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Reconstruction of Networks of Cyclostationary Processes

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***Abstract*— Many complex systems can be described by agents that can be modeled as a network of dynamically interacting cyclo-stationary processes. Such systems arise in areas like power systems and climate sciences. For many of these systems a key objective is to understand mutual influences between various subsystems without altering the natural behavior of the system. Such an objective translates to unveiling the interconnection of the topology of the network using only passive means. Most existing related works have emphasized correlation based methods where interdependencies over different time-instants can be missed. Recent work where dynamic influences are incorporated assuming stationary statistics cannot accommodate applications that arise in many areas such as power and climate sciences. In this article an algorithm based on Wiener filtering is devised for the reconstruction of interconnectivity of dynamically related cyclo-stationary processes. It is shown that all existing interdependencies are detected and spurious detection remains local. Application to a microgrid power network is shown to yield useful insights.**

I. INTRODUCTION

Cyclostationary processes arise naturally from periodic phenomenon [1]. In telecommunications, telemetry, radar and sonar applications; modulation, sampling, multiplexing, and coding operations give rise to periodicity. Periodicity in mechanical systems is due to rotation and reciprocation of gears, belts, chains, shafts, propellers, pistons, and so on [2]. In power generation the rotational motion of turbines and generators is responsible for periodic AC signals. In inverters PWM switching is responsible for the generation of periodic AC signals. In astronomy, periodicity arises due to rotation and revolution of celestial bodies. In econometrics and climate systems, sesonality and periodic demand - supply cycles of markets respectively give rise to periodic behavior [3], [4].

A common feature of the time series data obtained in these examples is that they exhibit nonstationary, but periodic statistics. In many cases it is of interest to understand the relationship among various components of these nonstationary systems via data driven methods by exploiting the periodic behavior of their dynamics.

In this article an algorithm is developed to reconstruct networks of cyclostationary processes which are dynamically related by transfer functions. It is shown that non causal Wiener filtering is capable of reconstructing the kin topology of the network which recovers the set of parents, children and spouses of every agent in the network. Furthermore the reconstruction algorithm is extended to accommodate poly-periodic processes. The algorithm developed is employed to characterize electrical dependence of various parts of a micro-grid. Further analysis of the results provides insights into the mutual electrical dependence of various subsystems of the micro-grid.

The paper is organized as follows. In Section II we present definitions and key results of cyclostationary processes. Section III describes a class of models that will suit our purpose of describing an interconnected structure of dynamical systems. Section IV presents basic concepts of non causal Wiener filtering. In Section V we show that non causal Wiener filtering is able to recover the parents, children and spouses (i.e., Markov blanket) of each node. In this section we also comment on the interconnections of this approach with graphical models. Section VI deals with simulation results to illustrate the developed algorithm. Section VII presents the conclusions.

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Notation:

The symbol $:=$ denotes a definition
$\|x\|$: 2 norm of a vector x
A_{j*}: $j-th$ row of matrix A
A_{*i}: $i-th$ column of matrix A
$\mathbf{0}$: zero matrix of appropriate dimension
$I_{T\times T}$: identity matrix of size T
A^*: the conjugate transpose of matrix A
A': transpose of a matrix A
$A \succ 0$: A is a positive definitie matrix

II. DEFINITIONS

In this section we present basic notions of graph theory and stochastic processes which are essential for the subsequent developments. Detailed description of graph theory is available in [5] and stochastic processes in [6].

Definition 1 (Directed and Undirected Graphs): An undirected graph G is a pair (V, A) where V is a set of vertices or nodes and A is a set of edges or arcs, which are unordered subsets of two distinct elements of V.

A directed (or oriented) graph G is a pair (V, A) where V is a set of vertices or nodes and A is a set of edges or arcs, which are ordered pairs of elements of V.
In the following, if not specified, oriented graphs are considered.

Definition 2 (Topology of a graph): Given an oriented graph $G = (V, A)$, its topology is defined as the undirected graph $G' = (V, A')$ such that $\{N_i, N_j\} \in A'$ if and only if $(N_i, N_j) \in A$ or $(N_j, N_i) \in A$, and $top(G) := G'$.
An example of a directed graph is represented in Figure 1(a) with its topology in Figure 1(b).

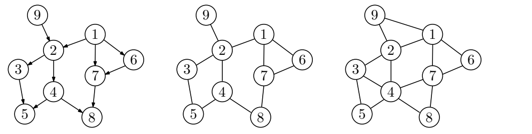

Fig. 1. (a) A directed graph, (b) its topology and (c) its kin-topology [7].

Definition 3 (Children and Parents): Given a graph $G = (V, A)$ and a node $N_j \in V$, the children of N_j are defined as $\mathcal{C}_G(N_j) := \{N_i | (N_j, N_i) \in A\}$ and the parents of N_j as $\mathcal{P}_G(N_j) := \{N_i | (N_i, N_j) \in A\}$.

Definition 4 (Kins): Given an oriented graph $G = (V, A)$ and a node $N_j \in V$, kins of N_j are defined as

$$\mathcal{K}_G(N_j) := \{N_i | N_i \neq N_j \text{ and } N_i \in \mathcal{C}_G(N_j) \cup \cup \mathcal{P}_G(N_j) \ \cup\ \mathcal{P}_G(\mathcal{C}_G(N_j))\}.$$

Note that the kin relation is symmetric, in the sense that $N_i \in \mathcal{K}_{\mathcal{G}}(N_j)$ if and only if $N_j \in \mathcal{K}_{\mathcal{G}}(N_i)$.

Definition 5 (Kin-graph): Given an oriented graph $G = (V, A)$, its kin-graph is the undirected graph $\tilde{G} = (V, \tilde{A})$ where

$$\tilde{A} := \{\{N_i, N_j\} | N_i \in \mathcal{K}_G(N_j) \text{ for all } j\}.$$

and it is denoted as $kin(G) = \tilde{G}$.
Figure 1(c) shows the kin-topology of the directed graph in Figure 1(a).

Definition 6 (Self-kin Graph): An oriented graph G is self-kin if $top(G) = kin(G)$.

Definition 7 (Wide Sense Stationary): A random process e_t is Wide Sense Stationary (WSS) if its mean function, $m(t)$ is constant and correlation function, $R_e(s,t)$ is a function of $s - t$.

Definition 8 (Cyclostationary Process): A random process x_t is said to be cyclostationary or periodically correlated with period $T > 0$ if for every $s, t \in \mathbb{Z}$ there is no value smaller than T such that the following hold

$$m(t) = m(t+T), \text{ and}$$
$$R_x(s,t) = E[x_s x_t] = E[x_{s+T} x_{t+T}] = R_x(s+T, t+T)$$

A WSS process is a cyclostationary processes with period 1. Any signal which can be described as a sum of a periodic signal and a wide sense stationary(WSS) noise results in a cyclostationary process

Definition 9 (Jointly Cyclostationary Processes): The random processes x_t and e_t are said to be jointly cyclostationary if x_t and e_t are cyclostationary with period T and the cross correlation function $R_{x,e}(s,t)$ is periodic with period T.

Definition 10 (Multivariate Stationarity): A 2^{nd} order q-variate random sequence $\boldsymbol{X}_t = [x_{1,t}, \cdots, x_{q,t}]', t \in \mathbb{Z}$ is said to be (weakly) stationary if

$$E[\boldsymbol{X}_{j,t}] = m_j,$$

where $\boldsymbol{X}_{j,t}$ is the j^{th} element of $\boldsymbol{X}_t$ and

$$\begin{aligned} R^{jk}_{\boldsymbol{X}}(s,t) &= E[\boldsymbol{X}_{j,s}\boldsymbol{X}_{k,t}] \\ &= E[x_{j,s}x_{k,t}] = E[x_{j,s-t}x_{k,0}] = R^{jk}_{\boldsymbol{X}}(s-t), \end{aligned}$$

for all $s, t \in \mathbb{Z}, \text{and} j, k \in \{1, 2, \cdots, q\}$. In this case denote $\boldsymbol{m} = [m_1, m_2, \cdots, m_q]'$ as the mean vector and $\boldsymbol{R_X}(\tau) = [R^{jk}_{\boldsymbol{X}}(\tau)]^q_{j,k=1}$as the correlation matrix of $\boldsymbol{X}_t$.

Blocking is a way of representing scalar time series as vectors (or blocks). For example - nT samples of a scalar periodic signal with period T can be represented as n blocks of size T each. In this example, size of each vector (or block) is T and hence the vectorized sequence is called T variate blocked sequence. Multivariate (or vector) sequences obtained from blocking of univariate (or scalar) sequences x_t will be indexed by n and denoted as $\boldsymbol{X}_n$. Thus, the univariate sequence x_t is related to the T variate blocked sequence $\boldsymbol{X}_n$ by $\boldsymbol{X}_{j,n} = x_{j+nT}, n \in \mathbb{Z}, j = 0, 1, \cdots, T-1$.

Theorem 2.1: A second order random sequence $x_t : t \in \mathbb{Z}$ is cyclostationary with period T if and only if T is the smallest integer for which the T-variate blocked sequence $\boldsymbol{X}_n$ is stationary [8].

Proof: See [8] for the proof. ■

Theorem 2.2: If x_n and e_n are jointly cyclostationary sequence with period T, Then $y_n = h_n * x_n + e_n$ is cyclostationary with period T.

Proof: The proof is left to the reader. ■

Note: If the output of linear filtering is the zero signal while the inputs being cyclostationary with perod T, though the zero signal is uniperiodic it is also T periodic. It should be clear from the context which period to use.

Definition 11: Let $\mathcal{E}$ be a set containing time-discrete vector jointly stationary processes such that for any $E_i, E_j \in \mathcal{E}$, the power spectral density $\Phi_{E_iE_j}(z)$ exists and each element of $\Phi_{E_iE_j}(z)$ is real rational with no poles on the unit circle and given by

$$[\Phi_{E_iE_j}(z)](m,n) = \frac{A(z)}{B(z)},$$

where $[\Phi_{E_iE_j}(z)](m,n)$ is the $(m,n)^{th}$ element of $\Phi_{E_iE_j}(z)$, $A(z)$ and $B(z)$ are polynomials with real coefficients such that $B(z) \neq 0$ for any $z \in \mathbb{C}$, with $|z| = 1$. Then, $\mathcal{E}$ is a set of rationally related random processes.

Definition 12: The set $\mathcal{F}$ is defined as the set of transfer matrices $\mathcal{H}$ comprised of real-rational single-input single-output (SISO) transfer functions that are analytic on the unit circle $\{z \in \mathbb{C} | \ |z| = 1\}$.

Definition 13: Let $\mathcal{E}$ be a set of rationally related vector stationary processes. The set $\mathcal{FE}$ is defined as

$$\mathcal{FE} := \left\{ x = \sum_{k=1}^{m} \mathcal{H}_k(z) E_k \mid E_k \in \mathcal{E}, \mathcal{H}_k(z) \in \mathcal{F} \right\}.$$

Definition 14 ((Inner Product)): Given two random vectors X_1 and X_2 in the space $\mathcal{FE}$ we define the inner product as $\langle X_1, X_2 \rangle := E[X_1' X_2] = E[tr[X_1 X_2']] = tr[E[X_1 X_2']] = tr[R_{X_1X_2}(0)] = tr[\int_{-\pi}^{\pi} \Phi_{X_1X_2}(e^{\iota\omega})]$, where tr is the trace operator.

Definition 15: For a finite number of elements $X_1, ..., X_m \in \mathcal{FE}$, tf-span is defined as

$$\textit{tf-span}\{X_1, ..., X_m\} := \left\{ x = \sum_{i=1}^{m} \boldsymbol{A}_i(z) X_i \mid \boldsymbol{A}_i(z) \in \mathcal{F} \right\}.$$

Lemma 2.1: The tf-span operator defines a subspace of $\mathcal{FE}$.

Proof: The proof is left to the reader. ■

III. LINEAR DYNAMIC GRAPHS

Now we introduce a class of models for the description of a network of jointly cyclostationary processes. This class of models also admits a visual representation where each process is a node in a graph. It is assumed that the dynamics of each agent (node) in the network is represented by a scalar cyclostationary random process $\{x_j\}_{j=1}^m$ that is given by the superposition of a noise component e_j and the "influences" of some other "parent nodes" through dynamic links. The noise component at each node e_j is assumed to be zero mean WSS and uncorrelated with $\{e_k\}_{k=1,k\neq j}^m$. Thus, the j^{th} node dynamics is given by

$$x_j = e_j + \sum_{i=1}^{n} H_{ji}(z)x_i.$$

Notice that this relation is implicit, thus, as it will be discussed later, well-posedness conditions will need to be imposed. Using Theorem 2.2 one can state that this description of node dynamics preserves the cyclostationary character of the time series. If a certain agent "influences" another one a directed edge can be drawn and a directed graph can be obtained. Using Theorem 2.1 each node x_j can be mapped into a T-variate stationary process X_j. Similarly, the node disturbence e_j can be mapped into a T variate vector stationary process E_j which is uncorrelated with X_j and $\{E_i\}_{i\neq j}$ and has the zero vector (of size $T\times 1$) as its mean vector. The node dynamics then is represented as

$$\begin{aligned}
X_j(k) &= \sum_{i=1}^{m} \mathcal{H}_{ji}(z)X_i(k) + E_j(k), \text{ where,}\\
X_j(k) &= [x_j(kT)\ x_j(kT+1)\ \cdots\ x_j(kT+T-1)]',\\
E_j &= [e_j(kT)\ e_j(kT+1)\ \cdots\ e_j(kT+T-1)]',\\
\mathcal{H}_{ji} &= I_{T\times T}H_{ji}(z)
\end{aligned}$$

where, $\mathcal{H}_{ji}$ is $T\times T$ transfer matrix. If there is no dynamical link from x_i to x_j then $H_{ji}=0 \Rightarrow \mathcal{H}_{ji}=\mathbf{0}$, which means that there is no dynamical link from the vector stationary process X_i to X_j. Similarly, if there is no dynamical link from X_i to X_j then $\mathcal{H}_{ji}=\mathbf{0} \Rightarrow H_{ji}=0$, which means there is no dynamical link from the scalar cyclostationary process x_i to x_j. Thus,we have the following Lemma.

Lemma 3.1: Two cyclostationary processes have a dynamical link (through a transfer function) if and only if there is a dynamical link between their stationary vector representations (through a transfer matrix).

Proof: The proof follows from the discussion above. ■

Theorem 2.1 enables us to treat a dynamical network of scalar cyclostationary processes as a dynamical network of vector stationary processes. The above lemma enables us to design reconstruction procedures for the stationary vector representation of cyclostationary processes and presence or absence of links in the vector representation can be directly mapped to the presence or absence of links in the scalar cyclostationary case. The discussion henceforth would be centered around reconstruction of topology of a dynamical network of $T\times 1$ vector stationary processes.

Definition 16 (Linear Dynamic Graph): A Linear Dynamic Graph $\mathcal{G}$ is defined as a pair $(\mathbb{H}(z), E)$ where

- $E = (E_1' \ \dots\ E_m')'$ is a vector of m rationally related uncorrelated vector stationary processes $\{E_j\}_{j=1}^m$. Thus, $\Phi_E(z)$ is block diagonal matrix of size $mT\times mT$ with size of each block $T\times T$.
- $\mathbb{H}(z)$ is a $mT\times mT$ matrix of block transfer matrices in $\mathcal{F}$ such that diagonal blocks $\mathcal{H}_{jj}(z)=\mathbf{0}$, for $j=1,\dots,m$ and $\mathbb{H}(j,i)=\mathcal{H}_{ji}, i\neq j$ is the (j,i) block of $\mathbb{H}(z)$.

The output processes $\{X_j\}_{j=1}^m$ of the LDG are defined in a compact way as

$$X(k) = \mathbb{H}(z)X(k) + E(k), \text{where } X = [X_1', \cdots, X_m']' \tag{1}$$

Let $V := \{x_1,\dots,x_m\}$ or equivalently $\{X_1,\cdots,X_m\}$ and let $A := \{(x_i,x_j)|H_{ji}(z)\neq 0\}$ or equivalently $\{(X_i,X_j)|\mathcal{H}_{ji}\neq\mathbf{0}\}$. The pair $G=(V,A)$ is the associated directed graph of the LDG. Nodes and edges of a LDG will mean nodes and edges of the graph associated with the LDG.

Observe that (1) defines a map from a vector of rationally related processes X to a vector of rationally related processes E. This map is well defined if the operator $(\mathbb{I}-\mathbb{H}(z))$ is invertible on the space of rationally related processes. It can then be guaranteed that, for any vector of rationally related processes E, there exists a vector X of processes in the space $\mathcal{FE}$. This leads us to the following definition.

Definition 17: A LDG $(\mathbb{H}(z), E)$ is well-posed if each entry of $(\mathbb{I}-\mathbb{H}(z))^{-1}$ belongs to $\mathcal{F}$. Thus, $X=(\mathbb{I}-\mathbb{H}(z))^{-1}E$ can be written.

Definition 18: A LDG $\mathcal{G}=(\mathbb{H}(z),E)$ is topologically detectable if $\Phi_{E_j}(e^{\iota\omega})$ is positive definite for any $\omega\in[-\pi,\pi]$ and for any $j=1,\dots,m$. This is written as $\Phi_{E_j}(e^{\iota\omega})\succ 0$ for any $\omega\in[-\pi,\pi]$ and for any $j=1,\dots,m$.

IV. NON CAUSAL WIENER FILTERING

Now we present a lemma which lists down the conditions for unique representation of any element in *tf-span*$\{X_i\}_{i=1,\dots,m}$.

Lemma 4.1: Let q and $X_1,\dots,X_m$ be processes in the space $\mathcal{FE}$. Define $X=[X_1'\ \dots\ X_m']'$. Suppose that $q\in$ *tf-span*$\{X_i\}_{i=1,\dots,m}$ and that $\Phi_X(e^{\iota\omega})\succ 0$ almost for any $\omega\in\ [-\pi,\pi]$. Then there exists a unique transfer matrix $\Lambda(z)\in\mathcal{F}^{T\times mT}$ such that $q=\Lambda(z)X$.

Proof: This *Lemma* is a direct extension of *Lemma* 23 in [7] for the case of vector stationary processes. ■

Now we present the non causal Wiener Filter for vector stationary processes.

Theorem 4.1 (Wiener Filter): Let V and $X_1,\dots,X_m$ be processes in the space $\mathcal{FE}$. Define $X:=[X_1',\dots,X_m']'$ and consider the problem

$$\begin{aligned}
\inf_{\boldsymbol{W}(z)} \|V-\boldsymbol{W}(z)X\|^2 = \inf_{\boldsymbol{W}(e^{\iota\omega})} tr[\int_{-\pi}^{\pi}\Phi_V(e^{\iota\omega})\\
-\Phi_{VX}(e^{\iota\omega})\boldsymbol{W}^*(e^{\iota\omega}) - \boldsymbol{W}(e^{\iota\omega})\Phi_{XV}(e^{\iota\omega})\\
+\boldsymbol{W}(e^{\iota\omega})\Phi_X(e^{\iota\omega})\boldsymbol{W}^*(e^{\iota\omega})d\omega]
\end{aligned}$$

If $\Phi_X(e^{\iota\omega}) \succ 0$, for $\omega \in [-\pi, \pi]$, then the solution $\hat{V} \in \mathcal{X}$ exists, is unique and is given by

$$\hat{V} = \boldsymbol{W}(z)X, \boldsymbol{W}(z) = \Phi_{VX}(z)\Phi_X(z)^{-1}.$$

Moreover, $\hat{V}$ is the only element in $\mathcal{X}$ such that, for any $q \in \mathcal{X}$,

$$\langle V - \hat{V}, q \rangle = 0. \tag{2}$$

Proof: This *Theorem* is a direct extension of *Proposition* 24 in [7] for the case of vector stationary processes. ■

Next we introduce the definition of conditional non causal Wiener orthogonality. The Wiener filter is a $T \times mT$ transfer matrix which is of the form $\boldsymbol{W}(z) = [\boldsymbol{W}_1 \boldsymbol{W}_2 ... \boldsymbol{W}_m]$ where $\{\boldsymbol{W}_j\}_{j=1}^m$ is a $T \times T$ transfer matrix and is the Wiener filter from X_j to V.

Definition 19 (Wiener Orthogonality): Let V, $X_1, ..., X_m$ be processes in the space $\mathcal{FE}$. Define $X := [X_1' \ ... \ X_m']'$ and $\mathcal{X} :=$ *tf-span*$\{X_1, ..., X_m\}$. The process V is said to be conditionally non-causally Wiener-orthogonal with X_i, for any $i \in \{1, ..., m\}$, given the processes $\{X_k\}_{k \neq i}$ if the i-th block of the Wiener filter to estimate V from X is zero, that is $\Phi_{VX}\Phi_X^{-1}\boldsymbol{B}_i = \boldsymbol{0}$, where $\boldsymbol{B}_i = [\boldsymbol{0} \ \boldsymbol{0} \ \cdots \ \boldsymbol{0} \ \boldsymbol{I}_{T\times T} \ \boldsymbol{0} \ \cdots \ \boldsymbol{0}]'$ is a matrix $\in \mathbb{R}^{mT\times T}$ that has $\boldsymbol{I}_{T\times T}$(identity matrix) as the $i-th$ block and $\boldsymbol{0}$ as all other blocks.

Lemma 4.2: Let $\mathcal{E}$ be a set of rationally related processes and let $X_1, ..., X_m$ be $T \times 1$ vector stationary processes in the space $\mathcal{FE}$. Define $X = [X_1' \ ... \ X_m']'$. Assume that Φ_X has full normal rank. The process X_i is non-causally Wiener-orthogonal with X_j given the processes $\{X_k\}_{k\neq i,j}$, if and only if the (i, j) block matrix, or equivalently the (j, i) block matrix, of $\Phi_X^{-1}(z)$ is $\boldsymbol{0}$, that is, for $i \neq j$,

$$\boldsymbol{B}_j'\Phi_X^{-1}\boldsymbol{B}_i = \boldsymbol{B}_i'\Phi_X^{-1}\boldsymbol{B}_j = \boldsymbol{0}. \tag{3}$$

Proof: The proof is an extension of the proof of *Lemma* 26 in [7] to vector stationary processes. ■

V. NETWORK RECONSTRUCTION

In this sectiion we present results which are sufficient conditions to determine if two nodes are kins. We will show that if two nodes are not kins in the generative LDG then the Wiener filter transfer matrix between them is a zero matrix.

Theorem 5.1: Consider a LDG $(\mathbb{H}(z), E)$ with associated graph G which is well-posed and topologically detectable. Let the output of the LDG be given by $X = (X_1', ..., X_m')'$. Define the space $\mathcal{X}_j =$ *tf-span*$\{X_i\}_{i\neq j}$. The approximation of the signal X_j with an element $\hat{X}_j \in \mathcal{X}_j$ (non-causal Wiener filtering)is given by the optimization problem

$$\min_{\hat{X}_j \in \mathcal{X}_j} \left\| X_j - \hat{X}_j \right\|^2. \tag{4}$$

Then a unique optimal solution to the above optimization problem $\hat{X}_j$ exists and is given by

$$\hat{X}_j = \sum_{i\neq j} \boldsymbol{W}_{ji}(z)X_i \tag{5}$$

where $\boldsymbol{W}_{ji}(z) \neq 0$ implies $\{X_i, X_j\} \in kin(G)$.

Proof: Dynamics of the LDG is given by $X = (\mathbb{I} - \mathbb{H}(z))^{-1}E$ which implies that

$$\Phi_X^{-1} = (\mathbb{I} - \mathbb{H})^*\Phi_E^{-1}(\mathbb{I} - \mathbb{H}). \tag{6}$$

Now we consider the (i, j) block matrix of Φ_X^{-1}, with $i \neq j$ and following the directions of the proof of *Theorem* 27 in [7] it is shown that,

$$\begin{aligned} \boldsymbol{B}_j'\Phi_X^{-1}\boldsymbol{B}_i &= (\boldsymbol{B}_j' - (\mathbb{H}_{*,j})^*)\Phi_E^{-1}(\boldsymbol{B}_i - \mathbb{H}\boldsymbol{B}_i) \\ &= -\Phi_{E_j}^{-1}\mathcal{H}_{ji} - \mathcal{H}_{ij}{}^*\Phi_{E_i}^{-1} + \sum_{k=1}^m (\mathcal{H}_{kj})^*\Phi_{E_k}^{-1}(\mathcal{H}_{ki}). \end{aligned}$$

The theorem is proved if it is shown that given (i, j) are not kins then the Wiener filter from i to j, $\boldsymbol{W}_{ji}(z) = \boldsymbol{0}$. If (i, j) are not kins then j cannot be the parent of i(which implies that $\mathcal{H}_{ij} = \boldsymbol{0}$) or child of i(which implies that $\mathcal{H}_{ji} = \boldsymbol{0}$) or spouse of i(which implies that there does not exist any $k \in \{1, ..., m\}|\mathcal{H}_{kj} \neq \boldsymbol{0}$ and $\mathcal{H}_{ki} \neq \boldsymbol{0}$). Thus, the entry (j, i) block of Φ_X^{-1} is null. Using Lemma (4.2), $\boldsymbol{W}_{ji}(z) = \boldsymbol{0}$ and the assertion is proved. ■

The sufficient conditions for the reconstruction of a link in a LDG of cyclostationary processes is listed in the following corollary.

Corollary 5.1: Consider a well-posed and topologically detectable LDG $\mathcal{G}$ with associated graph G. Let $X = (X_1', ..., X_m')'$ be its output. Let $\boldsymbol{W}_{ji}(z)$ be the entry of the non-causal Wiener filter estimating X_j from $\{X_k\}_{k\neq j}$ corresponding to the process X_i. If $\mathcal{G}$ is self-kin, then $\boldsymbol{W}_{ji}(z) \neq 0$ implies $(X_j, X_i) \in top(G)$.

Proof: Using Theorem 5.1, $\boldsymbol{W}_{ji}(z) \neq \boldsymbol{0} \Rightarrow (X_i, X_j) \in kin(G)$. Since, G is self kin, so $top(G) = kin(G)$. Thus, $(X_i, X_j) \in top(G)$. ■

The steps involved to unravel the structure of a LDG of cyclostationary process are summarized below.

1) Perform a periodogram analysis of the time series data to determine the period T.
2) Arrange each cyclostationary time series (or node in the graph) as vectors of size T.
3) For any node x_j compute $\boldsymbol{W}_{ji}(z)$.
4) If $\boldsymbol{W}_{ji}(z) \neq \boldsymbol{0}$ then add $\{x_i, x_j\}$ to the set of edges.
5) Repeat steps 1-4 for all nodes $x_j, j = 1, 2, \cdots, n$.

Suppose that the mean and correlation function of $\{x_j\}_{j=1}^{j=m}$ are periodic with period $T_1, T_2, \cdots, T_n$, i.e., the second order statistics of the time series are poly periodic. The network reconstruction procedure described above for cyclostationary process with period T is applicable to poly-periodic processes with $T = T_{eff} = LCM\{T_1, T_2, \cdots, T_n\}$ where poly-periodic processes can be treated as cyclostationary processes with period T_{eff}. Here LCM stands for the Least Common Multiple.

It is important to note that non causal Wiener filtering provides information on the presence or absence of the link and no conclusion on the direction of the arrows can be drawn from it. This approach returns the kin topology or the structure of the graph of the generative model and not the exact topology of the generative graph. Nonetheless the structure captures all parent child relationships and hence the knowledge of the structure is useful in various applications.

In this entire discussion it is assumed that the time period T, i.e., the number of samples after which the second order statistics show periodicity is an integer. The steps mentioned above are also applicable to the reconstruction of the underlying kin topology of a LDG of vector stationary processes.

Dynamically related time series have been studied by various researchers under the WSS time series framework. It is shown in [7] that non causal Wiener filtering is capable of identifying the Markov blanket of each node of a LDG from the WSS observations. Most of the development here is a generalization of the results presented in [7] and the methods for establishing results are similar. However, the domain of applications is significantly larger. We have shown above that non causal Wiener filtering is capable of identifying the Markov blanket of each node of a LDG of cyclostationary processes too. It is important to note that Wiener filtering based framework developed for WSS processes (see [7]) is not applicable to the reconstruction problem considered in this paper. However, the approach presented here for cyclostationary processes can be applied to obtain the kin topology of a LDG of WSS processes because WSS processes are cyclostationary processes with period 1.

VI. RESULTS

A. Network Reconstruction for LDG of Cyclostationary and Poly-Periodic Processes

A simulation is performed in MATLAB to illustrate the reconstruction procedure using non causal Wiener filtering for networks of cyclostationary processes. A five node network as shown in Figure 2 is used for the illustration of the key results derived earlier. In this network, node 2 and node 3 are spouses. Each link in Figure 2(a) is a 4th order finite impulse response transfer function $1+0.9z^{-1}+0.5z^{-2}+0.3z^{-3}$. The sequence e_i used is simulated as zero mean white gaussian noise. Node 1 has a sinusoid of frequency $\pi/3$ rad/sample which is an exogenous input and is responsible for the time series of all nodes being cyclostationary with period $T = 6$ samples. The network is simulated and about initial 600 samples are removed to ensure that the observed data is cyclostationary and transient effects are removed. The time period T is estimated from the periodogram of the observed data by rounding it off to the nearest integer. The presence or absence of a link is decided by comparing the 2 norm of the Wiener filter matrix $\boldsymbol{W}_{ij}(z)$ to a threshold of 0.1. If the 2 norm of $\boldsymbol{W}_{ij}(z) < 0.1$ then it is decided that $\boldsymbol{W}_{ij}(z)$ is negligible and there is no edge between node i and node j. It is seen in Figure 2(b) that the Wiener reconstruction provides the kin topology of the original graph associated with the true linear dynamic graph. The Wiener reconstruction introduces a spurious link between nodes 2 and 3 because these nodes share a common child and hence are kins.

Next we show the extension of the algorithm to poly-periodic processes. The setup is the same as in Figure 2(a) with the difference being that the exogenous inputs to Node 1 are now multiple sinusoids with frequency $2\pi/9, \pi/3$ and $2\pi/3$ rad/sample. These multiple frequencies are responsible for the poly-periodicity of the 2^{nd} order

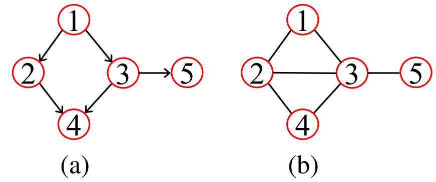

Fig. 2. The associated graph of a Linear Dynamic Graph of cyclostationary processes (a) and the reconstructed graph obtained using the Wiener projection technique suggested by Theorem 5.1 (b). Spurious links are introduced between the spouses 2-3.

statistics with periods $T = 9, 6$ and 3 samples. These periods are being estimated from the data using periodogram analysis which now shows three peaks corresponding to the respective frequencies. The reconstruction procedure developed for cyclostationary processes is applied to the poly-periodic data with $T_{eff} = LCM\{9, 6, 3\} = 18$ samples. The reconstructed topology is the same as shown in Figure 2(b). This example illustrates that non causal Wiener filtering is capable of inferring the kin topology of LDGs of poly-periodic processes.

B. Application in Power Distribution Networks

We demonstrate the concept of Wiener Orthogonality for cyclostationary processes in the context of a power distribution network shown in Figure 3.

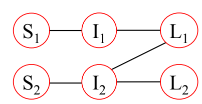

Fig. 3. Power distribution network for illustration of Wiener Orthogonality. Here S_1, S_2, I_1, I_2, L_1 and L_2 are the observed time series.

S_1 and S_2 represent the voltage time series from DC power sources (solar photovoltaic sources) which is modeled as a constant with additive zero mean WSS disturbances. I_1 and I_2 represent the output voltage across Inverter 1 and Inverter 2 respectively. L_1 and L_2 are the voltage time series across the heater (rated at 1700 W) and lights (rated at 1000 W) respectively. The models used for heater and lights are representative of the realistic behavior of heater and lights. These time series are cyclostationary because of the presence of periodic signals on the AC side and WSS disturbances on the DC side. The network reconstruction algorithm developed for cyclostationary processes is applied and the reconstructed topology obtained is shown in Figure 5 by the black edges only. An interesting observation is that the reconstructed topology has no edges between the sources and the loads. This observation has been verified on other power network simulations as well, the discussion of which has been omitted due to space constraints. Thus, it can be said that the loads and the sources are "Wiener Orthogonal conditioned on the inverters" (i.e., the AC side and DC side of the network are separated by the inverters). It is important to realize that although there is an electrical power flow path from the sources to the loads via the inverters, the absence of an edge in the reconstructed topology as inferred by Wiener filtering suggests that the sources do not have any direct influence on the loads. In the language of graphical models this observation can be interpreted as d-separation [9] of the loads and the sources by the inverters. It is important to understand whether the d-separation provides any insights into electrical separation of various subsystems; here all components are connected via electrical pathways and thus separation as suggested by our algorithm is not expected. Remarkably, motivated by the results of the algorithm when the dynamics is more closely analyzed such a separation is natural indeed. We now present the reason why the sources do not influence the loads directly.

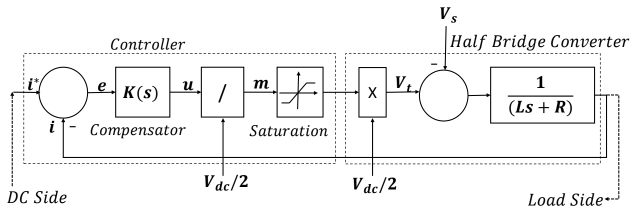

Fig. 4. Closed Loop Control of Half Bridge Converter for isolation of DC side and AC side dynamics [10].

We present a terse overview of the dynamic model of inverters and the control employed to appreciate the applicability of the proposed network reconstruction algorithm in the context of power distribution networks and interpret the above observations from a physical perspective. A four quadrant DC/AC half bridge converter is a building block for DC/AC voltage-sourced inverter. The modeling of converter takes into account the switching dynamics and the system accounts for the linkage between the DC side sources and AC side inverter output. Consider the closed loop control model of half bridge converter system with the objective of tracking the output current i and regulating it at a prespecified reference level i^* as illustrated in Figure 4. The compensator works on the error signal $e := i^* - i$ to generate the control input u which is then divided by half the DC side voltage, $V_{dc}/2$ to compensate for converter voltage gain. The controller output is limited before feeding into the converter Pulse Width Modulation (PWM) generator in order to ensure that $|m| \leq 1$, where $m := 2d - 1$ and d is the switch duty ratio of the half bridge converter. As outlined in Figure 4, the generation of m is followed by the half-bridge converter where the terminal voltage V_t is generated according to converter averaged dynamics as $V_t = mV_{dc}/2$. The disturbance input is represented as V_s which can be thought of as a voltage source interfaced on the AC side. The interface reactor smooths out the voltage ripple on the AC side. Under the assumption that the PWM switching of converter is achieved with carrier signal frequency being much higher than that of the modulating signal, the converter control model shown in Figure 4 is valid. Almost stiff voltage across the inverter input can be achieved with this control structure which imparts robustness to the inverter input against certain degree of oscillations in DC side generation voltage and thereby, effectively decoupling the DC side (e.g. photovoltaics) dynamics from the AC side (loads) [10]. Thus, the uncertainty in the power generation is effectively blocked from appearing at the loads directly by the inverters and the changes in the loads does appear at the DC side but only via the inverter measured signals. The above example indicates that the method developed for cyclostationary data can be used to understand which parts are effectively isolated from each other and in-turn can help in making decision on which subsystems can be islanded. We also implemented the Wiener filtering based reconstruction algorithm for WSS processes [7] in this example. The obtained topology is shown in Figure 5 by the network formed by the black as well as dashed blue edges. It is seen that in this case the reconstructed topology has edges connecting the sources and the loads. This illustrates that reconstruction procedures developed in earlier works for network of WSS time series data cannot be applied to networks of cyclostationary processes.

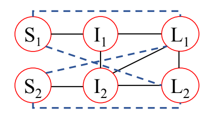

Fig. 5. Reconstructed topology using Wiener filtering.

VII. CONCLUSIONS

In this paper we demonstrate that non causal Wiener filtering is capable of recovering the kin topology of a LDG of cyclostationary processes. This is based on the fact that a cyclostationary process with period T has an equivalent T dimensional vector stationary process representation. This approach is restricted to integer periodicity of the second order statistics of the cyclostationary processes. This approach can also be applied to poly-periodic processes with the effective period being the LCM of all the periods present in the time series. It has been shown that Wiener filtering suggests "d-separation" of the sources and the loads by the inverters and this observation has also been justified from a power system dynamics viewpoint. It is important to note that Wiener filtering based reconstruction of WSS processes is not applicable to cyclostationary processeswhile the approach presented here can be applied to both the stationary and the cyclostationary scenario.

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