\begin{aligned} \Phi_{P_i}(z) &= S_i(z) \Phi_{E_i}(z) (S_i(z))^* \\ \Rightarrow \Phi_{E_i}^{-1}(z) &= \Phi_{P_i}^{-1}(z) |S_i(z)|^2 \end{aligned} \quad (5)$$

Lemma II.1. *A directed edge from a cyclostationary process x_i to x_j in the LDG of (1) exists if and only if there exists a directed edge from X_i to X_j*

Proof: Given that a directed edge exists from cyclostationary process x_i to x_j then $H_{ji} \neq 0$ in (2). It follows from (4) that $\mathcal{H}_{ji}(z) \neq 0$ which implies that there is a directed edge from vector stationary process X_i to X_j . To prove the converse, if suppose there is a directed edge from X_i to X_j then $\mathcal{H}_{ji}(z) \neq 0$ in (3). This implies that $H_{ji} \neq 0$ from (4) and there is a directed edge from x_i to x_j .

Remark 1. *The above lemma concludes that kin relationship's in the LDG of $\{x_j\}_{j=1}^m$ identical to the kin relationship's in the LDG of $\{X_j\}_{j=1}^m$. This enables us to reconstruct the topology of LDG for the cyclostationary processes $\{x_j\}_{j=1}^m$ by reconstructing the topology for their equivalent vector stationary processes $\{X_j\}_{j=1}^m$.*

The rest of the article focuses on the reconstruction of topology of a dynamical network of $T \times 1$ vector stationary processes that are jointly wide sense stationary (in contrast to x_i which are jointly wide sense cyclostationary).

III. LEARNING THE STRUCTURE FROM TIME SERIES DATA

Here we assume that $x_i(t)$ is available as a measured time series for all subsystems i . Given the time series data of the dynamical system with noise modeled as cyclostationary, the main aim of this article is to reconstruct the topology of the

LDG of the system. The output processes $\{X_j\}_{j=1}^m$ of (3) are defined in a compact way as:

$$X(z) = \mathbb{H}(z)X(z) + E(z), \quad (6)$$

where, $X(z) = [X_1'(z) \ \dots \ X_m'(z)]'$, $E(z) = [E_1'(z) \ \dots \ E_m'(z)]'$. The transfer function block matrix $\mathbb{H}(z)$ is $mT \times mT$ matrix with size of each block $T \times T$ and its diagonal entries are $0_{T \times T}$. The matrix $\mathbb{H}(z)$ in (6) characterizes the interconnection between the processes $X_i(k)$ and $X_j(k)$ with the $(i, j)^{th}$ block of $\mathbb{H}(z)$ being $\mathcal{H}_{ij}(z)$. The relation (6) defines a map from a vector processes E to a vector processes X . The LDM described by (6) is well-posed if the operator $(\mathbb{I} - \mathbb{H}(z))$ is invertible almost surely. Therefore for any vector processes E there exists a vector X of processes and the LDM described by (6) is said to be topologically detectable if $\Phi_{E_j}(e^{j\omega}) \succ 0$ for any $\omega \in [-\pi, \pi]$ and for any $j = 1, \dots, m$.

Consider the following least square optimization problem on the Hilbert space of $\mathcal{L}_2$ random variables,

$$\inf_{\{W_{ji}\}_{i=1, \dots, m, i \neq j}} \mathbb{E}(X_j(k) - \sum_{i=1, i \neq j}^m (W_{ji} * X_i)(k))^2, \quad (7)$$

where, W_{ji} is a $T \times T$ impulse response matrix filtering $X_i(k)$. In z -domain, (7) can be formulated as

$$\min_{\{W_{ji}\}_{i=1, \dots, m, i \neq j}} \|X_j(z) - \sum_{i=1, i \neq j}^m W_{ji}(z) X_i(z)\|^2. \quad (8)$$

where, $W_{ji}(z)$ is the z -transform of W_{ji} . The solution to (7) can be formulated as T independent optimization problems. For each $q \in \{1, 2, \dots, T\}$, consider the q^{th} row of (7):

$$\begin{aligned} \inf_{\{[W_{ji}]_{q,:}\}_{i=1, \dots, m, i \neq j}} \mathbb{E}(x_j(k+q-1) - \\ \sum_{i=1, i \neq j}^m ([W_{ji}]_{q,:} * X_i)(k))^2. \end{aligned} \quad (9)$$

where $[W_{ji}]_{q,:}$ is the q^{th} row of W_{ji} matrix. Let $X_j := [X_1' \dots X_{j-1}' X_{j+1}' \dots X_m']'$ and $X_{j,q} = [X_j]_q$. The solution for (9) is the *non-causal multivariate Wiener filter* [33] given by:

$$\begin{aligned} [W_j]_{q,:}(z) &= \Phi_{X_{j,q} X_j}(z) \Phi_{X_j}^{-1}(z) \\ &= [[W_{j1}]_{q,:}(z) \ \dots \ [W_{j(j-1)}]_{q,:}(z) \ [W_{j(j+1)}]_{q,:}(z) \ \dots \ [W_{jm}]_{q,:}(z)]. \end{aligned} \quad (10)$$

where $[W_{ji}]_{q,:}(z)$ is a $1 \times T$ matrix/ row vector, such that

$$[W_{ji}]_{q,p}(z) = \sum_{L=-\infty}^{\infty} h_{ji,q,p}^L z^{-L}. \quad (11)$$

where $h_{ji,q,p}$ is the filter coefficient. The proposed algorithm in this article for exact reconstruction of the true topology is based on non-causal multivariate wiener filtering described above.

A. Reconstruction of Moral Graph

In this section, we present an algorithm for reconstruction of the network structure of LDG $\mathcal{G}$ of (3), which is well-posed and topologically detectable. Consider any two vertices X_i, X_j with a solution $W_{ji}(z)$ of (8). If $W_{ji}(z) \neq 0$, add an edge between X_i, X_j . The process is repeated for all j and i . Let the resulting undirected graph be G . Then it is shown in [28] that G is the moral graph of the LDG associated with the LDM (6) (and thus the LDG associated

with the LDM (2)). The (j, i) entry block matrix ([28]) of $\Phi_X^{-1}(z)$ is

$$B'_j \Phi_X^{-1} B_i = -\Phi_{\mathcal{E}_j}^{-1} \mathbf{W}_{ji}(z) \quad (12)$$

$$\text{where } \mathcal{E}_j(k) = X_j(k) - \sum_{i=1, i \neq j}^m (\mathbf{W}_{ji} * X_i)(k).$$

where $B_j = [\mathbf{0} \ \mathbf{0} \ \dots \ \mathbf{0} \ \mathbf{I}_{T \times T} \ \mathbf{0} \ \dots \ \mathbf{0}]'$ is a matrix $\in \mathbb{R}^{mT \times T}$ that has an identity matrix $\mathbf{I}_{T \times T}$ as the j^{th} block and other blocks as $\mathbf{0}$. From (6) $X = (\mathbb{I} - \mathbb{H}(z))^{-1} E$ which implies that $\Phi_X^{-1} = (\mathbb{I} - \mathbb{H})^* \Phi_E^{-1} (\mathbb{I} - \mathbb{H})$. We now consider the (i, j) block matrix of Φ_X^{-1} , for $i \neq j$. Using the block diagonal structure of Φ_E , we have

$$\begin{aligned} B'_j \Phi_X^{-1} B_i &= B'_j (\mathbb{I} - \mathbb{H})^* \Phi_E^{-1} (\mathbb{I} - \mathbb{H}) B_i \\ &= -\Phi_{E_j}^{-1} \mathcal{H}_{ji} - \mathcal{H}_{ij}^* \Phi_{E_i}^{-1} + \sum_{k=1}^m (\mathcal{H}_{kj})^* \Phi_{E_k}^{-1} (\mathcal{H}_{ki}). \end{aligned} \quad (13)$$

If (i, j) are not kins then j cannot be the parent of i ($\mathcal{H}_{ij} = \mathbf{0}$) or child of i ($\mathcal{H}_{ji} = \mathbf{0}$) or spouse of i (which implies that there does not exist any $k \in \{1, \dots, m\}$ with $\mathcal{H}_{kj} \neq \mathbf{0}$ and $\mathcal{H}_{ki} \neq \mathbf{0}$). Therefore from (13), (j, i) entry block matrix of $\Phi_X^{-1}(z)$ is zero and thus using (12) it follows that $\mathbf{W}_{ji}(z) = \mathbf{0}$.

This gives a sufficient criterion to identify kins which include parents, children, spouses and the moral graph is obtained (see Fig. 1(d)). The limitation with the moral graph is that it may include substantial spurious links between two-hop neighbors as shown in Fig. 1(d). Thus, it is important to devise a method for the detection and removal of spurious links from the moral graph and to deduce the topology of the LDG; which forms the focus of the next section.

B. Pruning the Spurious Strict Two Hop Neighbor Links

The following theorem will provide a sufficient condition for elimination of all strict two-hop neighbors in the moral graph $\mathcal{G}_M$ which results in true topology $\mathcal{G}_T$

Theorem III.1. Consider a LDM $(\mathbb{H}(z), E)$ which is well-posed and topologically detectable, with its associated LDG $\mathcal{G}$ and topology $\mathcal{G}_T$. Let the output of the LDM be given by $X = [X'_1 \ \dots \ X'_m]'$ according to (6). Let the vertices i and j in $\mathcal{G}_T$ be such that, $i \in N_{\mathcal{G}_T}(j, 2)$ but $i \notin N_{\mathcal{G}_T}(j)$, that is, i, j are strict two-hop neighbors. Then, $\mathbf{W}_{ji}(e^{j\omega}) \prec 0$, for all $\omega \in [0, 2\pi)$.

Proof: Given that $i \notin N_{\mathcal{G}_T}(j)$ which implies that $\mathcal{H}_{ji}(\omega) = \mathcal{H}_{ij}^*(\omega) = \mathbf{0}_{T \times T}$. Further, $i \in N_{\mathcal{G}_T}(j, 2)$, so $N_{\mathcal{G}_T}(j, 2) \cap N_{\mathcal{G}_T}(j)$ is non-empty. It follows from (12), (13) through algebraic expansions that $\mathbf{W}_{ji}(e^{j\omega}) \prec 0$, for all $\omega \in [0, 2\pi)$.

Remark 2. The consequence of Theorem III.1 is that if i is a strict two hop neighbor of j , then all the eigenvalues of the wiener filter matrix $\mathbf{W}_{ji}$ are negative. Unlike the case for scalar WSS [29], where the wiener filter transfer function W_{ji} is negative (phase π), the $\mathbf{W}_{ji}$ matrix is negative definite. This theorem does not guarantee on its converse, but such cases are pathological as it will be evident in next theorem.

Theorem III.2. Given a well-posed and topologically detectable LDM $(\mathbb{H}(z), E)$ with associated graph $\mathcal{G}$ and its topology $\mathcal{G}_T$, the following holds:

- 1) Suppose nodes i and j are such that $i \in N_{\mathcal{G}_T}(j)$, $i \notin N_{\mathcal{G}_T}(j, 2)$, $\mathbf{W}_{ji}(e^{j\omega}) \prec 0$ for all $\omega \in [0, 2\pi)$. Then $[-b_{ji} \Phi_{P_j}^{-1} S_j^*(e^{j\omega}) - b_{ij} \Phi_{P_i}^{-1} S_i(e^{j\omega})] \succ 0$
- 2) Suppose i and j are such that $i \in N_{\mathcal{G}_T}(j)$ and $i \in N_{\mathcal{G}_T}(j, 2)$ (both one hop and two hop neighbor), with $\mathbf{W}_{ji}(e^{j\omega}) \prec 0$ for all $\omega \in [0, 2\pi)$. Then $[-b_{ji} \Phi_{P_j}^{-1} S_j^*(e^{j\omega}) - b_{ij} \Phi_{P_i}^{-1} S_i(e^{j\omega}) + \sum_{k \in \mathcal{C}_G(j) \cap \mathcal{C}_G(i)} b_{kj} b_{ki} \Phi_{P_k}^{-1}(e^{j\omega})] \succ 0$

Remark 3. The consequence of theorem III.2 is that for nodes i and j that are neighbors but not two hop neighbors, or, nodes i and j that are neighbors and two hop neighbors, the inequalities in theorem III.2 holds for all $\omega \in [0, 2\pi)$ when the system parameters satisfy a restrictive and specific set of conditions. In other words, aside for pathological cases, the converse of Theorem III.2 holds. So if the $\mathbf{W}_{ji}(e^{j\omega}) \prec 0$ then $i \in N_{\mathcal{G}_T}(j, 2)$. We use this condition to prune out spurious two hop neighbor edges using the algorithm discussed next.

C. Topology Learning Algorithm

The steps involved to unravel the structure of a LDG of cyclostationary process are summarized in Algorithm table 1. In practice, the (9) is implemented by allowing lags upto order F and is formulated below

$$\inf_{\{\mathbf{W}_{ji}\}_{q, \cdot:} \substack{i=1, \dots, m, i \neq j}} \mathbb{E} \left(x_j(k+q-1) - \sum_{i=1, i \neq j}^m [\mathbf{W}_{ji}]_{q, \cdot:} * X_i(k) \right)^2 \quad (14)$$

where $[\mathbf{W}_{ji}]_{pq} = [h_{ji,pq}^{-F}, \dots, h_{ji,pq}^0, \dots, h_{ji,pq}^F]$, $[\mathbf{W}_{ji}(z)]_{pq} = \sum_{L=-F}^F h_{ji,pq}^L z^{-L}$.

Remark 4. Note that this algorithm can also be applied by lifting the cyclostationary processes to nT -variate WSS process, where $n \in \mathbb{N}$. In particular, when the sample size is low, lifting to nT -variate WSS process has the potential to improve the algorithm robustness and accuracy of the topology reconstruction. However, the computational complexity increases due to the increase in the size of matrices ($nT \times nT$ instead of $T \times T$) involved. The involved trade-off will be examined in subsequent work.

IV. RESULTS

A. Topology Reconstruction of RC network with cyclostationary input

In this section, a 7 node RC network is considered as shown in Fig. 1(a), its topology in 1(c) and its moral graph in 1(d). The spurious edges are shown in red color and are substantial. RC networks are widely used to model power distribution networks, thermal dynamics in processors, commercial buildings and many thermal systems. [28] proposed a method to reconstruct the kin topology of cyclostationary processes. But this is insufficient to eliminate the spurious edges which are absent in the true topology of the network. Moral graph contains two hop neighbor links which are not the true influence pathways between the nodes. The

Algorithm 1 Learning Algorithm for reconstructing the topology of LDG with cyclostationary inputs

Input: Nodal time series $X_i(k)$ for each node $i \in \{1, 2, \dots, m\}$ which is WSCS. Thresholds ρ, τ . Frequency points Ω .

Output: Reconstruct the true topology with an edge set $\mathcal{E}_T$

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1: Perform a periodogram analysis of the time series data
   to determine the period  $T$ . Arrange each cyclostationary
   time series as vectors of size  $T$ 
2: for all  $l \in \{1, 2, \dots, m\}$  do
3:   Compute the Wiener filter  $\mathbf{W}_{lp}(z)$  using the voltage
   time series  $\forall p \in \{1, 2, \dots, m\} \setminus l$ 
4: end for
5: Edge set  $\bar{\mathcal{E}}_K \leftarrow \{\}$ 
6: for all  $l, p \in \{1, 2, \dots, m\}, l \neq p$  do
7:   if  $\|\mathbf{W}_{pl}(z)\|_\infty > \rho$  then
8:      $\bar{\mathcal{E}}_K \leftarrow \bar{\mathcal{E}}_K \cup \{(l, p)\}$ 
9:   end if
10: end for
11: Edge set  $\bar{\mathcal{E}} \leftarrow \bar{\mathcal{E}}_K$ 
12: for all  $l, p \in \{1, 2, \dots, m\}, l \neq p$  do
13:   Compute the eigenvalues  $\{\lambda_{pl}(t)\}_{t=1}^T$  of the matrix
    $0.5(\mathbf{W}_{pl}(e^{j\omega}) + \mathbf{W}_{pl}^*(e^{j\omega}))$ 
14:   if  $\lambda_{pl}(t) \leq \tau, \forall \omega \in \Omega, \forall t$  then
15:      $\bar{\mathcal{E}} \leftarrow \bar{\mathcal{E}} - \{(l, p)\}$ 
16:   end if
17: end for

```

algorithm presented in this article is able to reconstruct the true topology by eliminating the false edges from the moral graph.

Each node of the RC network has an input (current) which is a cyclostationary process generated by superimposing a periodic signal with WGN. Thus the mean and correlation functions are periodic with time period T . In this illustration T is taken as 3 with the periodic signal being $u(k) = 5 \sin \frac{2\pi k}{T}$. The differential equation for voltage at each node is given by Kirchoff's Voltage Law (KVL). The time step Δt used for discretization is 0.001. The voltage values at each node are computed and arranged as time series. Given this voltage time series, the algorithmic steps mentioned in the previous sections is implemented to obtain the moral graph $\mathcal{G}_M$.

First a periodogram analysis on the given voltage time series is performed. The voltage time series and the periodogram are shown in Fig. 2 (a) and (b) respectively. From the periodogram, its clear that the highest frequency is 333.3 Hz. Therefore the least time period is given by $3\Delta t$ and $T = 3$. All the wiener filters are computed using (14) by allowing lags upto order 20. Based on the infinity norm of wiener filters being greater than a threshold ρ , the kin set for each node is computed and the moral graph $\mathcal{G}_M$ is constructed (See Fig. 1(d)). Lastly, based on the negative definiteness of the wiener filters associated with kin set, the edges are pruned to obtain the topology $\mathcal{G}_T$. The reconstructed topology is shown in Fig. 1(c).

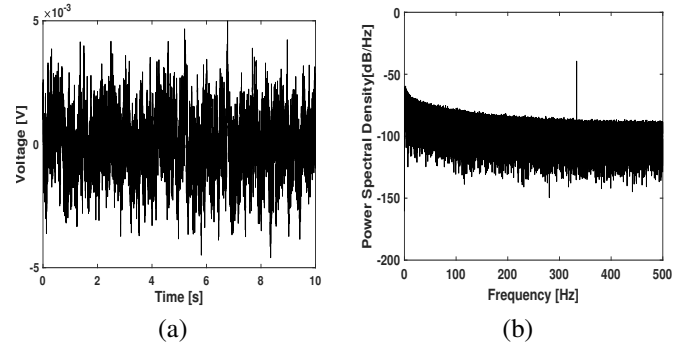


Fig. 2. (a) Voltage time series of node 1, (b) Its periodogram (peaks at 333Hz).

To check the negative definiteness of the Wiener filter matrix as mentioned in the algorithm, the eigenvalues of the Wiener filter matrix $\{\lambda_i\}_{i=1}^T$ are compared with a threshold ϵ , where ϵ is a small positive real number. It is observed that, for false edges, almost all eigenvalues are negative. An illustration is shown with node 4 as output and rest as input to compute $\hat{X}_4$. Let the thresholds ρ and ϵ be 0.2 and 0.006 (1% of the maximum eigenvalue) respectively. For $\omega \in [0, 0.16\pi]$:

- W_{41} : the $\|\mathbf{W}_{41}(z)\|_\infty$ is 0.715 $> \rho$ and its eigenvalues are $\{-0.098, -0.029, 0.006\}$ less than ϵ , deemed negative. Therefore vertex 1 is a strict two hop neighbor
- W_{42} : the $\|\mathbf{W}_{42}(z)\|_\infty$ is 0.936 $> \rho$ and its eigenvalues are $\{-0.4, 0.0004, 0.0004\}$ less than ϵ , deemed negative. Therefore vertex 2 is a strict two hop neighbor
- W_{43} : the $\|\mathbf{W}_{43}(z)\|_\infty$ is 0.636 $> \rho$ and its eigenvalues are $\{0.861, -0.06, 0.0018\}$. The wiener filter $\mathbf{W}_{43}(z)$ is not negative definite. Therefore vertex 3 is not a strict two hop neighbor
- W_{45} : the $\|\mathbf{W}_{45}(z)\|_\infty$ is 1 $> \rho$ and its eigenvalues are $\{1.347, 0.003, -0.012\}$. The wiener filter $\mathbf{W}_{45}(z)$ is not negative definite. Therefore vertex 5 is not a strict two hop neighbor
- W_{46} : the $\|\mathbf{W}_{46}(z)\|_\infty$ is 0.643 $> \rho$ and its eigenvalues are $\{-0.463, -0.009, -0.002\}$ less than ϵ , deemed negative. Therefore vertex 6 is a strict two hop neighbor
- W_{47} : the $\|\mathbf{W}_{47}(z)\|_\infty$ is 0.97 $> \rho$ and its eigenvalues are $\{-0.217, -0.002, 0.005\}$ less than ϵ , deemed negative. Therefore vertex 7 is a strict two hop neighbor

The performance of the algorithm is evaluated at different sample sizes as shown in Fig. 3. Note that a sample size of 4e5 seems sufficient for exact recovery.

V. CONCLUSIONS

In summary, the topology of the Linear Dynamic Graph of a Linear Dynamic Model with cyclostationary inputs is constructed from the time series data with provable guarantees. It is a data driven approach useful for exact identification of network structure with applications to power grids, thermal networks, network of rotating mechanical systems amongst many others. Our approach doesn't impose any structural restrictions on the network topology and doesn't use any knowledge of the system parameters.

In our future work, we will apply this algorithm on experimental data with a wide test cases for the parameters.

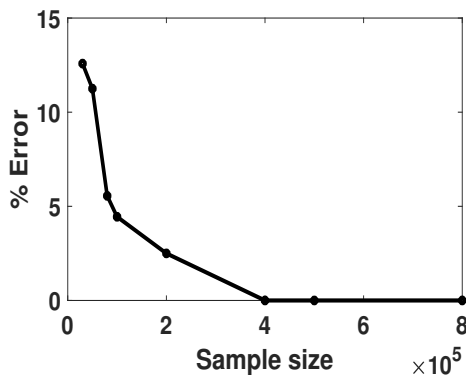


Fig. 3. Percentage error variation with the sample size

Regularization methods will be used to improve the performance with less samples and to promote the sparseness. This algorithm can also be used to learn the topology of a network of WSS processes more efficiently than the current state of the art and will be the focus of our future work.

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